Research Article

$q$-Analogues of the Bernoulli and Genocchi Polynomials and the Srivastava-Pintér Addition Theorems

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The main purpose of this paper is to introduce and investigate a new class of generalized Bernoulli and Genocchi polynomials based on the $q$-integers. The $q$-analogues of well-known formulas are derived. The $q$-analogue of the Srivastava-Pintér addition theorem is obtained.

1. Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

The $q$-shifted factorial is defined by

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - qa^j), \quad n \in \mathbb{N}, \quad (a; q)_\infty = \prod_{j=0}^{\infty} \left(1 - qa^j\right), \quad |q| < 1, \quad a \in \mathbb{C}.
\]

(1.1)

The $q$-numbers and $q$-numbers factorial is defined by

\[
[a]_q = \frac{1 - q^a}{1 - q} \quad (q \neq 1); \quad [0]_q = 1; \quad [n]_q! = [1]_q [2]_q \cdots [n]_q \quad n \in \mathbb{N}, \quad a \in \mathbb{C},\quad (1.2)
\]
respectively. The $q$-polynomial coefficient is defined by

$$\begin{align*}
\left[\frac{n}{k}\right]_q &= \frac{(q; q)_n}{(q; q)_{n-k}(q; q)_k},
\end{align*}$$

(1.3)

The $q$-analogue of the function $(x + y)^n$ is defined by

$$(x + y)^n := \sum_{k=0}^{n} \left[\frac{n}{k}\right]_q q^{(1/2)(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0.\quad (1.4)$$

In the standard approach to the $q$-calculus two exponential function are used:

$$e_q(z) = \sum_{n=0}^{\infty} z^n / [n]_q! = \prod_{k=0}^{\infty} \left(1 - (1-q)q^k z\right), \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} q^{(1/2)n(n-1)} z^n / [n]_q! = \prod_{k=0}^{\infty} \left(1 + (1-q)q^k z\right), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.\quad (1.5)$$

From this form we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(qz),\quad (1.6)$$

where $D_q$ is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}.\quad (1.7)$$

The previous $q$-standard notation can be found in [1].

Carlitz has introduced the $q$-Bernoulli numbers and polynomials in [2]. Srivastava and Pintér proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [3]. They also gave some generalizations of these polynomials. In [4–6], Kim et al. investigated some properties of the $q$-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [7], Cenkci et al. gave the $q$-extension of Genocchi numbers in a different manner. In [5], Kim gave a new concept for the $q$-Genocchi numbers and polynomials. In [8], Simsek et al. investigated the $q$-Genocchi zeta function and $l$-function by using generating functions and Mellin transformation. We also recall the definitions of the $q$-Bernoulli and the $q$-Genocchi polynomials of higher order (see [2, 9–12]):

$$(-t)^n \sum_{n=0}^{\infty} \left[\frac{\alpha}{n}\right]_q \frac{q^{n+x} e^{[n+x]t}}{[n]_q!} = \sum_{n=0}^{\infty} B_{n,q}(\alpha) \frac{t^n}{n!},$$

(1.8)

$$\left(2t\right)^n \sum_{n=0}^{\infty} \left[\frac{\alpha}{n}\right]_q \frac{(-1)^n q^{n-x} e^{[n-x]t}}{[n]_q!} = \sum_{n=0}^{\infty} C_{n,q}(\alpha) \frac{t^n}{n!}.$$
We propose the following definitions. We define the $q$-Bernoulli and the $q$-Genocchi polynomials of higher order in two variables $x$ and $y$, using two $q$-exponential functions, which helps us easily prove some properties of these polynomials and $q$-analogue of the Srivastava and Pintér addition theorem.

*Definition 1.1.* The $q$-Bernoulli numbers $\mathcal{B}_{n,q}^{(a)}$ and polynomials $\mathcal{B}_{n,q}^{(a)}(x,y)$ in $x, y$ of order $\alpha$ are defined by means of the generating function:

\[
\left( \frac{t}{e_q(t) - 1} \right)^\alpha = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(a)} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi, \tag{1.9}
\]

\[
\left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.
\]

*Definition 1.2.* The $q$-Genocchi numbers $\mathcal{G}_{n,q}^{(a)}$ and polynomials $\mathcal{G}_{n,q}^{(a)}(x,y)$ in $x, y$ are defined by means of the generating functions:

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(a)} \frac{t^n}{[n]_q!}, \quad |t| < \pi, \tag{1.10}
\]

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.
\]

It is obvious that

\[
\begin{align*}
\mathcal{B}_{n,q}^{(a)} & = \mathcal{B}_{n,q}^{(a)}(0,0), & \lim_{q \to 1^{-1}} \mathcal{B}_{n,q}^{(a)}(x,y) & = B_n^{(a)}(x+y), & \lim_{q \to 1^{-1}} \mathcal{B}_{n,q}^{(a)} & = B_n^{(a)}, \\
\mathcal{G}_{n,q}^{(a)} & = \mathcal{G}_{n,q}^{(a)}(0,0), & \lim_{q \to 1^{-1}} \mathcal{G}_{n,q}^{(a)}(x,y) & = G_n^{(a)}(x+y), & \lim_{q \to 1^{-1}} \mathcal{G}_{n,q}^{(a)} & = G_n^{(a)}. \tag{1.11}
\end{align*}
\]

Here $B_n^{(a)}(x)$ and $E_n^{(a)}(x)$ denote the classical Bernoulli, and Genocchi polynomials of order $\alpha$ are defined by

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{tx} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!}, \quad \left( \frac{2}{e^t + 1} \right)^\alpha e^{tx} = \sum_{n=0}^{\infty} G_n^{(a)}(x) \frac{t^n}{n!}. \tag{1.12}
\]

The aim of the present paper is to obtain some results for the $q$-Genocchi polynomials (properties of the $q$-Bernoulli polynomials are studied in [13]). The $q$-analogue of well-known results, for example, Srivastava and Pintér [3], can be derived from these $q$-identities. It should be mentioned that probabilistic proofs the Srivastava-Pintér addition theorems were given recently in [14]. The formulas involving the $q$-Stirling numbers of the second kind, $q$-Bernoulli polynomials and $q$-Bernstein polynomials, are also given. Furthermore some special cases are also considered.

The following elementary properties of the $q$-Genocchi polynomials $\mathcal{G}_{n,q}^{(a)}(x,y)$ of order $\alpha$ are readily derived from Definition 1.2. We choose to omit the details involved.
Property 1.3. Special values of the $q$-Genocchi polynomials of order $\alpha$:

$$E_{n,q}^{(0)}(x,0) = x^n, \quad E_{n,q}^{(0)}(0,y) = q^{(1/2)n(n-1)}y^n.$$  \hfill (1.13)

Property 1.4. Summation formulas for the $q$-Genocchi polynomials of order $\alpha$:

$$E_{n,q}^{(a)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q E_{k,q}^{(a)}(x+y)^{n-k}, \quad E_{n,q}^{(a)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q E_{n-k,q}^{(a-1)}E_{k,q}(x,y),$$

$$G_{n,q}^{(a)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} G_{k,q}^{(a)}(x,0)y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q G_{k,q}^{(a)}(0,y)x^{n-k},$$ \hfill (1.14)

$$F_{n,q}^{(a)}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q F_{k,q}^{(a)}x^{n-k}, \quad F_{n,q}^{(a)}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} F_{k,q}^{(a)}y^{n-k}.$$  

Property 1.5. Difference equations:

$$E_{n,q}^{(a)}(1,y) + E_{n,q}^{(a)}(0,y) = 2[n]_q E_{n-1,q}^{(a-1)}(0,y),$$

$$E_{n,q}^{(a)}(x,0) + E_{n,q}^{(a)}(x,-1) = 2[n]_q E_{n-1,q}^{(a-1)}(x,-1).$$  \hfill (1.15)

Property 1.6. Differential relations:

$$D_{q,x}E_{n,q}^{(a)}(x,y) = [n]_q E_{n-1,q}^{(a)}(x,y), \quad D_{q,y}E_{n,q}^{(a)}(x,y) = [n]_q E_{n-1,q}^{(a)}(x,0).$$ \hfill (1.16)

Property 1.7. Addition theorem of the argument:

$$E_{n,q}^{(a+b)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q E_{n-k,q}^{(a)}(x,0)E_{k,q}^{(b)}(0,y).$$ \hfill (1.17)

Property 1.8. Recurrence relationships:

$$E_{n,q}^{(a)}\left(\frac{1}{m},y\right) + \sum_{k=0}^{n} \begin{bmatrix} n \atop k \end{bmatrix}_q \left(\frac{1}{m}-1\right)^{n-k} E_{q,k}^{(a)}(0,y) = 2[n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \atop k \end{bmatrix}_q \left(\frac{1}{m}-1\right)^{n-1-k} E_{k,q}^{(a-1)}(0,y).$$ \hfill (1.18)

2. Explicit Relationship between the $q$-Genocchi and the $q$-Bernoulli Polynomials

In this section we prove an interesting relationship between the $q$-Genocchi polynomials $E_{n,q}^{(a)}(x,y)$ of order $\alpha$ and the $q$-Bernoulli polynomials. Here some $q$-analogues of known results will be given. We also obtain new formulas and their some special cases in the following.
Theorem 2.1. For \( n \in \mathbb{N}_0 \), the following relationship

\[
\mathcal{G}_{n,q}^{(a)}(x, y) = \sum_{k=0}^{n} \frac{1}{m^{n-k}[k+1]_q} \left[ 2[k+1]_q \sum_{j=0}^{k} \frac{1}{j!} m^{k-j} \mathcal{G}^{(a-1)}_{j,q}(x, -1) \right. \\
\left. - \sum_{j=0}^{k+1} \frac{1}{j!} m^{k+1-j} \mathcal{G}^{(a)}_{j,q}(x, -1) - \mathcal{G}^{(a)}_{k+1,q}(x, 0) \right] \mathcal{B}_{n-k,q}(0, my).
\]

holds true between the \( q \)-Genocchi and the \( q \)-Bernoulli polynomials.

Proof. Using the following identity:

\[
\left( \frac{2t}{e_q(t) + 1} \right)^a e_q(tx) E_q(ty) = \left( \frac{2t}{e_q(t) + 1} \right)^a e_q(tx) \cdot \frac{e_q(t/m) - 1}{e_q(t/m) - 1} \cdot E_q \left( \frac{t}{m}my \right).
\]

we have

\[
\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(a)}(x, y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{[k]_q} \left[ m^{n-k} \mathcal{G}^{(a)}_{k,q}(x, 0) \right. \\
\left. - m \mathcal{G}^{(a)}_{n,q}(x, 0) \right] \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(0, my) \frac{t^n}{m^n[n]_q!} \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{1}{[k]_q} m^n \mathcal{G}^{(a)}_{n,q}(x, 0) \\
\sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \frac{1}{[k]_q} m^n \mathcal{G}^{(a)}_{n+1,q}(x, 0) \frac{t^n}{m^n[n+1]_q!} \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(0, my) \frac{t^n}{m^n[n]_q!} \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{m^{n-k}[k+1]_q} \left[ \sum_{j=0}^{k+1} \frac{1}{j!} m^j \mathcal{G}^{(a)}_{j,q}(x, 0) \\
- m \mathcal{G}^{(a)}_{k+1,q}(x, 0) \right] \mathcal{B}_{n-k,q}(0, my) \frac{t^n}{[n]_q!}.
\]

It remains to use Property 1.8.
Since $\mathcal{G}^{(a)}_{n,q}(x,y)$ is not symmetric with respect to $x$ and $y$, we can prove a different form of the previously mentioned theorem. It should be stressed out that Theorems 2.1 and 2.2 coincide in the limiting case when $q \to 1^{-}$.

**Theorem 2.2.** For $n \in \mathbb{N}_0$, the following relationship

$$
\mathcal{G}^{(a)}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} q^{k} \left[ \frac{1}{m^{n-k-1}[k+1]_q} \right] 2^{[k+1]}_q \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \\ \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{k-j}_q \mathcal{G}^{(a-1)}_{j,q}(0,y) \\
- \sum_{j=0}^{k+1} \left[ \frac{k+1}{j} \right]_q \left( \frac{1}{m} - 1 \right)^{k+1-j}_q \mathcal{G}^{(a)}_{j,q}(0,y) - \mathcal{G}^{(a)}_{k+1,q}(0,y) \\
\times \mathcal{B}_{n-k,q}(mx,0)
$$

holds true between the $q$-Genocchi and the $q$-Bernoulli polynomials.

**Proof.** The proof is based on the following identity:

$$
\left( \frac{2t}{e_q(t)+1} \right)^a e_q(tx)E_q(ty) = \left( \frac{2t}{e_q(t)+1} \right)^a E_q(ty) \cdot e_q(t/m) - \frac{1}{t} \cdot e_q(t/m) - 1 \cdot e_q \left( \frac{t}{m} mx \right).
$$

(2.5)

Next we discuss some special cases of Theorems 2.1 and 2.2. By noting that

$$
\mathcal{G}^{(0)}_{j,q}(0,y) = q^{(1/2)(j-1)}^j, \quad \mathcal{G}^{(0)}_{j,q}(x,-1) = (x-1)^j_q,
$$

we deduce from Theorems 2.1 and 2.2 Corollary 2.3 below.

**Corollary 2.3.** For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship

$$
\mathcal{G}_{n,q}(x,y) = \sum_{k=0}^{n} \binom{n}{k} q^{k} \left[ \frac{1}{m^{n-k-1}[k+1]_q} \right] 2^{[k+1]}_q \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \\ \end{array} \right]_q \left( \frac{1}{m} - 1 \right)^{k-j}_q q^{(1/2)(j-1)}^j y^j \\
- \sum_{j=0}^{k+1} \left[ \frac{k+1}{j} \right]_q \left( \frac{1}{m} - 1 \right)^{k+1-j}_q \mathcal{G}_{j,q}(0,y) - \mathcal{G}_{k+1,q}(0,y) \\
\times \mathcal{B}_{n-k,q}(mx,0),
$$

(2.6)
\[ \mathcal{G}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}[k+1]_q} \left[ 2[k+1]_q \sum_{j=0}^{k} \binom{k}{j}_q \frac{1}{m^j} (x-1)_q^j \right. \\
\left. - \sum_{j=0}^{k+1} \binom{k+1}{j}_q \frac{1}{m^{k+1-j}} \mathcal{G}_{j,q}(x, -1) - \mathcal{G}_{k+1,q}(x, 0) \right] \times \mathcal{B}_{n-k,q}(0, my) \]

holds true between the \(q\)-Bernoulli polynomials and \(q\)-Euler polynomials.

**Corollary 2.4.** For \(n \in \mathbb{N}_0\), \(m \in \mathbb{N}\) the following relationship holds true:

\[ G_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \frac{2}{k+1} \left( (k+1)y^k - G_{k+1,q}(y) \right) B_{n-k}(x), \] (2.8)

\[ G_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^{n-k-1}(k+1)} \left[ 2(k+1)G_k \left( y + \frac{1}{m} - 1 \right) \right. \\
\left. - G_{k+1} \left( y + \frac{1}{m} - 1 \right) - G_{k+1}(y) \right] B_{n-k,q}(mx) \] (2.9)

between the classical Genocchi polynomials and the classical Bernoulli polynomials.

Note that the formula (2.9) is new for the classical polynomials.

In terms of the \(q\)-Genocchi numbers \(\mathcal{G}^{(a)}_{k,q}\), by setting \(y = 0\) in Theorem 2.1, we obtain the following explicit relationship between the \(q\)-Genocchi polynomials \(\mathcal{G}^{(a)}_{k,q}\) of order \(\alpha\) and the \(q\)-Bernoulli polynomials.

**Corollary 2.5.** The following relationship holds true:

\[ \mathcal{G}^{(a)}_{n,q}(x, 0) = \sum_{k=0}^{n} \binom{n}{k}_q \frac{1}{m^{n-k-1}[k+1]_q} \left[ 2[k+1]_q \sum_{j=0}^{k} \binom{k}{j}_q \left( \frac{1}{m} - 1 \right)^{k-j} \mathcal{G}^{(a-1)}_{j,q} \right. \\
\left. - \sum_{j=0}^{k+1} \binom{k+1}{j}_q \left( \frac{1}{m} - 1 \right)^{k+1-j} \mathcal{G}^{(a)}_{j,q} - \mathcal{G}^{(a)}_{k+1,q} \right] \mathcal{B}_{n-k,q}(mx, 0). \] (2.10)

**Corollary 2.6.** For \(n \in \mathbb{N}_0\) the following relationship holds true:

\[ \mathcal{G}_{n,q}(x, y) = \sum_{k=0}^{n} \binom{n}{k}_q \frac{2}{[k+1]_q} \left[ k+1]_q y^{(1/2)(k-1)} - \mathcal{G}_{k+1,q}(0, y) \right] \mathcal{B}_{n-k,q}(x, 0). \] (2.11)
Corollary 2.7. For $n \in \mathbb{N}_0$ the following relationship holds true:

$$G_{n,q}(x,0) = -\sum_{k=0}^{n} \binom{n}{k} \frac{2}{[k+1]_q} G_{k+1,q} B_{n-k,q}(x,0),$$

$$G_{n,q} = -\sum_{k=0}^{n} \binom{n}{k} \frac{2}{[k+1]_q} G_{k+1,q} B_{n-k,q}.$$

(2.12)

References


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