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Research Article

Stability and Local Hopf Bifurcation for a Predator-Prey Model with Delay

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A predator-prey system with disease in the predator is investigated, where the discrete delay τ is regarded as a parameter. Its dynamics are studied in terms of local analysis and Hopf bifurcation analysis. By analyzing the associated characteristic equation, it is found that Hopf bifurcation occurs when τ crosses some critical values. Using the normal form theory and center manifold argument, the explicit formulae which determine the stability, direction, and other properties of bifurcating periodic solutions are derived.

1. Introduction

Many models in ecology can be formulated as system of differential equations with time delays. The effect of the past history on the stability of system is also an important problem in population biology. Recently, the properties of periodic solutions arising from the Hopf bifurcation have been considered by many authors [1–4].

May [5] first proposed and discussed the delayed predator-prey system

$$\begin{aligned} \frac{dx}{dt} &= x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t)\right], \\ \frac{dy}{dt} &= y(t)\left[-r_2 + a_{21}y(t) - a_{22}y(t)\right], \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ can be interpreted as the population densities of prey and predator at time t, respectively; $\tau \geq 0$ is the feedback time delay of the prey to the growth of the species itself; $r_1 > 0$ denotes the intrinsic growth rate of the prey, and $r_2 > 0$ denotes the death rate of the predator; the parameter $a_{ij}(i, j = 1, 2)$ are all positive constants. System (1,1) shows that,

in the absence of predator species, the prey species are governed by the well-known delayed logistic equation $dx/dt = x(t)[r_1 - a_{11}x(t-\tau)]$ and the predator species will decrease in the absence of the prey species.There has been an extensive literature dealing with system (1,1) or the system similar to (1.1), regarding boundedness of solutions, persistence, local and global stabilities of equilibria, and existence of nonconstant periodic solutions [6–9].

Recently, Faria [7] investigated the stability and Hopf bifurcation of the following system with instantaneous feedback control and two different discrete delays:

$$\begin{aligned} \frac{dx}{dt} &= x(t)\left[r_1 - a_{11}x(t) - a_{12}y(t-\tau_1)\right] \\ \frac{dy}{dt} &= y(t)\left[-r_2 + a_{21}x(t-\tau_2) - a_{22}y(t)\right], \end{aligned} \tag{1.2}$$

where $\tau_1 > 0$ and $\tau_2 > 0$. But, as pointed out by Kuang [8], in view of the fact that in real situations, instantaneous responses are rare, and thus, more realistic models should consist of delay differential equations without instantaneous feedbacks. Based on this idea, in the present paper, we combine the model (1.1) and (1.2) and consider the following delayed prey-predator system with a single delay:

$$\begin{aligned} \frac{dX}{dt} &= X(t)\left[r_1 - r_2X(t-\tau) - pS(t-\tau)\right] \\ \frac{dS}{dt} &= S(t)\left[-c_1 + kpX(t-\tau) - \sigma I(t)\right] + \gamma I(t) \\ \frac{dI}{dt} &= I(t)\left[\sigma S(t) - c_2 - \gamma\right], \end{aligned} \tag{1.3}$$

where X, S, I denote, respectively, the population of prey species, susceptible predator species and infected predator species. In addition, the coefficients $r1$, $r2$, p, k, σ, c_1, c_2 in model (1.3) are all positive constants and their ecological meaning are interpreted as follows: r_1 denotes the intrinsic growth rate of prey and r_1/r_2 denotes the carrying capacity of prey; p, k, c_1 and c_2 represent the predating coefficient of predator to prey, absorbing rate of predator to prey, and the death rate of susceptible and infected predator, respectively.

The main purpose of this paper is to investigate the effects of the delay on the dynamics of model (1.3) with the following initial conditions:

$$\begin{gathered} X(t) = \phi_1(t) > 0, \qquad S(t) = \phi_2(t) > 0, \qquad I(t) = \phi_3(t) > 0, \quad t \in [-\tau, 0] \\ \left(\phi_1(t), \phi_2(t), \phi_3(t)\right) \in C\left([-\tau, 0], \Re^3_{0,+}\right), \end{gathered} \tag{1.4}$$

where $\Re^3_{0,+} = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$. We will take the delay τ as the bifurcation parameter and show that when τ passes through a certain critical value, the positive equilibrium loses its stability and a Hopf bifurcation will take place. Furthermore, when τ takes a sequence of critical values containing the above critical value, the positive equilibrium of system (1.3) will undergo a Hopf bifurcation. In particular, by using the normal form theory and the center manifold, the formulae determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are also obtained.

The organization of this paper is as follows. In Section 2, we discuss the stability of the positive solutions and the existence of the Hopf bifurcations. In Section 3, the direction of the Hopf bifurcation and the stability of bifurcated periodic solutions are obtained by using the normal form theory and the center manifold theorem. In Section 4, we do some numerical simulations to validate our theoretical results.

2. Stability of Positive Equilibrium and Hopf Bifurcation

System (1.3) has a unique positive equilibrium $E^+(X^*, S^*, I^*)$ provided that the condition

(H_1) $\sigma r_1 > (c_2+\gamma)p, kp[\sigma r_1 - p(c_2+\gamma)] > \sigma c_1 r_2$

is satisfied, where

$$X^* = \frac{\sigma r_1 - p(c_2+\gamma)}{\sigma r_2}, \qquad S^* = \frac{c_2+\gamma}{\sigma}, \qquad I^* = (c_2+\gamma)\frac{kp[\sigma r_1 - p(c_2+\gamma)] - \sigma c_1 r_2}{\sigma^2 r_2 (c_2+2\gamma)}. \tag{2.1}$$

Linearizing system (1.2) at E^+ gives the following linear system:

$$\begin{aligned} \frac{dU}{dt} &= -r_2 X^* U(t-\tau) - pX^* V(t-\tau) \\ \frac{dV}{dt} &= kpS^* U(t-\tau) - (c_1 + \sigma I^* - kpX^*)V(t) - (\sigma S^* - \gamma)W \\ \frac{dW}{dt} &= \sigma I^* V. \end{aligned} \tag{2.2}$$

The characteristic matrix of this system (2.2) is

$$\begin{pmatrix} \lambda + r_2 X^* e^{-\lambda\tau} & pX^* e^{-\lambda\tau} & 0 \\ -kpS^* e^{-\lambda\tau} & \lambda + (c_1 + \sigma I^* - kpX^*) & \sigma S^* - \gamma \\ 0 & -\sigma I^* & \lambda \end{pmatrix}. \tag{2.3}$$

Thus, the characteristic equation of system (2.2) is given by

$$\begin{aligned} &\lambda^3 + (c_1 + \sigma I^* - kpX^*)\lambda^2 + \left(\sigma^2 S^* I^* - \sigma\gamma I^*\right)\lambda + r_2 X^* e^{-\lambda\tau}\lambda^2 + (c_1 + \sigma I^* - kpX^*)\lambda r_2 X^* e^{-\lambda\tau} \\ &\quad + \left(\sigma^2 S^* I^* - \sigma\gamma I^*\right) r_2 X^* e^{-\lambda\tau} + kp^2 S^* X^* \lambda e^{-2\lambda\tau} = 0. \end{aligned} \tag{2.4}$$

Let

$$\begin{gathered} d_1 = c_1 + \sigma I^* - kpX^*, \qquad d_2 = \sigma^2 S^* I^* - \sigma\gamma I^*, \qquad d_3 = r_2 X^*, \qquad d_4 = (c_1 + \sigma I^* - kpX^*) r_2 X^*, \\ d_5 = \left(\sigma^2 S^* I^* - \sigma\gamma I^*\right) r_2 X^*, \qquad d_6 = kp^2 S^* X^*. \end{gathered} \tag{2.5}$$

Then we rewrite (2.4) as:

$$\left(\lambda^3 + d_1\lambda^2 + d_2\lambda\right)e^{\lambda\tau} + d_3\lambda^2 + d_4\lambda + d_5 + d_6\lambda e^{-\lambda\tau} = 0. \tag{2.6}$$

Obviously, $i\omega(\omega > 0)$ is a root of (2.6) if and only if ω satisfies

$$\left(-i\omega^3 - d_1\omega^2 + d_2 i\omega\right)(\cos\omega\tau + i\sin\omega\tau) - d_3\omega^2 + d_4 i\omega + d_5 + d_6 i\omega(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.7}$$

Separating the real and imaginary parts, we have

$$\begin{gathered} -d_1\omega^2\cos\omega\tau + \left(\omega^3 - d_2\omega + d_6\omega\right)\sin\omega\tau = d_3\omega^2 - d_5 \\ \left(-\omega^3 + d_2\omega + d_6\omega\right)\cos\omega\tau - d_1\omega^2\sin\omega\tau = -d_4\omega. \end{gathered} \tag{2.8}$$

By calculating, we have obtained

$$\begin{gathered} \sin\omega\tau = \frac{d_3\omega^5 + (d_1d_4 - d_2d_3 - d_3d_6 - d_5)\omega^3 + (d_2d_5 + d_5d_6)\omega}{\omega^6 + (d_1^2 - 2d_2)\omega^4 + (d_2^2 - d_6^2)\omega^2} \\ \cos\omega\tau = \frac{(d_4 - d_1d_3)\omega^4 + (d_1d_5 + d_4d_6 - d_2d_4)\omega^2}{\omega^6 + (d_1^2 - 2d_2)\omega^4 + (d_2^2 - d_6^2)\omega^2}. \end{gathered} \tag{2.9}$$

Let $e_1 = d_1^2 - 2d_2$, $e_2 = d_2^2 - d_6^2$, $e_3 = d_3$, $e_4 = d_1d_4 - d_2d_3 - d_3d_6 - d_5$, $e_5 = d_2d_5 + d_5d_6$, $e_6 = d_4 - d_1d_3$, $e_7 = d_1d_5 + d_4d_6 - d_2d_4$. Then $\sin\omega\tau$, $\cos\omega\tau$ can be written as

$$\sin\omega\tau = \frac{\omega\left(e_3\omega^4 + e_4\omega^2 + e_5\right)}{\omega^6 + e_1\omega^4 + \mathrm{e}_2\omega^2} \tag{2.10}$$

$$\cos\omega\tau = \frac{e_6\omega^4 + e_7\omega^2}{\omega^6 + e_1\omega^4 + e_2\omega^2}. \tag{2.11}$$

As $\sin^2\omega\tau + \cos^2\omega\tau = 1$, so we have

$$\omega^{10} + f_1\omega^8 + f_2\omega^6 + f_3\omega^4 + f_4\omega^2 + f_5 = 0, \tag{2.12}$$

where $f_1 = 2e_1 - e_3^2$, $f_2 = e_1^2 + 2e_2 - 2e_3e_4 - e_6^2$, $f_3 = 2e_1e_2 - 2e_3e_5 - 2e_6e_7 - e_4^2$, $f_4 = e_2^2 - 2e_4e_5 - e_7^2$, $f_5 = -e_5^2$.

Denote $z = \omega^2$, then (2.12) becomes

$$z^5 + f_1z^4 + f_2z^3 + f_3z^2 + f_4z + f_5 = 0. \tag{2.13}$$

Let

$$G(z) = z^5 + f_1 z^4 + f_2 z^3 + f_3 z^2 + f_4 z + f_5. \tag{2.14}$$

Since $\lim_{z\to\infty} G(z) = +\infty$ and $f_5 < 0$, then we can get the following conclusion.

(H_2) Equation (2.13) has at least one positive real root.

Without loss of generality, we assume that it has five positive roots, defined by z_1, z_2, z_3, z_4, z_5, respectively. Then (2.13) has five positive roots

$$\omega_1 = \sqrt{z_1}, \qquad \omega_2 = \sqrt{z_2}, \qquad \omega_3 = \sqrt{z_3}, \qquad \omega_4 = \sqrt{z_4}, \qquad \omega_5 = \sqrt{z_5}. \tag{2.15}$$

By (2.11), we have

$$\cos \omega_k \tau = \frac{e_6 \omega_k^2 + e_7}{\omega_k^4 + e_1 \omega_k^2 + e_2}. \tag{2.16}$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k} \left\{ \arccos\left(\frac{e_6 \omega_k^2 + e_7}{\omega_k^4 + e_1 \omega_k^2 + e_2} \right) + 2j\pi \right\}, \tag{2.17}$$

where $k = 1, \ldots, 5$; $j = 0, 1, \ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of (2.6) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,\ldots,5\}} \left\{ \tau_k^{(0)} \right\}, \qquad \omega_0 = \omega_{k_0}. \tag{2.18}$$

Note that when $\tau = 0$, (2.6) becomes

$$\lambda^3 + b\lambda^2 + (a + d)\lambda + c = 0. \tag{2.19}$$

By Routh-Hurwitz criterion, we know that all the roots of (2.19) have negative real parts, that is, the positive equilibrium E^+ is locally asymptotically stable for $\tau = 0$.

In order to give the main results, it is necessary to make the following assumption:

(H_3) $[d(\operatorname{Re}\lambda)/d\tau]_{\tau=\tau_0} \neq 0$.

Differentiating two sides of (2.6) in respect to τ, we get

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{(3\lambda^2+2d_1\lambda+d_2\lambda)e^{\lambda\tau}+\tau(\lambda^3+d_1\lambda^2+d_2\lambda)e^{\lambda\tau}+2d_3\lambda+d_4+d_6e^{-\lambda\tau}-d_6\lambda\tau e^{-\lambda\tau}}{-\lambda(\lambda^3+d_1\lambda^2+d_2\lambda)e^{\lambda\tau}+d_6\lambda^2e^{-\lambda\tau}} \\
&= \frac{(3\lambda^2+2d_1\lambda+d_2\lambda)e^{\lambda\tau}+2d_3\lambda+d_4+d_6e^{-\lambda\tau}}{-\lambda(\lambda^3+d_1\lambda^2+d_2\lambda)e^{\lambda\tau}+d_6\lambda^2e^{-\lambda\tau}}-\frac{\tau}{\lambda} \\
&= \frac{(3\lambda^2+2d_1\lambda+d_2\lambda)e^{\lambda\tau}+2d_3\lambda+d_4+d_6e^{-\lambda\tau}}{d_3\lambda^3+d_4\lambda^2+d_5\lambda+2d_6\lambda^2e^{-\lambda\tau}}-\frac{\tau}{\lambda} \\
&= \frac{(-3\omega^2+d_2+d_6)\cos\omega\tau-2d_1\omega\sin\omega\tau+d_4}{(-d_4\omega^2-2d_6\omega^2\cos\omega\tau)+i(-d_3\omega^3+d_5\omega+2d_6\omega^2\sin\omega\tau)} \\
&\quad+\frac{[(-3\omega^2+d_2-d_6)\sin\omega\tau+2d_1\omega\cos\omega\tau+2d_3\omega]}{(-d_4\omega^2-2d_6\omega^2\cos\omega\tau)+i(-d_3\omega^3+d_5\omega+2d_6\omega^2\sin\omega\tau)}.
\end{aligned} \tag{2.20}$$

Let

$$Q=\left(-d_4\omega^2-2d_6\omega^2\cos\omega\tau\right)^2+\left(-d_3\omega^3+d_5\omega+2d_6\omega^2\sin\omega\tau\right)^2>0 \tag{2.21}$$

$$\begin{aligned}
&Q\,\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&\quad=\left[\left(-3\omega^2+d_2+d_6\right)\cos\omega\tau-2d_1\omega\sin\omega\tau+d_4\right]\left[-d_4\omega^2-2d_6\omega^2\cos\omega\tau\right] \\
&\quad+\left[\left(-3\omega^2+d_2-d_6\right)\sin\omega\tau+2d_1\omega\cos\omega\tau+2d_3\omega\right]\left[-d_3\omega^3+d_5\omega+2d_6\omega^2\sin\omega\tau\right].
\end{aligned} \tag{2.22}$$

Noting that

$$\mathrm{sgn}\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right]_{\tau=\tau_0}=\mathrm{sgn}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0}. \tag{2.23}$$

Now, we can employ a result from Ruan and Wei [10] to analyze (2.6), which is, for the convenience of the reader, stated as follows.

Lemma 2.1 (see [2]). *Consider the exponential polynomial*

$$\begin{aligned}
P\left(\lambda,e^{-\lambda\tau_1},\ldots,e^{-\lambda\tau_m}\right) &= \lambda^n+p_1^{(0)}\lambda^{n-1}+\cdots+p_{n-1}^{(0)}\lambda+p_n^{(0)}+\left[p_1^{(1)}\lambda^{n-1}+\cdots+p_{n-1}^{(1)}\lambda+p_n^{(1)}\right]e^{-\lambda\tau_1} \\
&\quad+\cdots+\left[p_1^{(m)}\lambda^{n-1}+\cdots+p_{n-1}^{(m)}\lambda+p_n^{(m)}\right]e^{-\lambda\tau_m},
\end{aligned} \tag{2.24}$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ *and* $p_j^{(i)}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, m)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Form Lemma 2.1, it is easy to obtain the following theorem.

Theorem 2.2. *Suppose the condition,* (H_1), (H_2), *and* (H_3) *are satisfied, then one has the following results:*

(i) *if* $\tau \in [0, \tau_0)$, *then the positive equilibrium* E^+ *of* (1.2) *is locally asymptotically stable and unstable when* $\tau > \tau_0$,

(ii) *system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium* E^+ *when* $\tau = \tau_k$ $(k = 0, 1, 2, \ldots)$, *where* τ_k *is defined by* (2.17).

3. Stability and Direction of Hopf Bifurcation

In this section, we will derive the explicit formulae determining the properties of the Hopf bifurcation at the critical value using the normal form theory and center manifold theorem introduced by Hassard et al. [11].

Without loss of generality, let $\tau = \tau_k + \mu$, where τ_k is defined by (2.17), $\mu \in R$, then system (1.3) can be rewritten as

$$u'(t) = L_\mu(u_t) + f(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T = (U(\tau t), V(\tau t), W(\tau t))^T \in R^3$, $u_t(\theta) = u(t + \theta)$ and $L_\mu : C \to R^3$, $f : R \times C \to R^3$ are given, respectively, by

$$\begin{aligned} L_\mu(u_t) = {} & (\tau_k + \mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & kpX^* - c_1 - \sigma I^* & -\sigma S^* + \gamma \\ 0 & \sigma I^* & 0 \end{pmatrix}\begin{pmatrix} u_{1t}(0) \\ u_{2t}(0) \\ u_{3t}(0) \end{pmatrix} \\ & + (\tau_k + \mu)\begin{pmatrix} -r_2X^* & -pX^* & 0 \\ kpS^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{1t}(-1) \\ u_{2t}(-1) \\ u_{3t}(-1) \end{pmatrix}, \end{aligned} \tag{3.2}$$

$$f(\mu, u_t) = (\tau_k + \mu)\begin{pmatrix} -r_2u_{1t}(0)u_{1t}(-1) - pu_{1t}(0)u_{2t}(-1) \\ kpu_{2t}(0)u_{1t}(-1) - \sigma u_{2t}(0)u_{3t}(0) \\ \sigma u_{2t}(0)u_{3t}(0) \end{pmatrix}. \tag{3.3}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in \left([-1, 0], R^3\right). \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_k+\mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & kpX^* - c_1 - \sigma I^* & -\sigma S^* + \gamma \\ 0 & \sigma I^* & 0 \end{pmatrix}\delta(\theta) - (\tau_k+\mu)\begin{pmatrix} -r_2X^* & -pX^* & 0 \\ kpS^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\delta(\theta+1), \tag{3.5}$$

where δ denote the Dirac delta function. For $\phi \in C([-1,0],R^3)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases}$$
$$R(\mu)(\phi) = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{3.6}$$

Then system (3.1) is equivalent to

$$u'(t) = A(\mu)u_t + R(\mu)u_t. \tag{3.7}$$

For $\psi \in C([0,1],(R^3)^*)$, define

$$A^* = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \psi(-t)d\eta(t,0), & s = 0 \end{cases} \tag{3.8}$$

and a bilinear inner product

$$\langle \psi(s),\phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.9}$$

where $\eta(\theta) = \eta(\theta,0)$.

Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of A^*. We first need to compute the eigenvectors of $A(0)$ and A^* corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively.

Suppose $q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_k\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_k$, then $A(0)q(\theta) = i\omega_0\tau_k q(\theta)$. It follows from the definition of $A(0)$ and (3.2), (3.4) and (3.5), we have

$$\tau_0\begin{pmatrix} i\omega_0 + r_2X^*e^{-i\omega_0\tau_0} & pX^*e^{-i\omega_0\tau_0} & 0 \\ -kpX^*e^{-i\omega_0\tau_0} & i\omega_0 - kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ 0 & -\sigma I^* & i\omega_0 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

For $q(-1) = q(0)e^{-i\omega_0\tau_k}$, then we obtain

$$\begin{aligned} a_1 &= \frac{i\omega_0 + r_2X^*e^{-i\omega_0\tau_0}}{pX^*e^{-i\omega_0\tau_0}}, \\ a_2 &= \frac{\sigma I^*\left(i\omega_0 + r_2X^*e^{-i\omega_0\tau_0}\right)}{i\omega_0\left(pX^*e^{-i\omega_0\tau_0}\right)}. \end{aligned} \tag{3.11}$$

On the other hand, suppose that $q^*(s) = J(1, a_1^*, a_2^*)e^{i\omega_0\tau_k S}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau_k$, by the similar method, we have

$$\begin{aligned} a_1^* &= \frac{-i\omega_0 + r_2X^*e^{i\omega_0\tau_0}}{kpX^*e^{i\omega_0\tau_0}}, \\ a_2^* &= \frac{-\sigma I^*\left(-i\omega_0 + r_2X^*e^{i\omega_0\tau_0}\right)}{i\omega_0\left(kpX^*e^{i\omega_0\tau_0}\right)}. \end{aligned} \tag{3.12}$$

In order to assure $\langle q^*(s), q(\theta)\rangle = 1$, we need to determine the value of J. From (3.9), we have

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \overline{J}\left(1, \overline{a}_1^*, \overline{a}_2^*\right)(1, a_1, a_2)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{J}\left(1, \overline{a}_1^*, \overline{a}_2^*\right)e^{-i\omega_0\tau_k}(\xi - \theta)d\eta(\theta)(1, a_1, a_2)^T e^{i\omega_0\tau_k\xi}d\xi \\ &= \overline{J}\left\{1 + a_1\overline{a}_1^* + a_2\overline{a}_2^* - \int_{-1}^{0}\left(1, \overline{a}_1^*, \overline{a}_2^*\right)\theta e^{i\omega_0\tau_k\theta}d\eta(\theta)(1, a_1, a_2)^T\right\} \\ &= \overline{J}\left\{1 + a_1\overline{a}_1^* + a_2\overline{a}_2^* + \tau_k\left(-r_2X^* - a_1pX^* + kpS^*\overline{a}_1^*\right)e^{-i\omega_0\tau_k}\right\}. \end{aligned} \tag{3.13}$$

Therefore, we can choose J as

$$\overline{J} = \frac{1}{1 + a_1\overline{a}_1^* + a_2\overline{a}_2^* + \tau_k\left(-r_2X^* - a_1pX^* + kpS^*\overline{a}_1^*\right)e^{-i\omega_0\tau_k}}. \tag{3.14}$$

Next we will compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (3.1) with $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, W(t,\theta) = u_t(\theta) - z(t)q(\theta) - \overline{z}(t)q^*(\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.15}$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t),\overline{z}(t),\theta), \tag{3.16}$$

where

$$W(z(t),\overline{z}(t),\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.17}$$

z and $\overline{z}$ are local coordinates of center manifold C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if u_t is real. We consider only real solutions. For solution $u_t \in C_0$ of (3.7), since $\mu = 0$, we have

$$\begin{aligned} z'(t) &= \langle q^*, u'(t)\rangle = \langle q^*, Au_t + Ru_t\rangle = \langle q^*, Au_t\rangle + \langle q^*, Ru_t\rangle = \langle A^*q^*, u_t\rangle + \langle q^*, Ru_t\rangle \\ &= i\omega_0\tau_k z + \overline{q}^*(0)f(0), W(z,\overline{z},0) + 2\operatorname{Re}\{zq(0)\} \triangleq i\omega_0\tau_k z + \overline{q}^*(0)f_0(z,\overline{z}). \end{aligned} \tag{3.18}$$

We rewrite this equation as

$$z'(t) = i\omega_0\tau_k z(t) + g(z,\overline{z}), \tag{3.19}$$

where

$$g(z,\overline{z}) = \overline{q}^*(0)f_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.20}$$

Noticing $u_t(\theta) = W(t,\theta) + zq(\theta) + \overline{z}\,\overline{q}(\theta)$ and $q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_k\theta}$, we have

$$\begin{aligned} u_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ u_{2t}(0) &= a_1 z + \overline{a}_1\overline{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ u_{3t}(0) &= a_2 z + \overline{a}_2\overline{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + \cdots, \\ u_{1t}(-1) &= e^{-i\omega_0\tau_k}z + e^{i\omega_0\tau_k}\overline{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots, \\ u_{2t}(-1) &= e^{-i\omega_0\tau_k}a_1 z + e^{i\omega_0\tau_k}\overline{a}_1\overline{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots. \end{aligned} \tag{3.21}$$

It follows together with (3.3), that

$$
\begin{aligned}
g(z,\overline{z}) &= \overline{q}^{*}(0)f_0(z,\overline{z}) = \overline{q}^{*}(0)f_0(0,u_t).\\
&= \tau_k\overline{J}\left(1,\overline{a}_1^{*},\overline{a}_2^{*}\right)\begin{pmatrix} -r_2u_{1t}(0)u_{1t}(-1)-pu_{1t}(0)u_{2t}(-1)\\ kpu_{2t}(0)u_{1t}(-1)-\sigma u_{2t}(0)u_{3t}(0)\\ \sigma u_{2t}(0)u_{3t}(0)\end{pmatrix}\\
&= -\tau_k\overline{J}\left\{r_2\left[z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\right.\\
&\qquad *\left[e^{-i\omega_0\tau_k}z+e^{i\omega_0\tau_k}\overline{z}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+\cdots\right]\\
&\qquad +p\left[z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\\
&\qquad \left.*\left[e^{-i\omega_0\tau_k}a_1z+e^{-i\omega_0\tau_k}\overline{a}_1\overline{z}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\overline{z}+W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2}+\cdots\right]\right\}\\
&\quad +\tau_k\overline{J}\overline{a}_1^{*}\left\{kp\left[a_1z+\overline{a}_1\overline{z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\right.\\
&\qquad *\left[e^{-i\omega_0\tau_k}z+e^{i\omega_0\tau_k}\overline{z}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+\cdots\right]\\
&\qquad -\sigma\left[a_1z+\overline{a}_1\overline{z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\\
&\qquad \left.*\left[a_2z+\overline{a}_2\overline{z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\right\}\\
&\quad +\tau_k\overline{J}\overline{a}_2^{*}\left\{\sigma\left[a_1z+\overline{a}_1\overline{z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\right.\\
&\qquad \left.*\left[a_2z+\overline{a}_2\overline{z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+\cdots\right]\right\}.
\end{aligned}
\tag{3.22}
$$

Comparing the coefficients with (3.20), we have

$$
\begin{aligned}
g_{20} &= 2\tau_k\overline{J}\left[-r_2e^{-i\omega_0\tau_k}-pa_1e^{-i\omega_0\tau_k}+\overline{a}_1^{*}kpa_1e^{-i\omega_0\tau_k}-\overline{a}_1^{*}\sigma a_1a_2+\overline{a}_2^{*}\sigma a_1a_2\right],\\
g_{11} &= \tau_k\overline{J}\left\{e^{-i\omega_0\tau_k}\left(-r_2-pa_1+\overline{a}_1^{*}kp\overline{a}_1\right)+e^{i\omega_0\tau_k}\left(-r_2-p\overline{a}_1+\overline{a}_1^{*}kpa_1\right)\right.\\
&\qquad \left.-\overline{a}_1^{*}\sigma a_1\overline{a}_2-\overline{a}_1^{*}\sigma\overline{a}_1a_2+\overline{a}_2^{*}\sigma a_1\overline{a}_2+\overline{a}_2^{*}\sigma\overline{a}_1a_2\right\},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
g_{02} &= 2\tau_k \overline{J}\left[-r_2 e^{i\omega_0\tau_k} - p\overline{a}_1 e^{i\omega_0\tau_k} + \overline{a}_1^* kp\overline{a}_1 e^{i\omega_0\tau_k} - \overline{a}_1^*\sigma\overline{a}_1\overline{a}_2 + \overline{a}_2^*\sigma\overline{a}_1\overline{a}_2\right],\\
g_{21} &= \tau_k\overline{J}\Big\{-r_2\left(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1) + W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + 2W_{11}^{(1)}(0)e^{-i\omega_0\tau_k}\right)\\
&\quad - p\left(2W_{11}^{(2)}(-1) + W_{20}^{(2)}(-1) + \overline{a}_1 W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + 2a_1 W_{11}^{(1)}(0)e^{-i\omega_0\tau_k}\right)\\
&\quad + \overline{a}_1^* kp\left(2a_1 W_{11}^{(1)}(-1) + \overline{a}_1 W_{20}^{(1)}(-1) + W_{20}^{(2)}(0)e^{i\omega_0\tau_k} + 2W_{11}^{(2)}(0)e^{-i\omega_0\tau_k}\right)\\
&\quad - \overline{a}_1^*\sigma\left(2a_1 W_{11}^{(3)}(0) + \overline{a}_1 W_{20}^{(3)}(0) + \overline{a}_2 W_{20}^{(2)}(0) + 2a_2 W_{11}^{(2)}(0)\right)\\
&\quad + \overline{a}_2^*\sigma\left(2a_1 W_{11}^{(3)}(0) + \overline{a}_1 W_{20}^{(3)}(0) + \overline{a}_2 W_{20}^{(2)}(0) + 2a_2 W_{11}^{(2)}(0)\right)\Big\}.
\end{aligned}
\tag{3.24}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we need to determine them.

From (3.7) and (3.15), we have

$$
\begin{aligned}
W' &= u_t' - z'q - \overline{z}'\overline{q}\\
&= \begin{cases} A(0)W - 2\operatorname{Re}\{\overline{q}^*(0)f_0 q(\theta)\}, & \theta \in [-1,0)\\ A(0)W - 2\operatorname{Re}\{\overline{q}^*(0)f_0 q(0)\} + f_0, & \theta = 0 \end{cases} \triangleq A(0)W + G(z,\overline{z},\theta),
\end{aligned}
\tag{3.25}
$$

where

$$
G(z,\overline{z},\theta) = G_{20}(\theta)\frac{z^2}{2} + G_{11}(\theta)z\overline{z} + G_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.26}
$$

Substituting the corresponding series into (3.25) and comparing the coefficients, we have

$$
(A(0) - 2i\omega_0\tau_k I)W_{20}(\theta) = -G_{20}(\theta), \qquad A(0)W_{11}(\theta) = -G_{11}(\theta). \tag{3.27}
$$

From (3.20) and (3.25), we have, for $\theta \in [-1,0)$

$$
G(z,\overline{z},\theta) = -\overline{q}^*(0)f_0 q(\theta) - q^*(0)\overline{f}_0\overline{q}(\theta) = -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta). \tag{3.28}
$$

Comparing the coefficients with (3.26), we have

$$
G_{20}(\theta) = -\, g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{3.29}
$$

$$
G_{11}(\theta) = -\, g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.30}
$$

From (3.27), (3.29) and the definition of $A(0)$, we have

$$
W_{20}'(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{3.31}
$$

Notice that $q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_k\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k} q(0)e^{i\omega_0\tau_k\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau_k}\overline{q}(0)e^{-i\omega_0\tau_k\theta} + E_1 e^{2i\omega_0\tau_k\theta}, \tag{3.32}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ is a constant vector. Similarly, we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k} q(0)e^{i\omega_0\tau_k\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau_k}\overline{q}(0)e^{-i\omega_0\tau_k\theta} + E_2, \tag{3.33}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ is a constant vector. In the following, we will seek the values of E_1 and E_2. From the definition of $A(0)$ and (3.27), we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - G_{20}(0), \tag{3.34}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -G_{11}(0), \tag{3.35}$$

where $\eta(\theta) = \eta(0, \theta)$.

By (3.25), when $\theta = 0$, we have

$$\begin{aligned} G(z, \overline{z}, 0) &= 2\operatorname{Re}\{\overline{q}^*(0)f_0 q(0)\} + f_0 \\ &= -\overline{q}^*(0)f_0 q(0) - q^*(0)\overline{f}_0\overline{q}(0) + f_0 = -g(z, \overline{z})q(0) - \overline{g}(z, \overline{z})\overline{q}(0) + f_0. \end{aligned} \tag{3.36}$$

So we obtain

$$\begin{aligned} G_{20}(\theta)\frac{z^2}{2} + G_{11}(\theta)z\overline{z} + G_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots = &-q(0)\left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots\right) \\ &-\overline{q}(0)\left(\overline{g}_{20}\frac{\overline{z}^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \cdots\right) + f_0. \end{aligned} \tag{3.37}$$

By (3.37), we have

$$G_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_k\begin{pmatrix} -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} \\ kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 \\ \sigma a_1 a_2 \end{pmatrix}, \tag{3.38}$$

$$G_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau_k\begin{pmatrix} -r_2 - p\operatorname{Re}\{a_1\} \\ kp\operatorname{Re}\{a_1\} - \sigma\operatorname{Re}\{a_1\overline{a}_2\} \\ \sigma\operatorname{Re}\{a_1\overline{a}_2\} \end{pmatrix}. \tag{3.39}$$

Noticing that

$$\left(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta)\right) q(0) = 0,$$
$$\left(-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta)\right) \overline{q}(0) = 0. \tag{3.40}$$

So, substituting (3.32) and (3.38) into (3.34), we have

$$\left(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta} d\eta(\theta)\right) E_1 = 2\tau_k \begin{pmatrix} -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} \\ kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 \\ \sigma a_1 a_2 \end{pmatrix}. \tag{3.41}$$

That is

$$\begin{pmatrix} 2i\omega_0 + r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & 2i\omega_0 - kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ 0 & -\sigma I^* & 2i\omega_0 \end{pmatrix} E_1 = 2\begin{pmatrix} -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} \\ kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 \\ \sigma a_1 a_2 \end{pmatrix}. \tag{3.42}$$

Let

$$L_1 = \begin{vmatrix} 2i\omega_0 + r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & 2i\omega_0 - kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ 0 & -\sigma I^* & 2i\omega_0 \end{vmatrix}. \tag{3.43}$$

It follows that

$$E_1^{(1)} = \frac{2}{L_1} \begin{vmatrix} -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & 0 \\ kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 & 2i\omega_0 - kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ \sigma a_1 a_2 & -\sigma I^* & 2i\omega_0 \end{vmatrix},$$
$$E_1^{(2)} = \frac{2}{L_1} \begin{vmatrix} 2i\omega_0 + r_2 X^* e^{-i\omega_0\tau_k} & -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 & \sigma S^* - \gamma \\ 0 & \sigma a_1 a_2 & 2i\omega_0 \end{vmatrix}, \tag{3.44}$$
$$E_1^{(3)} = \frac{2}{L_1} \begin{vmatrix} 2i\omega_0 + r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & -r_2 e^{-i\omega_0\tau_k} - pa_1 e^{-i\omega_0\tau_k} \\ -kpS^* e^{-i\omega_0\tau_k} & 2i\omega_0 - kpX^* + c_1 + \sigma I^* & kpa_1 e^{-i\omega_0\tau_k} - \sigma a_1 a_2 \\ 0 & -\sigma I^* & \sigma a_1 a_2 \end{vmatrix}.$$

Similarly, we can get

$$\begin{pmatrix} r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & -kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ 0 & -\sigma I^* & 0 \end{pmatrix} E_2 = \begin{pmatrix} -r_2 - p\,\mathrm{Re}\{a_1\} \\ kp\,\mathrm{Re}\{a_1\} - \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} \\ \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} \end{pmatrix}, \tag{3.45}$$

and hence

$$E_2^{(1)} = \frac{1}{L_2}\begin{vmatrix} -r_2 - p\,\mathrm{Re}\{a_1\} & pX^* e^{-i\omega_0\tau_k} & 0 \\ kp\,\mathrm{Re}\{a_1\} - \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} & -kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} & -\sigma I^* & 0 \end{vmatrix},$$

$$E_2^{(2)} = \frac{1}{L_2}\begin{vmatrix} r_2 X^* e^{-i\omega_0\tau_k} & -r_2 - p\,\mathrm{Re}\{a_1\} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & kp\,\mathrm{Re}\{a_1\} - \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} & \sigma S^* - \gamma \\ 0 & \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} & 0 \end{vmatrix}, \tag{3.46}$$

$$E_2^{(3)} = \frac{1}{L_2}\begin{vmatrix} r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & -r_2 - p\,\mathrm{Re}\{a_1\} \\ -kpS^* e^{-i\omega_0\tau_k} & 0 & kp\,\mathrm{Re}\{a_1\} - \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} \\ 0 & -\sigma I^* & \sigma\,\mathrm{Re}\{a_1\overline{a}_2\} \end{vmatrix},$$

where

$$L_2 = \begin{vmatrix} r_2 X^* e^{-i\omega_0\tau_k} & pX^* e^{-i\omega_0\tau_k} & 0 \\ -kpS^* e^{-i\omega_0\tau_k} & -kpX^* + c_1 + \sigma I^* & \sigma S^* - \gamma \\ 0 & -\sigma I^* & 0 \end{vmatrix} = r2X^*\left(\sigma^2 S^* I^* - \sigma I^* \gamma\right) e^{-i\omega_0\tau_k}. \tag{3.47}$$

Therefore, we can determine $W_{20}(0)$ and $W_{11}(0)$, hence we can obtain g_{21}. Thus, we can compute these values

$$\begin{gathered} c_1(0) = \frac{i}{2\omega_0\tau_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k)\}}, \\ \beta_2 = 2\,\mathrm{Re}\{c_1(0)\}\ , \\ T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_k)\}}{\omega_0\tau_k}, \end{gathered} \tag{3.48}$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical values τ_k, so we have the following results.

Theorem 3.1. *(i) μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_k$ ($\tau < \tau_k$).*

(ii) β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

(iii) T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

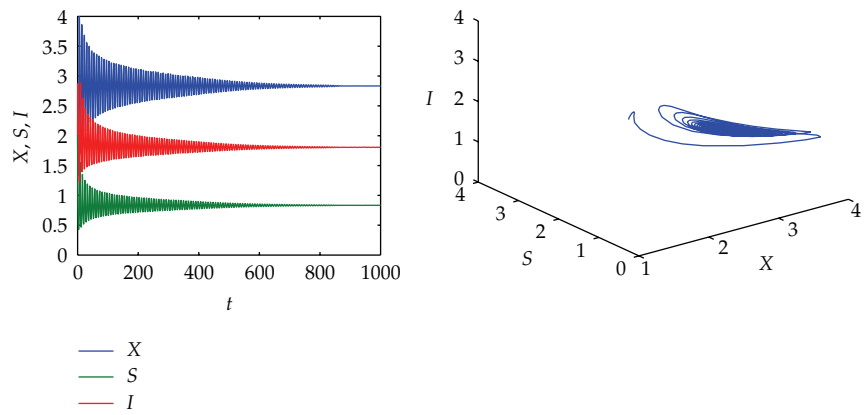

Figure 1: When $\tau = 1.11 < \tau_0$, the positive equilibrium $E^+ = (2.8333, 0.8333, 1.8056)$ is asymptotically stable. (a) shows the trajectories graphs of the system (4.1) with initial data $X(t) = 2$, $S_1(t) = 2$, $I_2(t) = 2$. (b) shows the phase portrait of system (4.1).

4. Discussion and Numerical Example

In this section,we present some numerical results of system (1.3) at different values of τ. Form Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcating periodic solutions. We consider the following system:

$$\begin{aligned} \frac{dX}{dt} &= X(t)[0.9 - 0.2X(t) - 0.4S(t)] \\ \frac{dS}{dt} &= S(t)[-0.2 + 0.3X(t) - 0.6I(t)] + 0.2I(t) \\ \frac{dI}{dt} &= I(t)[0.6S(t) - 0.3 - 0.2], \end{aligned} \tag{4.1}$$

which has a positive equilibrium $E^+ = (2.8333, 0.8333, 1.8056)$. Form (2.13) and (2.14), we are easy to get at least one positive real root 0.5977. In addition, it is easy to show that $[d(\operatorname{Re}\lambda)/d\tau]_{\tau=\tau_k} = 2.4377$, the hypothesis of H_3 holds. Hence, E^+ satisfies the condition of Theorem (2.2). When $\tau = 0$, the positive equilibrium $E^+ = (2.8333, 0.8333, 1.8056)$ is asymptotically stable. According to (2.18), we obtain $\tau_0 = 1.124$, $\omega_0 = 0.7731$, $\lambda'(\tau_0) = 0.3504 - 0.1448i$. Form the formulae (3.48) in Section 3, it follows that $c_1(0) = -15.6822 + 2.8655i$, $\mu_2 = 44.7551 > 0$, $\beta_2 = -31.3764 < 0$ and $T_2 = 4.1602 > 0$. Thus, E^+ is stable when $\tau < \tau_0$ as is illustrated by the computer simulations (see Figures 1(a) and 1(b)).

When τ passes through the critical value τ_0. E^+ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from E^+. Since $\mu_2 > 0$ and $\beta_2 < 0$, the Hopf bifurcation is supercritical and the direction of the bifurcation is $\tau > \tau_0$ and these bifurcating periodic solutions from E^+ at τ_0 are stable,which are depicted in Figures 2(a) and 2(b).

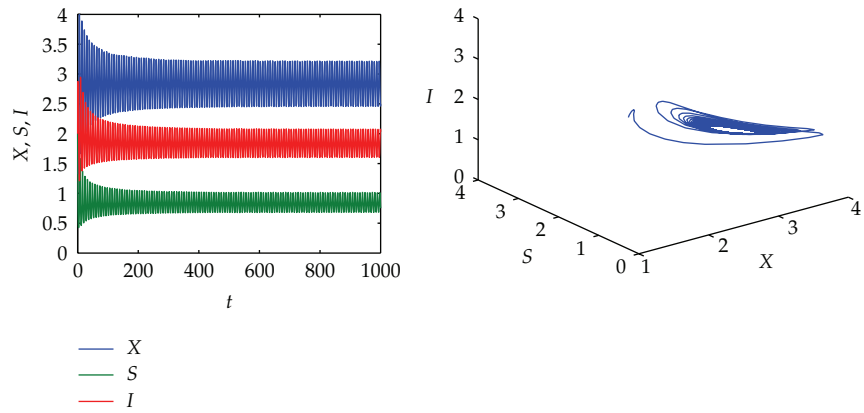

Figure 2: When $\tau = 1.13 > \tau_0$, bifurcation periodic solutions form E^+. (a) shows the trajectory graphs of system (4.1) with initial data $x(t) = 2,\ y_1(t) = 2,\ y_2(t) = 2$. (b) shows the phase portrait of system (4.1).

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