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*Research Article*

# Robust Filtering for Linear Equality Constrained Systems

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This paper deals with the robust filtering problem for linear discrete-time constrained systems. The purpose is the design of a linear filter such that the resulting error system is bounded. An orthogonal factorization is used to decompose the original robust filtering problem into stochastic and deterministic parts, which are then solved separately. Finally, a numerical example is presented to demonstrate the applicability of the proposed method.

## 1. Introduction

Kalman filtering is one of the well-known $H_2$ filtering methods that is widely used in the fields of signal processing and automatic control [1]. It is noted that the Kalman filtering method is based on the assumption that the system has known model and its disturbances are Gaussian white noises with known statistics. In some applications, however, the statistics of the noises are not exactly known, and the standard Kalman filtering algorithms will generally not guarantee satisfactory performance and only can obtain the estimate value with great error. Also, the strict assumptions limit the application scope of this filtering especially when there are uncertainties in either the state model or the measurement model. To handle the above problem, an alternative regularized estimate method based on least square design technique has been proposed recently. The objective is to find a filter such that the resulting estimate error is bounded and the main idea of this method is to reduce the vector optimization problem to an equivalent scalar minimization problem [2, 3]. Compared with earlier studies, such as $H_\infty$ method and guaranteed-cost method, the new method simultaneously uses regularization and weighting to deal with a class of uncertainties [3–5].

On the other hand, constrained filtering and control problem has drawn considerable attention over the past decades due to extensive application backgrounds. Actually, constraint formulation arises naturally in many fields such as target tracking, manufacturing production, engine health estimation, and vehicle motion [6, 7]. One of the features of these systems is that some components of the state are affected by some equations without noises. In conventional linear stochastic models with additive white process noise, filtering method for constrained systems has been investigated by many scholars. For example, Wen and Durrant-Whyte have considered the constrained problem by treating the set of constraint equations as additional accurate observations without noises [8]. Simon and Chia have shown that the solution of constrained problem can be obtained by treating the constraint equation as a constrained condition and solving a Lagrangian equation [9]. Moreover, Hewett et al. have presented a reduced null space method based on the null space decomposition to solve such problems [10].

Among the previous works on constrained estimation, the most popular approach is the projection method. This method enforces linear equality constraints on state space estimation, and the constrained estimate is merely a correction that forces the unconstrained estimate onto the constraint space [9]. In actual estimate and in value of objective function, the null space method often produces similar results as that of the project method. For the problem of robust filtering for constrained systems, however, there are still no results available in the literature. This motivates the present study. The regularized robust filtering is originally developed by Sayed to deal with the regularized dynamic system [2, 3] and Ishihara et al. use this method to present the robust filtering for uncertain singular system [11]. In this paper, we will give the regularized robust design method for uncertain constrained system.

In this paper, we deal with the robust filtering problem for uncertain constrained systems. Attention is focused on solving the least square problem, and the robust Kalman type recursion is developed. The remainder of this paper is organized as follows. Section 2 formulates the constrained systems and the problems to be solved. We review the filtering method for accurate constrained model in Section 3. In Section 4, the QR factorization is used to gain a new reduced system and the robust filtering is presented. We show numerical example that illustrates the new method performance in Section 5 and offer conclusion in Section 6.

The notation used in this paper is standard. $A^{\mathrm{T}}$ and $A^{\dagger}$ are the transpose and the pseudoinverse of the matrix $A$, respectively. $P > 0$ ($P \geq 0$) denotes a positive-definite (semidefinite) matrix. For a column vector $x$ and a positive matrix $W$, $\|x\|^2$ is the Euclidean norm of $x$, and $\|x\|_W$ is the weighted form. $\mathrm{diag}\{x, y\}$ denotes a block diagonal matrix with entries $x$ and $y$.

## 2. Problem Formulation and Analysis

### 2.1. Problem Formation

Consider a uncertain linear constrained system described by following model:

$$x_k = (A_k + \delta A_k)x_{k-1} + w_{k-1}, \tag{2.1}$$

$$y_k = (H_k + \delta H_k)x_k + v_k, \tag{2.2}$$

$$D_k x_k = d_k, \tag{2.3}$$

where $x_k \in R^n$ is the state vector satisfying equality constraints and $y_k \in R^m$ is the measurement output. $A_k$ is a $n \times n$ state update matrix, $H_k$ is an $m \times n$ observation matrix, $D_k$ is a $s \times n$ constraint matrix, and $d_k$ is a known vector. $\delta A_k$ and $\delta H_k$ are time-varying uncertainties to the nominal system matrices. The initial state $x_0$, process noises sequence $w_k$, and measurement noises sequence $v_k$ are uncorrelated zero mean white noises with variance

$$E\left(\begin{bmatrix} x_0 \\ w_k \\ v_k \end{bmatrix}\begin{bmatrix} x_0 \\ w_l \\ v_l \end{bmatrix}^{\mathrm{T}}\right) = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & \Omega_k\delta_{kl} & 0 \\ 0 & 0 & V_k\delta_{kl} \end{bmatrix}, \tag{2.4}$$

where $\delta_{kl}$ is the Kronecker function, $\Pi_0 > 0$, $\Omega_k > 0$, and $V_k > 0$. The uncertainties are assumed with the following structure:

$$\begin{bmatrix} \delta A_k \\ \delta H_k \end{bmatrix} = \begin{bmatrix} M_{ak} \\ M_{hk} \end{bmatrix} \Delta_k N_k, \quad \Delta_k \leq 1, \tag{2.5}$$

where $M_{ak}$, $M_{hk}$, and $N_k$ are known matrices, and $\Delta_k$ is a bounded matrix but otherwise arbitrary. We allow $M_{ak}$, $M_{hk}$, and $N_k$ to vary with time.

The purpose of this paper is to find a recursive robust state estimate algorithm for this constrained system with modeling uncertainties. With the constrained condition (2.3), the system (2.1)–(2.3) is not a standard form and the robust filtering presented in [2] is not applicable, so we cannot directly use them to present the analysis. The key to solving this problem is to transfer the constrained system into some new systems without constraint.

On the other hand, the final estimate result of the state $x_k$ should satisfy the additional constraint (2.3), which means that the estimate belongs to the space, denoted as $\Theta_k$, composed by the solutions of (2.3). The constraint matrix $D_k$ and vector $d_k$ are assumed to satisfy $\mathrm{Rank}[D_k \;\; d_k] = \mathrm{Rank}[D_k]$ to make $\Theta_k \neq \varnothing$. We assume that the constraint matrix $D_k$ has full column row rank and $s < n$.

## 3. Standard Constrained Filter

The constrained filter algorithm has some advantages compared with the standard Kalman filter, which are given in [9, 10]. In this section, we will review the constrained filtering method for accurate state-space model.

The accurate constrained system is

$$\begin{aligned} x_k &= A_k x_{k-1} + w_{k-1}, \\ y_k &= H_k x_k + v_k, \\ D_k x_k &= d_k. \end{aligned} \tag{3.1}$$

In [10], it uses orthogonal factorization to decompose the original state into stochastic and deterministic parts. The $QR$ factorization of $D_k^{\mathrm{T}}$

$$D_k^{\mathrm{T}} = \begin{bmatrix} Q_{1,k} & Q_{2,k} \end{bmatrix} \begin{bmatrix} R_{11,k} \\ 0 \end{bmatrix} \tag{3.2}$$

and the initial state can be rewritten as

$$x_k = x_{d,k} + Q_{2,k} z_{s,k}, \tag{3.3}$$

where

$$x_{d,k} = Q_{1,k} R_{11,k}^{-\mathrm{T}} d_k. \tag{3.4}$$

It also gives a new reduced constrained system

$$\begin{gathered} z_{s,k} = Q_{2,k}^{\mathrm{T}} A_k x_{d,k-1} + Q_{2,k}^{\mathrm{T}} A_k Q_{2,k-1} z_{s,k-1} + Q_{2,k}^{\mathrm{T}} w_{k-1}, \\ y_k - H_k x_{d,k} = H_k Q_{2,k} z_{s,k} + v_k. \end{gathered} \tag{3.5}$$

The recursive estimate algorithm for accurate constrained system can be summarized as

*Step 0.* Initialization

$QR$ decomposition:

$$\begin{bmatrix} Q_{1,0} & Q_{2,0} \end{bmatrix} \begin{bmatrix} R_{11,0} \\ 0 \end{bmatrix} = QR\left(D_0^{\mathrm{T}}\right). \tag{3.6}$$

The deterministic part is

$$x_{d,0} = Q_{1,0} R_{11,0}^{-\mathrm{T}} d_0. \tag{3.7}$$

Set

$$\begin{gathered} \hat{z}_{s,0|-1} = -Q_{2,0}^{\mathrm{T}} x_{d,0}, \\ P_{s,0|-1} = Q_{2,0}^{\mathrm{T}} \Pi_0 Q_{2,0}. \end{gathered} \tag{3.8}$$

*Step 1.* Prediction

$QR$ decomposition:

$$\begin{bmatrix} Q_{1,k} & Q_{2,k} \end{bmatrix} \begin{bmatrix} R_{11,k} \\ 0 \end{bmatrix} = QR\left(D_k^{\mathrm{T}}\right). \tag{3.9}$$

The deterministic part is

$$x_{d,k} = Q_{1,k} R_{11,k}^{-\mathrm{T}} d_k. \tag{3.10}$$

According to (3.5), it gives

$$\begin{aligned} \widehat{z}_{s,k|k-1} &= Q_{2,k}^{\mathrm{T}} \left[ A_k Q_{2,k-1} \widehat{z}_{s,k-1|k-1} + A_k x_{d,k} \right], \\ P_{s,k|k-1} &= Q_{2,k}^{\mathrm{T}} \left[ A_k Q_{2,k-1} P_{s,k-1|k-1} Q_{2,k-1}^{\mathrm{T}} A_k^{\mathrm{T}} + \Omega_{k-1} \right] Q_{2,k}. \end{aligned} \tag{3.11}$$

*Step 2.* Measurement Update

One has

$$\begin{aligned} K_{z,k} &= P_{s,k|k-1} (H_k Q_{2,k})^{\mathrm{T}} \left[ H_k Q_{2,k} P_{s,k|k-1} (H_k Q_{2,k})^{\mathrm{T}} + V_k \right]^{-1}, \\ \widehat{z}_{s,k|k} &= \widehat{z}_{s,k|k-1} + K_{z,k} \left[ y_k - H_k x_{d,k} - H_k Q_{2,k} \widehat{z}_{s,k|k-1} \right], \\ P_{s,k|k} &= P_{s,k|k-1} - K_{z,k} H_k Q_{2,k} P_{s,k|k-1}. \end{aligned} \tag{3.12}$$

*Step 3.* Reconstruction

Prediction reconstruction:

$$\begin{aligned} \widehat{x}_{k|k-1} &= x_{d,k} + Q_{2,k} \widehat{z}_{s,k|k-1}, \\ P_{k|k-1} &= Q_{2,k} P_{s,k|k-1} Q_{2,k}^{\mathrm{T}}. \end{aligned} \tag{3.13}$$

Estimate reconstruction:

$$\begin{aligned} \widehat{x}_{k|k} &= x_{d,k} + Q_{2,k} \widehat{z}_{s,k|k}, \\ P_{k|k} &= Q_{2,k} P_{s,k|k} Q_{2,k}^{\mathrm{T}}. \end{aligned} \tag{3.14}$$

The key for the above recursive estimation algorithm is finding the optimal estimation of $\widehat{z}_{s,k|k}$. With (3.5), the optimal estimate of $\widehat{z}_{s,k|k}$ can be derived by solving the following regularized least-square problem:

$$\begin{aligned} \min_{z_{s,k-1}, z_{s,k}} \Bigg[ & \left\| z_{s,k-1} - \widehat{z}_{s,k-1|k-1} \right\|_{P_{s,k-1|k-1}^{-1}}^2 + \left\| z_{s,k} - Q_{2,k}^{\mathrm{T}} A_k Q_{2,k-1} z_{s,k-1} - Q_{2,k}^{\mathrm{T}} A_k x_{d,k} \right\|_{\Omega_{k-1}^{-1}}^2 \\ & + \left\| y_k - H_k x_{d,k} - H_k Q_{2,k} z_{s,k} \right\|_{V_k^{-1}}^2 \Bigg]. \end{aligned} \tag{3.15}$$

Next, we will present the robust filter for the uncertain constrained system also by solving a uncertain least-square problem.

## 4. Robust Filtering

Referring again to the state-space model (2.1)–(2.3), the optimum robust filtering method will be presented in this section. Firstly, we will decompose the original uncertain constrained system into two parts, and then solve them separately.

### 4.1. New Dimension Reduced Uncertain Model

With the state evolution equation (2.1) and measurement equation (2.3), we will give the optimal estimate $\hat{x}_{k|k}$ for state $x_k$. Similar to the approach described in Section 3, we will use the null space method to deal with the uncertain model.

According to the uncertain model (2.1)–(2.3), we define the $QR$ factorization of $D_k^{\mathrm{T}}$ and rewrite the initial state equation as

$$x_k = x_{d,k} + x_{s,k} = Q_{1,k}\xi_k + Q_{2,k}\eta_k, \tag{4.1}$$

where $Q_{1,k}^{\mathrm{T}}$ is an $s \times n$ matrix whose columns form a basis for $\mathrm{Span}(A)$, and $Q_{2,k}^{\mathrm{T}}$ is an $(n-s)\times n$ matrix whose columns form a orthogonal basis for $\mathrm{Span}(A)^{\perp} = \mathrm{Null}(A)$.

Substituting (4.1) into (2.3) gives

$$\begin{bmatrix} R_{11,k}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} Q_{1,k}^{\mathrm{T}} \\ Q_{2,k}^{\mathrm{T}} \end{bmatrix} \left(Q_{1,k}\xi_k + Q_{2,k}\eta_k\right) = d_k, \tag{4.2}$$

then we have

$$\xi_k = R_{11,k}^{-\mathrm{T}} d_k. \tag{4.3}$$

Also substituting (4.1) into (2.1), we have

$$Q_{1,k}\xi_k + Q_{2,k}\eta_k = (A_k + \delta A_k)\left(Q_{1,k-1}\xi_{k-1} + Q_{2,k-1}\eta_{k-1}\right) + w_{k-1}. \tag{4.4}$$

Both sides of above equation multiplying $Q_{2,k}^{\mathrm{T}}$ gives

$$Q_{2,k}^{\mathrm{T}}Q_{1,k}\xi_k + Q_{2,k}^{\mathrm{T}}Q_{2,k}\eta_k = Q_{2,k}^{\mathrm{T}}(A_k + \delta A_k)\left(Q_{1,k-1}\xi_{k-1} + Q_{2,k-1}\eta_{k-1}\right) + Q_{2,k}^{\mathrm{T}}w_{k-1}. \tag{4.5}$$

Since

$$\begin{aligned} Q_{2,k}^{\mathrm{T}}Q_{1,k} &= 0, \\ Q_{2,k}^{\mathrm{T}}Q_{2,k} &= I. \end{aligned} \tag{4.6}$$

We have

$$\begin{aligned}\eta_k &= Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{1,k-1}\xi_{k-1}\\ &\quad + Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{2,k-1}\eta_{k-1}+Q_{2,k}^{\mathrm{T}}w_{k-1}\\ &= Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{1,k-1}R_{11,k-1}^{-\mathrm{T}}d_{k-1}\\ &\quad + Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{2,k-1}\eta_{k-1}+Q_{2,k}^{\mathrm{T}}w_{k-1}.\end{aligned} \tag{4.7}$$

Similarly, substituting (4.1) into (2.2) gives

$$\begin{aligned}y_k &= (H_k+\delta H_k)x_k+v_k\\ &= (H_k+\delta H_k)\left(Q_{1,k}\xi_k+Q_{2,k}\eta_k\right)+v_k\\ &= (H_k+\delta H_k)Q_{1,k}R_{11,k}^{-\mathrm{T}}d_k+(H_k+\delta H_k)Q_{2,k}\eta_k+v_k,\end{aligned} \tag{4.8}$$

that is,

$$y_k-H_kQ_{1,k}R_{11,k}^{-\mathrm{T}}d_k=\delta H_kQ_{1,k}R_{11,k}^{-\mathrm{T}}d_k+(H_k+\delta H_k)Q_{2,k}\eta_k+v_k. \tag{4.9}$$

The uncertain constrained state space model in $x_k$ is converted into an unconstrained uncertain state space model in $\eta_k$. Written together, (4.7) and (4.9) yield a new uncertain unconstrained state space model.

$$\begin{gathered}\eta_k = Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{1,k-1}R_{11,k-1}^{-\mathrm{T}}d_{k-1}+Q_{2,k}^{\mathrm{T}}(A_k+\delta A_k)Q_{2,k-1}\eta_{k-1}+Q_{2,k}^{\mathrm{T}}w_{k-1},\\ y_k-H_kQ_{1,k}R_{11,k}^{-\mathrm{T}}d_k=\delta H_kQ_{1,k}R_{11,k}^{-\mathrm{T}}d_k+(H_k+\delta H_k)Q_{2,k}\eta_k+v_k.\end{gathered} \tag{4.10}$$

### *4.2. Robust Filtering for the Uncertain Model*

Reference [2] develops the framework for state estimation when the parameters of the state equations are subject to uncertainties. However, both the system matrix and measurement matrix in the system (4.7) and (4.9) have uncertainties, and the matrix defined in [2] cannot directly be used. In order to present the robust filtering for this system, some new matrices will be defined in next subsection.

Let us first introduce a lemma.

**Lemma 4.1** (see [2]). *Consider the following optimization problem:*

$$\min_{x}\max_{\delta A,\delta b}\left[\|x\|_Q^2+\|(A+\delta A)x-(b-\delta b)\|_W^2\right], \tag{4.11}$$

*where $A$ denotes the data matrix, $\delta A$ denotes a perturbation matrix, $b$ denotes the measurement vector, and $\delta b$ denotes a perturbation vector. $x$ is the unknown vector, $Q = Q^{\mathrm{T}} > 0$ and $W = W^{\mathrm{T}} > 0$ is a weighting matrix. $\delta A$ and $\delta b$ are assumed to satisfy a model*

$$\left[\delta A \quad \delta b\right] = H\Delta\left[E_a \quad E_b\right], \tag{4.12}$$

*where $\Delta$ is an arbitrary contraction satisfying $\|\Delta\| \leq 1$. $H$, $E_a$, and $E_b$ are known quantities of appropriate dimensions.*

*The problem* (4.11) *has a unique solution, which is given by*

$$\hat{x} = \left[\widehat{Q} + A^{\mathrm{T}}\widehat{W}A\right]^{-1}\left[A^{\mathrm{T}}\widehat{W}b + \hat{\lambda}E_a^{\mathrm{T}}E_b\right], \tag{4.13}$$

*where the modified weighting matrix $\{\widehat{Q}, \widehat{W}\}$ is defined by*

$$\begin{gathered}\widehat{Q} := Q + \hat{\lambda}E_a^{\mathrm{T}}E_a, \\ \widehat{W} := W + WH\left(\hat{\lambda}I - H^{\mathrm{T}}WH\right)^{\dagger}H^{\mathrm{T}}W,\end{gathered} \tag{4.14}$$

*and $\hat{\lambda}$ is a nonnegative scalar parameter obtained by following optimization problem:*

$$\hat{\lambda} = \arg\min_{\lambda \geq \|H^{\mathrm{T}}WH\|} G(\lambda), \tag{4.15}$$

*where*

$$G(\lambda) := \|x(\lambda)\|_Q^2 + \lambda\|E_a x(\lambda) - E_b\|^2 + \|Ax(\lambda) - b\|_{W(\lambda)}^2. \tag{4.16}$$

*The auxiliary function are defined by*

$$\begin{gathered}W(\lambda) := W + WH\left(\lambda I - H^{\mathrm{T}}WH\right)^{\dagger}H^{\mathrm{T}}W, \\ Q(\lambda) := Q + \lambda E_a^{\mathrm{T}}E_a, \\ x(\lambda) := \left[Q(\lambda) + A^{\mathrm{T}}W(\lambda)A\right]^{-1}\left[A^{\mathrm{T}}W(\lambda)b + \lambda E_a^{\mathrm{T}}E_b\right].\end{gathered} \tag{4.17}$$

As mentioned in Section 3, the optimal estimate problem can be solved by minimizing the cost function (3.15). Similarly, the robust filtering problem for the dimension reduced model (4.7) and (4.9) can be turn to solve following least-square problem:

$$\begin{aligned}
&\min_{\eta_{k-1},\eta_k}\max_{\delta A_k}\left\|\eta_{k-1}-\widehat{\eta}_{k-1|k-1}\right\|^2_{P^{-1}_{\eta,k-1|k-1}}\\
&\quad+\left\|\eta_k-Q^{\mathrm{T}}_{2,k}(A_k+\delta A_k)Q_{1,k-1}R^{-\mathrm{T}}_{11,k-1}d_{k-1}-Q^{\mathrm{T}}_{2,k}(A_k+\delta A_k)Q_{2,k-1}\eta_{k-1}\right\|^2_{\overline{\Omega}^{-1}_{k-1}}\\
&\quad+\left\|y_k-H_kQ_{1,k}R^{-\mathrm{T}}_{11,k}d_k-\delta H_kQ_{1,k}R^{-\mathrm{T}}_{11,k}d_k-(H_k+\delta H_k)Q_{2,k}\eta_k\right\|^2_{V^{-1}_k},
\end{aligned}\tag{4.18}$$

where

$$\overline{\Omega}_{k-1}=Q^{\mathrm{T}}_{2,k}\Omega_{k-1}Q_{2,k}.\tag{4.19}$$

In (4.18), the parameters $A_k$ and $H_k$ contain uncertainties. With appropriate definition, the lest-square problem can be rewritten more compactly. Let us define

$$x\longleftarrow\begin{bmatrix}\eta_{k-1}-\widehat{\eta}_{k-1|k-1}\\ \eta_k\end{bmatrix},$$

$$A\longleftarrow\begin{bmatrix}Q^{\mathrm{T}}_{2,k}A_kQ_{2,k-1} & -I\\ 0 & H_kQ_{2,k}\end{bmatrix},$$

$$\delta A\longleftarrow\begin{bmatrix}Q^{\mathrm{T}}_{2,k}\delta A_kQ_{2,k-1} & 0\\ 0 & \delta H_kQ_{2,k}\end{bmatrix},$$

$$b\longleftarrow\begin{bmatrix}Q^{\mathrm{T}}_{2,k}A_kQ_{1,k-1}R^{-\mathrm{T}}_{11,k-1}d_{k-1}+Q^{\mathrm{T}}_{2,k}A_kQ_{2,k-1}\widehat{\eta}_{k-1|k-1}\\ H_kQ_{1,k}R^{-\mathrm{T}}_{11,k}d_k-y_k\end{bmatrix},$$

$$\delta b\longleftarrow\begin{bmatrix}Q^{\mathrm{T}}_{2,k}\delta A_kQ_{1,k-1}R^{-\mathrm{T}}_{11,k-1}d_{k-1}+Q^{\mathrm{T}}_{2,k}\delta A_kQ_{2,k-1}\widehat{\eta}_{k-1|k-1}\\ \delta H_kQ_{1,k}R^{-\mathrm{T}}_{11,k}d_k\end{bmatrix},$$

$$Q\longleftarrow\begin{bmatrix}P^{-1}_{\eta,k-1|k-1} & 0\\ 0 & 0\end{bmatrix},$$

$$W\longleftarrow\begin{bmatrix}\left(Q^{\mathrm{T}}_{2,k}\Omega_{k-1}Q_{2,k}\right)^{-1} & 0\\ 0 & V^{-1}_k\end{bmatrix},$$

$$H\longleftarrow\begin{bmatrix}Q^{\mathrm{T}}_{2,k}M_{hk} & 0\\ 0 & Q_{2,k}\end{bmatrix},$$

$$E_a \longleftarrow \begin{bmatrix} N_k Q_{2,k-1} & 0 \\ 0 & Q_{2,k} \end{bmatrix},$$

$$E_b \longleftarrow \begin{bmatrix} N_k \left( Q_{1,k-1} R_{11,k-1}^{-\mathrm{T}} d_{k-1} + Q_{2,k-1} \widehat{\eta}_{k-1|k-1} \right) \\ N_k Q_{1,k} R_{11,k}^{-\mathrm{T}} d_k \end{bmatrix}. \tag{4.20}$$

Let $\eta_{k|k}$ and $P_{\eta,k|k}$ $k = 1, 2, 3, \ldots$ be the estimate result and estimate error covariance of the stochastic vector $\eta_k$, respectively. With Lemma 4.1 and above definition, the robust filter for $\eta_k$ can be summarized as in the following theorem.

**Theorem 4.2.** *Assume that the estimate $\widehat{\eta}_{k-1|k-1}$ and the estimate error covariance $P_{\eta,k-1|k-1}$ of $\eta_{k-1}$ have been known. At time index $k$, the robust filter of $x$ can be given by solving the following equation:*

$$\left( \widehat{Q} + A^{\mathrm{T}} \widehat{W} A \right) \widehat{x} = A^{\mathrm{T}} \widehat{W} b + \widehat{\lambda}_k E_a^{\mathrm{T}} E_b, \tag{4.21}$$

*where*

$$\widehat{Q} = \begin{bmatrix} P_{\eta,k-1|k-1}^{-1} + \widehat{\lambda}_k (N_k Q_{2,k-1})^{\mathrm{T}} N_k Q_{2,k-1} & 0 \\ 0 & Q_{2,k}^{\mathrm{T}} Q_{2,k} \end{bmatrix},$$

$$\widehat{W} = \begin{bmatrix} \widehat{\Omega}_{k-1} & 0 \\ 0 & \widehat{V}_k^{-1} \end{bmatrix}^{-1}, \tag{4.22}$$

$$\widehat{\Omega}_{k-1} = Q_{2,k}^{\mathrm{T}} \left( \Omega_{k-1} - \widehat{\lambda}_{k-1}^{-1} M_{ak-1} M_{ak-1}^{\mathrm{T}} \right) Q_{2,k},$$

$$\widehat{V}_k = V_k - \widehat{\lambda}_{k-1}^{-1} H_k Q_{2,k} M_{hk} M_{hk}^{\mathrm{T}} (H_k Q_{2,k})^{\mathrm{T}},$$

*and $\widehat{\lambda}_k$ is determined by minimizing the function $G(\lambda)$ of* (4.16) *in the interval $(\lambda_{l,k}, \infty)$, where*

$$\lambda_{l,k} := \left\| H^{\mathrm{T}} W H \right\| = \left\| \mathrm{diag} \left\{ M_{ak}^{\mathrm{T}} Q_{2,k} \left( Q_{2,k}^{\mathrm{T}} \Omega_{k-1} Q_{2,k} \right)^{-1} Q_{2,k}^{\mathrm{T}} M_{ak}, Q_{2,k}^{\mathrm{T}} V_k^{-1} Q_{2,k} \right\} \right\|. \tag{4.23}$$

*Proof.* Analogous to [2, 3, 11], using Lemma 4.1 yields (4.21). □

Theorem 4.2 gives the robust filter of $\eta_k$, then, the robust constrained estimate of the full state $x_k$ can be constructed by using the relationship

$$\widehat{x}_{k|k} = Q_{1,k} \xi_k + Q_{2,k} \widehat{\eta}_{k|k}. \tag{4.24}$$

It is easily verified that $\hat{x}_{k|k}$ satisfies the constraint equation

$$\begin{aligned} D_k \hat{x}_{k|k} &= D_k \left[ Q_{1,k} \xi_k + Q_{2,k} \hat{\eta}_{k|k} \right] \\ &= D_k Q_{1,k} \xi_k + D_k Q_{2,k} \hat{\eta}_{k|k} \\ &= d_k. \end{aligned} \tag{4.25}$$

Similarly, the constrained error covariance $P_{k|k}$ can be computed by using

$$P_{k|k} = Q_{2,k} P_{\eta,k|k} Q_{2,k}^{\mathrm{T}}, \tag{4.26}$$

where $P_{\eta,k|k}$ of (4.35) is the estimate error covariance of $\eta_k$.

### 4.3. Recursive Form of Constrained Robust Filter

After some considerable algebra, similar to [2], the recursive robust estimate $\hat{x}_{k|k}$ can be summarized as follows.

*Step 0.* Initialization

$QR$ decomposition:

$$\begin{bmatrix} R_{11,0}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} Q_{1,0}^{\mathrm{T}} \\ Q_{2,0}^{\mathrm{T}} \end{bmatrix} = QR(D_0). \tag{4.27}$$

The deterministic part is

$$\xi_0 = R_{11,0}^{\mathrm{T}} d_0. \tag{4.28}$$

Set

$$\begin{aligned} \hat{\eta}_{0|0} &= P_{\eta,0|0} (H_0 Q_{2,0})^{\mathrm{T}} V_0^{-1} y_0, \\ P_{\eta,0|0} &= \left[ \left( Q_{2,0}^{\mathrm{T}} \Pi_0 Q_{2,0} \right)^{-1} + (H_0 Q_{2,0})^{\mathrm{T}} V_0^{-1} H_0 Q_{2,0} \right]^{-1}. \end{aligned} \tag{4.29}$$

*Step 1.* Determining $\hat{\lambda}_k$

$QR$ decomposition:

$$\begin{bmatrix} R_{11,k}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} Q_{1,k}^{\mathrm{T}} \\ Q_{2,k}^{\mathrm{T}} \end{bmatrix} = QR(D_k). \tag{4.30}$$

It gives the deterministic part at time index $k$

$$\xi_k = R_{11,k}^{\mathrm{T}} d_k \tag{4.31}$$

and the new dimension reduced robust system (4.7) and (4.9).

If $M_{ak} \neq 0$, then set $\widehat{\lambda}_k = 0$. Otherwise, with the definitions of (4.20), determine the scalar parameter $\widehat{\lambda}_k$ by minimizing $G(\lambda)$ over the interval $(\lambda_{l,k}, \infty)$.

*Step 2.* Replace Parameters

If $\widehat{\lambda}_k \neq 0$, the original parameters $\{\Omega_{k-1}, V_k, P_{\eta,k-1|k-1}, A_k\}$ are replaced by

$$\begin{aligned}
\widehat{\Omega}_{k-1} &= Q_{2,k}^{\mathrm{T}}\left(\Omega_{k-1} - \widehat{\lambda}_{k-1}^{-1} M_{ak-1} M_{ak-1}^{\mathrm{T}}\right) Q_{2,k}, \\
\widehat{V}_k &= V_k - \widehat{\lambda}_{k-1}^{-1} H_k Q_{2,k} M_{hk} M_{hk}^{\mathrm{T}} (H_k Q_{2,k})^{\mathrm{T}}, \\
\widehat{P}_{\eta,k-1|k-1} &= P_{\eta,k-1|k-1} - P_{\eta,k-1|k-1} (N_k Q_{2,k-1})^{\mathrm{T}} \\
&\quad \times \left[\widehat{\lambda}_{k-1}^{-1} I + N_k Q_{2,k-1} P_{\eta,k-1|k-1} (N_k Q_{2,k-1})^{\mathrm{T}}\right]^{-1} \\
&\quad \times N_k Q_{2,k-1} P_{\eta,k-1|k-1}, \\
\widehat{A}_k &= A_k \left[I - \widehat{\lambda}_{k-1} \widehat{P}_{k-1|k-1} (N_k Q_{2,k-1})^{\mathrm{T}} N_k Q_{2,k-1}\right].
\end{aligned} \tag{4.32}$$

*Step 3.* Prediction and Update

Prediction:

$$\begin{aligned}
\widehat{\eta}_{k|k-1} &= \widehat{A}_k \widehat{\eta}_{k-1|k-1} + Q_{2,k}^{\mathrm{T}} A_k Q_{1,k-1} R_{11,k-1}^{-\mathrm{T}} d_{k-1}, \\
P_{\eta,k|k-1} &= \widehat{A}_k \widehat{P}_{\eta,k|k-1} \widehat{A}_k^{\mathrm{T}}.
\end{aligned} \tag{4.33}$$

Update:

$$\widehat{\eta}_{k|k} = \widehat{\eta}_{k|k-1} + P_{\eta,k|k-1} (H_k Q_{2,k})^{\mathrm{T}} \widehat{V}_k^{-1} e_k, \tag{4.34}$$

$$P_{\eta,k|k} = P_{\eta,k|k-1} - P_{\eta,k|k-1} (H_k Q_{2,k})^{\mathrm{T}} V_{e,k}^{-1} H_k Q_{2,k} P_{\eta,k|k-1}, \tag{4.35}$$

where

$$\begin{aligned}
e_k &= y_k - H_k Q_{1,k} R_{11,k-1}^{-\mathrm{T}} d_{k-1} - H_k Q_{2,k} \widehat{\eta}_{k|k-1}, \\
V_{e,k} &= \widehat{V}_k + (H_k Q_{2,k})^{\mathrm{T}} P_{\eta,k|k-1} H_k Q_{2,k}.
\end{aligned} \tag{4.36}$$

*Step 4.* Reconstruction

Prediction reconstruction:

$$\begin{aligned} \hat{x}_{k|k-1} &= Q_{1,k}\xi_k + Q_{2,k}\hat{\eta}_{k|k-1}, \\ P_{k|k-1} &= Q_{2,k}P_{\eta,k|k-1}Q_{2,k}^{\mathrm{T}}. \end{aligned} \tag{4.37}$$

Estimate reconstruction:

$$\begin{aligned} \hat{x}_{k|k} &= Q_{1,k}\xi_k + Q_{2,k}\hat{\eta}_{k|k}, \\ P_{k|k} &= Q_{2,k}P_{\eta,k|k}Q_{2,k}^{\mathrm{T}}. \end{aligned} \tag{4.38}$$

Steps 1–4 give the robust estimate of the full state $x_k$.

*Remark 4.3.* From the definition of $\{\hat{\Omega}_{k-1}, \hat{V}_k, \hat{P}_{\eta,k-1|k-1}, \hat{A}_k\}$ and the prediction and update process in Step 3, it is easy to verify that for the constrained system without uncertainties, the robust filtering algorithm reduces to the filtering result introduced in [10].

*Remark 4.4.* If $D_k \in R^{n\times n}$ and $\mathrm{Rank}\{D_k\} = n$, the matrix $Q_{2,k}$ and the dimensional reduced model (4.7) will disappear, then we have $\hat{x}_{k|k} = D_k^{-1}d_k$.

## 5. Numerical Example

In this section, simulations are presented to verify the performance of the new algorithm. We consider an example described by (2.1)–(2.3), with $x_k = [x_k^1, x_k^2, x_k^3]^{\mathrm{T}}$. The parameters are given as follows:

$$\begin{gathered} A_k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad H_k = \begin{bmatrix} 0.8 & 1 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \qquad \Omega_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ V_k = \begin{bmatrix} 1.2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_k = [1, 0.3, 0.2], \qquad d(k) = 1, \\ M_{ak} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \qquad M_{hk} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad N_k = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}. \end{gathered} \tag{5.1}$$

The initial state is $x_0 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$, $P_0 = \left[\begin{smallmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{smallmatrix}\right]$, and we will take $L = 1000$ sampling points. Figures 1 and 2 display the estimate error variance of $x_k^1$ and $x_k^2$, respectively. The variance curves are computed via the ensemble-average

$$\varepsilon\|x_k - \hat{x}_k\| \approx \frac{1}{T}\sum_{i=1}^{T}\left\|x_k^i - \hat{x}_k^i\right\|. \tag{5.2}$$

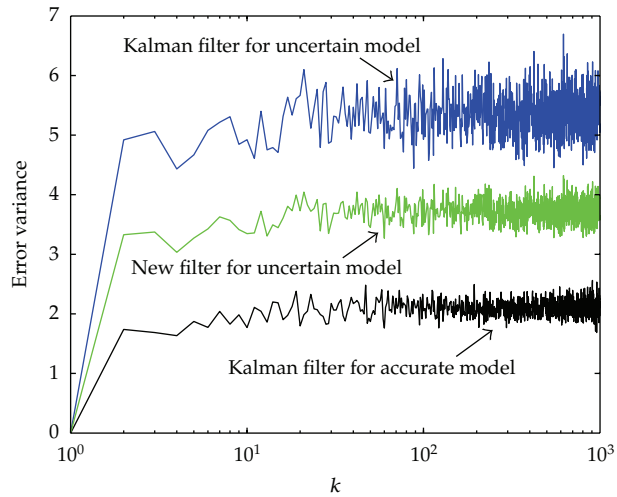


**Figure 1:** Comparison of the estimate variance $x_k^1$ with different methods.

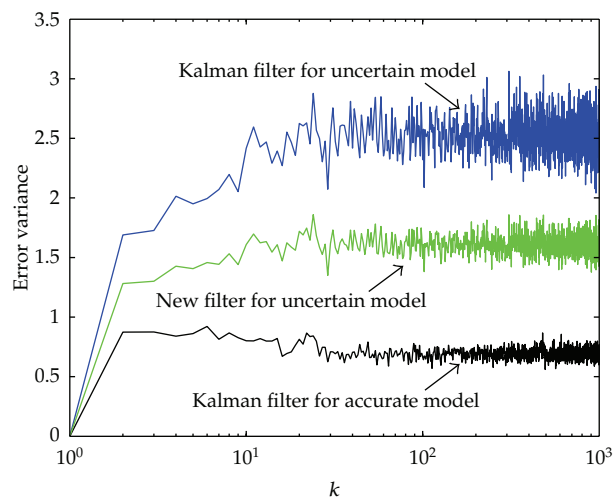


**Figure 2:** Comparison of the estimate variance $x_k^2$ with different methods.

Each point at instant $k$ in each variance curve is the ensemble-average calculated over $T = 500$ experiments. For each experiment $i$, $\Delta_k^i$ with norm less or equal than one is selected randomly.

To demonstrate the performance of the new robust filter more clearly, we also present the variance curves of the Kalman filter for uncertain model and the system without uncertainties. The variances of these two filters are also shown in Figures 1 and 2.

From Figures 1 and 2, we see that the performance of new filter is better than that of Kalman filter when they are used to deal with the uncertain model, this is because Kalman filter does not consider the uncertain parameters. The variance of Kalman filter dealing with accurate model is smaller than that of new filter dealing with uncertain model.

**Table 1:** Variation of error variance with different measurement noises variance.

| Precision | Measurement noises variance | | |
|---|---|---|---|
| | $V(k) = 0.7 * I2$ | $V(k) = 1.2 * I2$ | $V(k) = 2.5 * I2$ |
| Error variance of $\tilde{x}_{1,100}$ | 0.9429 | 1.3377 | 2.7273 |
| Error variance of $\tilde{x}_{2,100}$ | 1.3217 | 1.8631 | 3.6547 |
| Error variance of $\tilde{x}_{3,100}$ | 1.6217 | 2.2641 | 3.9199 |

Furthermore, the performance of an algorithm is often affected by the measurement noises, and larger noises variance always bring larger estimate error variance. Table 1 lists the error variance for $x_{100}$ with three different measurement noises variance to show the variation of performance.

From Table 1, we see that, for $x_k^1$ and $x_k^2$, the larger is the noise variance, the larger is estimation error variance.

## 6. Conclusions

This paper has studied the robust constrained filtering problem for linear discrete uncertain systems. The original constrained system is transformed into a new uncertain unconstrained system. The state of the new system is derived by the least square method and then the optimal estimate is obtained similar to the update process of the robust Kalman filter. A numerical example is presented to show the effectiveness of the new filter. Next, we will consider the regularized filtering method for the case when network-induced phenomena are taken into account [12–15].

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