Research Article

On (α, β) -Derivations in BCI-Algebras

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The notion of (regular) (α, β) -derivations of a BCI-algebra X is introduced, some useful examples are discussed, and related properties are investigated. The condition for a (α, β) -derivation to be regular is provided. The concepts of a $d_{(\alpha, \beta)}$ -invariant (α, β) -derivation and α -ideal are introduced, and their relations are discussed. Finally, some results on regular (α, β) -derivations are obtained.

1. Introduction

BCK-algebras and BCI-algebras are two classes of nonclassical logic algebras which were introduced by Imai and Iséki in 1966 [1, 2]. They are algebraic formulation of BCK-system and BCI-system in combinatory logic. However, these algebras were not studied any further until 1980. Iséki published a series of notes in 1980 and presented a beautiful exposition of BCI-algebras in these notes (see [3–5]). The notion of a BCI-algebra generalizes the notion of a BCK-algebra in the sense that every BCK-algebra is a BCI-algebra but not vice versa (see [6]). Later on, the notion of BCI-algebras has been extensively investigated by many researchers (see [7–9] and references therein).

Throughout our discussion, X will denote a BCI-algebra unless otherwise mentioned. In the year 2004, Jun and Xin [10] applied the notion of derivation in ring and near-ring theory to BCI-algebras, and as a result they introduced a new concept, called a (regular) derivation, in BCI-algebras. Using this concept as defined, they investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a p -semisimple BCI-algebra. For a self map d of a BCI-algebra, they defined a d -invariant ideal and gave conditions for an ideal to be d -invariant. According to Jun and Xin, a self-map $d : X \rightarrow X$ is called a left-right derivation (briefly (l, r) -derivation) of X if $d(x * y) = d(x) * y \wedge x * d(y)$ holds for all $x, y \in X$. Similarly, a self-map $d : X \rightarrow X$ is called a right-left derivation (briefly (r, l) -derivation) of X if $d(x * y) = x * d(y) \wedge d(x) * y$ holds for all $x, y \in X$.

Moreover, if d is both (l, r) - and (r, l) -derivation, it is a derivation on X . After the work of Jun and Xin [10], many research articles have been appeared on the derivations of BCI-algebras and a greater interest has been devoted to the study of derivation in BCI-algebras on various aspects (see [11–15]).

Several authors [16–19] have studied derivations in rings and near-rings. Inspired by the notions of σ -derivation [20], left derivation [21] and generalized derivation [19, 22] in rings and near rings theory, many authors have applied these notions in a similar way to the theory of BCI-algebras (see [11, 14, 15]). For instant, in 2005 [15], Zhan and Liu have given the notion of f -derivation of BCI-algebras as follows: a self-map $d_f : X \rightarrow X$ is said to be a left-right f -derivation or (l, r) - f -derivation of X if it satisfies the identity $d_f(x * y) = d_f(x) * f(y) \wedge f(x) * d_f(y)$ for all $x, y \in X$. Similarly, a self map $d_f : X \rightarrow X$ is said to be a right-left f -derivation or (r, l) - f -derivation of X if it satisfies the identity $d_f(x * y) = f(x) * d_f(y) \wedge d_f(x) * f(y)$ for all $x, y \in X$. Moreover, if d_f is both (l, r) and (r, l) - f -derivation, it is said that d_f is an f -derivation where f is an endomorphism. In the year 2007, Abujabal and Al-Shehri [11] defined and studied the notion of left derivation of BCI-algebras as follows: a self-map $D : X \rightarrow X$ is said to be a left derivation of X if satisfying $D(x * y) = x * D(y) \wedge y * D(x)$ for all $x, y \in X$. Furthermore, in 2009 [14], Öztürk et al. have introduced the notion of generalized derivation in BCI-algebras. A self map $D : X \rightarrow X$ is called a generalized (l, r) -derivation if there exists an (l, r) -derivation $d : X \rightarrow X$ such that $D(x * y) = D(x) * y \wedge x * d(y)$ for all $x, y \in X$. If there exists an (r, l) -derivation $d : X \rightarrow X$ such that $D(x * y) = x * D(y) \wedge d(x) * y$ for all $x, y \in X$, the mapping $D : X \rightarrow X$ is called generalized (r, l) -derivation. Moreover, if D is both a generalized (l, r) - (r, l) -derivation, D is a generalized derivation on X .

In fact, the notion of derivation in ring theory is quite old and plays a significant role in analysis, algebraic geometry, and algebra. In his famous book “Structures of Rings” Jacobson [23] introduced the notion of (s_1, s_2) -derivation which was later more commonly known as (σ, τ) or (θ, ϕ) -derivation. After that a number of research articles have been appeared on (σ, τ) or (θ, ϕ) -derivations in the theory of rings (see [16, 24, 25] and references therein).

Motivated by the notion of (σ, τ) or (θ, ϕ) -derivation in the theory of rings, in the present paper, we introduce the notion of (α, β) -derivation in a BCI-algebra X and investigate related properties. We provide a condition for a (α, β) -derivation to be regular. We also introduce the concepts of a $d_{(\alpha, \beta)}$ -invariant (α, β) -derivation and α -ideal, and then we investigate their relations. Furthermore, we obtain some results on regular (α, β) -derivations.

2. Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set X with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x, y, z \in X$ the following conditions hold:

- (I) $((x * y) * (x * z)) * (z * y) = 0$,
- (II) $(x * (x * y)) * y = 0$,
- (III) $x * x = 0$,
- (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on X by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra.

A BCI-algebra X has the following properties: for all $x, y, z \in X$

- (a1) $x * 0 = x$,
- (a2) $(x * y) * z = (x * z) * y$,
- (a3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
- (a4) $(x * z) * (y * z) \leq x * y$,
- (a5) $x * (x * (x * y)) = x * y$,
- (a6) $0 * (x * y) = (0 * x) * (0 * y)$,
- (a7) $x * 0 = 0$ implies $x = 0$.

For a BCI-algebra X , denote by X_+ (resp. $G(X)$) the BCK-part (resp. the BCI-G part) of X , that is, X_+ is the set of all $x \in X$ such that $0 \leq x$ (resp. $G(X) := \{x \in X \mid 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$ (see [26]). If $X_+ = \{0\}$, then X is called a p -semisimple BCI-algebra. In a p -semisimple BCI-algebra X , the following hold:

- (a8) $(x * z) * (y * z) = x * y$,
- (a9) $0 * (0 * x) = x$ for all $x \in X$,
- (a10) $x * (0 * y) = y * (0 * x)$,
- (a11) $x * y = 0$ implies $x = y$,
- (a12) $x * a = x * b$ implies $a = b$,
- (a13) $a * x = b * x$ implies $a = b$,
- (a14) $a * (a * x) = x$.

Let X be a p -semisimple BCI-algebra. We define addition “+” as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p -semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [9]).

For a BCI-algebra X we denote $x \wedge y = y * (y * x)$, in particular $0 * (0 * x) = a_x$, and $L_p(X) := \{a \in X \mid x * a = 0 \Rightarrow x = a, \text{ for all } x \in X\}$. We call the elements of $L_p(X)$ the p -atoms of X . For any $a \in X$, let $V(a) := \{x \in X \mid a * x = 0\}$, which is called the branch of X with respect to a . It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X \mid a_x = x\}$, which is the p -semisimple part of X , and X is a p -semisimple BCI-algebra if and only if $L_p(X) = X$ (see [27, Proposition 3.2]). Note also that $a_x \in L_p(X)$, that is, $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. A BCI-algebra X is said to be torsion free if $x + x = 0 \Rightarrow x = 0$ for all $x \in X$ [14]. For more details, refer to [7–10, 26, 27].

3. (α, β) -Derivations in BCI-Algebras

In what follows, α and β are endomorphisms of a BCI-algebra X unless otherwise specified.

Definition 3.1. Let X be a BCI-algebra. Then a self map $d_{(\alpha, \beta)} : X \rightarrow X$ is called a (α, β) -derivation of X if it satisfies:

$$(\forall x, y \in X) \quad (d_{(\alpha, \beta)}(x * y) = (d_{(\alpha, \beta)}(x) * \alpha(y)) \wedge (d_{(\alpha, \beta)}(y) * \beta(x))). \quad (3.1)$$

Example 3.2. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$$\begin{array}{c|ccc}
 * & 0 & a & b \\
 \hline
 0 & 0 & 0 & b \\
 a & a & 0 & b \\
 b & b & b & 0
 \end{array} \tag{3.2}$$

(1) Define a map

$$d_{(\alpha, \beta)} : X \longrightarrow X, \quad x \longmapsto \begin{cases} b & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x = b, \end{cases} \tag{3.3}$$

and define two endomorphisms

$$\begin{aligned}
 \alpha : X \longrightarrow X, \quad x &\longmapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ b & \text{if } x = b, \end{cases} \\
 \beta : X \longrightarrow X, \quad x &\longmapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a. \end{cases}
 \end{aligned} \tag{3.4}$$

It is routine to verify that $d_{(\alpha, \beta)}$ is a (α, β) -derivation of X .

(2) Define a map

$$d_{(\alpha, \beta)} : X \longrightarrow X, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a, \end{cases} \tag{3.5}$$

and define two endomorphisms

$$\begin{aligned}
 \alpha : X \longrightarrow X, \quad x &\longmapsto \begin{cases} 0 & \text{if } x \in \{a, b\}, \\ b & \text{if } x = 0, \end{cases} \\
 \beta : X \longrightarrow X, \quad x &\longmapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ a & \text{if } x = b. \end{cases}
 \end{aligned} \tag{3.6}$$

It is routine to verify that $d_{(\alpha, \beta)}$ is a (α, β) -derivation of X .

Lemma 3.3 (see [8]). *Let X be a BCI-algebra. For any $x, y \in X$, if $x \leq y$, then x and y are contained in the same branch of X .*

Lemma 3.4 (see [8]). *Let X be a BCI-algebra. For any $x, y \in X$, if x and y are contained in the same branch of X , then $x * y, y * x \in X_+$.*

Proposition 3.5. *Let X be a commutative BCI-algebra. Then every (α, β) -derivation $d_{(\alpha, \beta)}$ of X satisfies the following assertion:*

$$(\forall x, y \in X) \quad (x \leq y \implies d_{(\alpha, \beta)}(x) \leq d_{(\alpha, \beta)}(y)), \quad (3.7)$$

that is, every (α, β) -derivation of X is isotone.

Proof. Let $x, y \in X$ be such that $x \leq y$. Since X is commutative, we have $x = x \wedge y$. Hence

$$\begin{aligned} d_{(\alpha, \beta)}(x) &= d_{(\alpha, \beta)}(x \wedge y) \\ &= (d_{(\alpha, \beta)}(y) * \alpha(y * x)) \wedge (d_{(\alpha, \beta)}(y * x) * \beta(y)) \\ &\leq (d_{(\alpha, \beta)}(y) * \alpha(y * x)). \end{aligned} \quad (3.8)$$

Since every endomorphism of X is isotone, we have $\alpha(x) \leq \alpha(y)$. It follows from Lemma 3.3 that $0 = \alpha(x) * \alpha(y) \in X_+$ and $\alpha(y) * \alpha(x) \in X_+$ so that there exists $a (\neq 0) \in X_+$ such that $\alpha(y * x) = \alpha(y) * \alpha(x) = a$. Hence (3.8) implies that $d_{(\alpha, \beta)}(x) \leq d_{(\alpha, \beta)}(y) * a$. Using (a3), (a2), and (III), we have

$$\begin{aligned} d_{(\alpha, \beta)}(x) * d_{(\alpha, \beta)}(y) &\leq (d_{(\alpha, \beta)}(y) * a) * d_{(\alpha, \beta)}(y) \\ &= (d_{(\alpha, \beta)}(y) * d_{(\alpha, \beta)}(y)) * a = 0 * a = 0, \end{aligned} \quad (3.9)$$

and so $d_{(\alpha, \beta)}(x) * d_{(\alpha, \beta)}(y) = 0$, that is, $d_{(\alpha, \beta)}(x) \leq d_{(\alpha, \beta)}(y)$ by (a7). $\square$

Example 3.6. In Example 3.2 (1), the (α, β) -derivation $d_{(\alpha, \beta)}$ does not satisfy the inequality (3.7).

Proposition 3.7. *Every (α, β) -derivation $d_{(\alpha, \beta)}$ of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X) \quad (d_{(\alpha, \beta)}(x) = d_{(\alpha, \beta)}(x) \wedge d_{(\alpha, \beta)}(0)). \quad (3.10)$$

Proof. Let $d_{(\alpha, \beta)}$ be an (α, β) -derivation of X . Using (a2) and (a4), we have

$$\begin{aligned} d_{(\alpha, \beta)}(x) &= d_{(\alpha, \beta)}(x * 0) = (d_{(\alpha, \beta)}(x) * \alpha(0)) \wedge (d_{(\alpha, \beta)}(0) * \beta(x)) \\ &= (d_{(\alpha, \beta)}(x) * 0) \wedge (d_{(\alpha, \beta)}(0) * \beta(x)) \\ &= d_{(\alpha, \beta)}(x) \wedge (d_{(\alpha, \beta)}(0) * \beta(x)) \\ &= (d_{(\alpha, \beta)}(0) * \beta(x)) * ((d_{(\alpha, \beta)}(0) * \beta(x)) * d_{(\alpha, \beta)}(x)) \\ &= (d_{(\alpha, \beta)}(0) * \beta(x)) * ((d_{(\alpha, \beta)}(0) * d_{(\alpha, \beta)}(x)) * \beta(x)) \\ &\leq d_{(\alpha, \beta)}(0) * (d_{(\alpha, \beta)}(0) * d_{(\alpha, \beta)}(x)) \\ &= d_{(\alpha, \beta)}(x) \wedge d_{(\alpha, \beta)}(0). \end{aligned} \quad (3.11)$$

Obviously $d_{(\alpha, \beta)}(x) \wedge d_{(\alpha, \beta)}(0) \leq d_{(\alpha, \beta)}(x)$ by (II). Therefore, the equality (3.10) is valid. $\square$

Theorem 3.8. Let $d_{(\alpha,\beta)}$ be a (α, β) -derivation on a BCI-algebra X . Then

$$(1) \text{ (for all } a \in L_p(X), x \in X) (d(a * x) = d(a) * \alpha(x)),$$

$$(2) \text{ (for all } a \in L_p(X), x \in X) (d(a + x) = d(a) + \alpha(x)),$$

$$(3) \text{ (for all } a, b \in L_p(X)) (d(a + b) = d(a) + \alpha(b)).$$

Proof. (1) For any $a \in L_p(X)$, we have $a * x \in L_p(X)$ for all $x \in X$. Thus $d(a * x) = d(a) * \alpha(x) \wedge d(x) * \beta(a) = d(a) * \alpha(x)$.

(2) For any $a \in L_p(X)$ and $x \in X$, it follows from (1) that

$$\begin{aligned} d(a + x) &= d(a * (0 * x)) = d(a) * \alpha(0 * x) \\ &= d(a) * (\alpha(0) * \alpha(x)) = d(a) * (0 * \alpha(x)) \\ &= d(a) + \alpha(x). \end{aligned} \tag{3.12}$$

(3) The proof follows directly from (2). $\square$

Definition 3.9. Let X be a BCI-algebra and $d_{(\alpha,\beta)}, d'_{(\alpha,\beta)}$ be two self maps of X , we define $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} : X \rightarrow X$ by $(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x))$ for all $x \in X$.

Theorem 3.10. Let X be a p -semisimple BCI-algebra. If $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ are two (α, β) -derivations on X such that $\alpha^2 = \alpha$. Then $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}$ is a (α, β) -derivation on X .

Proof. For any $x, y \in X$, it follows from (a14) that

$$\begin{aligned} \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} \right) (x * y) &= d_{(\alpha,\beta)} \left(d'_{(\alpha,\beta)} (x * y) \right) \\ &= d_{(\alpha,\beta)} \left(\left(d'_{(\alpha,\beta)}(x) * \alpha(y) \right) \wedge \left(d'_{(\alpha,\beta)}(y) * \beta(x) \right) \right) \\ &= d_{(\alpha,\beta)} \left(d'_{(\alpha,\beta)}(x) * \alpha(y) \right) \\ &= \left(d_{(\alpha,\beta)} \left(d'_{(\alpha,\beta)}(x) \right) * \alpha(\alpha(y)) \right) \wedge \left(d_{(\alpha,\beta)}(\alpha(y)) * \beta \left(d'_{(\alpha,\beta)}(x) \right) \right) \\ &= d_{(\alpha,\beta)} \left(d'_{(\alpha,\beta)}(x) \right) * \alpha(y) \end{aligned}$$

$$\begin{aligned}
&= \left(d_{(\alpha, \beta)} \left(d'_{(\alpha, \beta)}(y) * \beta(x) \right) \right) * \left(\left(d_{(\alpha, \beta)} \left(d'_{(\alpha, \beta)}(y) * \beta(x) \right) \right) \right. \\
&\quad \left. * \left(d_{(\alpha, \beta)} \left(d'_{(\alpha, \beta)}(x) \right) * \alpha(y) \right) \right) \\
&= \left(d_{(\alpha, \beta)} \left(d'_{(\alpha, \beta)}(x) \right) * \alpha(y) \right) \wedge \left(d_{(\alpha, \beta)} \left(d'_{(\alpha, \beta)}(y) * \beta(x) \right) \right) \\
&= \left(\left(d_{(\alpha, \beta)} \circ d'_{(\alpha, \beta)} \right)(x) * \alpha(y) \right) \wedge \left(\left(d_{(\alpha, \beta)} \circ d'_{(\alpha, \beta)} \right)(y) * \beta(x) \right).
\end{aligned} \tag{3.13}$$

This completes the proof. $\square$

Theorem 3.11. Let α, β be two endomorphisms and $d_{(\alpha, \beta)}$ be a self map on a p -semisimple BCI-algebra X such that $d_{(\alpha, \beta)}(x) = \alpha(x)$ for all $x \in X$. Then $d_{(\alpha, \beta)}$ is a (α, β) -derivation on X .

Proof. Let us take $d_{(\alpha, \beta)}(x) = \alpha(x)$ for all $x \in X$. Since $x, y \in X \Rightarrow x * y \in X$. Using (a14), we have

$$\begin{aligned}
d_{(\alpha, \beta)}(x * y) &= \alpha(x * y) = \alpha(x) * \alpha(y) = d_{(\alpha, \beta)}(x) * \alpha(y) \\
&= (d_{(\alpha, \beta)}(y) * \beta(x)) * ((d_{(\alpha, \beta)}(y) * \beta(x)) * (d_{(\alpha, \beta)}(x) * \alpha(y))) \\
&= (d_{(\alpha, \beta)}(x) * \alpha(y)) \wedge (d_{(\alpha, \beta)}(y) * \beta(x)).
\end{aligned} \tag{3.14}$$

This completes the proof. $\square$

Definition 3.12. A (α, β) -derivation $d_{(\alpha, \beta)}$ of a BCI-algebra X is said to be regular if $d_{(\alpha, \beta)}(0) = 0$.

Example 3.13. (1) The (α, β) -derivation $d_{(\alpha, \beta)}$ of X in Example 3.2 (1) is not regular.

(2) The (α, β) -derivation $d_{(\alpha, \beta)}$ of X in Example 3.2 (2) is regular.

We provide conditions for a (α, β) -derivation to be regular.

Theorem 3.14. Let $d_{(\alpha, \beta)}$ be a (α, β) -derivation of a BCI-algebra X . If there exists $a \in X$ such that $d_{(\alpha, \beta)}(x) * \alpha(a) = 0$ for all $x \in X$, then $d_{(\alpha, \beta)}$ is regular.

Proof. Assume that there exists $a \in X$ such that $d_{(\alpha, \beta)}(x) * \alpha(a) = 0$ for all $x \in X$. Then

$$\begin{aligned}
0 &= d_{(\alpha, \beta)}(x * a) * a = ((d_{(\alpha, \beta)}(x) * \alpha(a)) \wedge (d_{(\alpha, \beta)}(a) * \beta(x))) * a \\
&= (0 \wedge (d_{(\alpha, \beta)}(a) * \beta(x))) * a = 0 * a,
\end{aligned} \tag{3.15}$$

and so $d_{(\alpha, \beta)}(0) = d_{(\alpha, \beta)}(0 * a) = (d_{(\alpha, \beta)}(0) * \alpha(a)) \wedge (d_{(\alpha, \beta)}(a) * \beta(0)) = 0$. Hence $d_{(\alpha, \beta)}$ is regular. $\square$

Definition 3.15. For a (α, β) -derivation $d_{(\alpha, \beta)}$ of a BCI-algebra X , we say that an ideal A of X is a α -ideal (resp. β -ideal) if $\alpha(A) \subseteq A$ (resp. $\beta(A) \subseteq A$).

Definition 3.16. For a (α, β) -derivation $d_{(\alpha, \beta)}$ of a BCI-algebra X , we say that an ideal A of X is $d_{(\alpha, \beta)}$ -invariant if $d_{(\alpha, \beta)}(A) \subseteq A$.

Example 3.17. (1) Let $d_{(\alpha, \beta)}$ be a (α, β) -derivation of X which is described in Example 3.2 (1). We know that $A := \{0, a\}$ is both a α -ideal and a β -ideal of X . But $A := \{0, a\}$ is an ideal of X which is not $d_{(\alpha, \beta)}$ -invariant.

(2) Let $d_{(\alpha, \beta)}$ be a (α, β) -derivation of X which is described in Example 3.2 (2). We know that $A := \{0, a\}$ is both a β -ideal and a $d_{(\alpha, \beta)}$ -invariant ideal of X . But $A := \{0, a\}$ is not a α -ideal of X .

Next, we prove some results on regular (α, β) -derivations in a BCI-algebra.

Theorem 3.18. Let $d_{(\alpha, \beta)}$ be a regular (α, β) -derivation of a BCI-algebra X . Then

- (1) (for all $a \in X$) $(a \in L_p(X) \Rightarrow d_{(\alpha, \beta)}(a) \in L_p(X))$,
- (2) (for all $a \in X$) $(a \in L_p(X) \Rightarrow \alpha(a), \beta(a) \in L_p(X))$,
- (3) (for all $a \in L_p(X)$) $(d_{(\alpha, \beta)}(a) = d_{(\alpha, \beta)}(0) + \alpha(a))$,
- (4) (for all $a, b \in L_p(X)$) $(d_{(\alpha, \beta)}(a + b) = d_{(\alpha, \beta)}(a) + d_{(\alpha, \beta)}(b) - d_{(\alpha, \beta)}(0))$.

Proof. (1) Let $d_{(\alpha, \beta)}$ be a regular (α, β) -derivation, that is, $d_{(\alpha, \beta)}(0) = 0$. Then the proof follows directly from Proposition 3.7.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$, and so $\alpha(a) = \alpha(0 * (0 * a)) = 0 * (0 * \alpha(a))$. Thus $\alpha(a) \in L_p(X)$. Similarly, $\beta(a) \in L_p(X)$.

(3) Let $a \in L_p(X)$. Using (2), (a1) and (a14), we have

$$\begin{aligned}
 d_{(\alpha, \beta)}(a) &= d_{(\alpha, \beta)}(0 * (0 * a)) \\
 &= (d_{(\alpha, \beta)}(0) * \alpha(0 * a)) \wedge (d_{(\alpha, \beta)}(0 * a) * \beta(0)) \\
 &= (d_{(\alpha, \beta)}(0) * \alpha(0 * a)) \wedge (d_{(\alpha, \beta)}(0 * a) * 0) \\
 &= (d_{(\alpha, \beta)}(0) * \alpha(0 * a)) \wedge d_{(\alpha, \beta)}(0 * a) \\
 &= d_{(\alpha, \beta)}(0 * a) * (d_{(\alpha, \beta)}(0 * a) * (d_{(\alpha, \beta)}(0) * \alpha(0 * a))) \\
 &= d_{(\alpha, \beta)}(0) * \alpha(0 * a) \\
 &= d_{(\alpha, \beta)}(0) * (0 * \alpha(a)) \\
 &= d_{(\alpha, \beta)}(0) + \alpha(a).
 \end{aligned} \tag{3.16}$$

(4) Let $a, b \in L_p(X)$. Then $a + b \in L_p(X)$. Using (3), we have

$$\begin{aligned}
 d_{(\alpha, \beta)}(a + b) &= d_{(\alpha, \beta)}(0) + \alpha(a + b) = d_{(\alpha, \beta)}(0) + \alpha(a) + \alpha(b) \\
 &= d_{(\alpha, \beta)}(0) + \alpha(a) + d_{(\alpha, \beta)}(0) + \alpha(b) - d_{(\alpha, \beta)}(0) \\
 &= d_{(\alpha, \beta)}(a) + d_{(\alpha, \beta)}(b) - d_{(\alpha, \beta)}(0).
 \end{aligned} \tag{3.17}$$

This completes the proof. □

Theorem 3.19. Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$ be a regular (α, β) -derivation on X such that $\alpha \circ d_{(\alpha,\beta)} = d_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)}^2 = 0$ on $L_p(X)$, then $d_{(\alpha,\beta)} = 0$ on $L_p(X)$.

Proof. Let us suppose $d_{(\alpha,\beta)}^2 = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (3) and (4), we have

$$\begin{aligned}
 0 &= d_{(\alpha,\beta)}^2(x + x) = d_{(\alpha,\beta)}(d_{(\alpha,\beta)}(x + x)) \\
 &= d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x + x)) = d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x + x) \\
 &= d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0) \\
 &= d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x).
 \end{aligned} \tag{3.18}$$

Since X is a torsion free. Therefore, $d_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ implying thereby $d_{(\alpha,\beta)} = 0$. This completes the proof. $\square$

Theorem 3.20. Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}, d'_{(\alpha,\beta)}$ be two regular (α, β) -derivations on X such that $\alpha \circ d'_{(\alpha,\beta)} = d'_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d'_{(\alpha,\beta)} = 0$ on $L_p(X)$.

Proof. Let us suppose $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (1) and (2), we have

$$\begin{aligned}
 0 &= \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} \right)(x + x) = d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(x + x) \right) = d_{(\alpha,\beta)}(0) + \alpha\left(d'_{(\alpha,\beta)}(x + x) \right) \\
 &= d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(x + x) = d_{(\alpha,\beta)}(0) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0) \right) \\
 &= \left(d_{(\alpha,\beta)}(0) - d'_{(\alpha,\beta)}(0) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= \left(\left(d_{(\alpha,\beta)}(0) * d'_{(\alpha,\beta)}(0) \right) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= \left(d_{(\alpha,\beta)}(0) * \left(0 * d'_{(\alpha,\beta)}(0) \right) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= \left(d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(0) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= \left(d_{(\alpha,\beta)}(0) + \alpha d'_{(\alpha,\beta)}(0) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= d_{(\alpha,\beta)}\left(d'_{(\alpha,\beta)}(0) \right) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) \\
 &= \left(d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} \right)(0) + \left(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) \right) = d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x).
 \end{aligned} \tag{3.19}$$

Since X is a torsion free. Therefore $d'_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ and so $d'_{(\alpha,\beta)} = 0$. This completes the proof. $\square$

Proposition 3.21. Let $d_{(\alpha,\beta)}$ be a regular (α, β) -derivation of a BCI-algebra X . If $d_{(\alpha,\beta)}^2 = 0$ on $L_p(X)$, then $(\alpha \circ d_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

Proof. Assume that $d_{(\alpha,\beta)}^2 = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.18 (3) and (4), we have

$$\begin{aligned} 0 &= d_{(\alpha,\beta)}^2(x + x) = d_{(\alpha,\beta)}(d_{(\alpha,\beta)}(x + x)) = d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x + x)) \\ &= d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0)) \\ &= d_{(\alpha,\beta)}(0) + 2\alpha(d_{(\alpha,\beta)}(x)) - \alpha(d_{(\alpha,\beta)}(0)). \end{aligned} \quad (3.20)$$

Hence $(\alpha \circ d_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

This completes the proof. $\square$

Proposition 3.22. Let $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ be two regular (α, β) -derivations of a BCI-algebra X . If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d'_{(\alpha,\beta)})(x) = (1/2)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

Proof. Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and so $d'_{(\alpha,\beta)}(x + x) \in L_p(X)$ by Theorem 3.18 (1). It follows from Theorem 3.18 (3) and (4) that

$$\begin{aligned} 0 &= (d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(x + x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x + x)) \\ &= d_{(\alpha,\beta)}(0) + \alpha(d'_{(\alpha,\beta)}(x + x)) \\ &= d_{(\alpha,\beta)}(0) + \alpha(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0)) \\ &= d_{(\alpha,\beta)}(0) + 2\alpha(d'_{(\alpha,\beta)}(x)) - \alpha(d'_{(\alpha,\beta)}(0)) \end{aligned} \quad (3.21)$$

so that $\alpha(d'_{(\alpha,\beta)}(x)) = (1/2)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

This completes the proof. $\square$

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