Research Article

The Form of the Solutions and Periodicity of Some Systems of Difference Equations

M. Mansour,^{1,2} M. M. El-Dessoky,^{1,2} and E. M. Elsayed^{1,2}

¹ *Mathematics Department, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

² *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to M. Mansour, mansour@mans.edu.eg

Received 20 May 2012; Accepted 6 July 2012

Academic Editor: Garyfalos Papaschinopoulos

Copyright © 2012 M. Mansour et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to get the form of the solutions and the periodic nature of the following systems of rational difference equations $x_{n+1} = x_{n-5}/(-1 + x_{n-5}y_{n-2})$, $y_{n+1} = y_{n-5}/(\pm 1 \pm y_{n-5}x_{n-2})$, where the initial conditions are real numbers.

1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential equations. They have many applications in biology, ecology, economy, and physics. So, recently, there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions, see [1–23] and the references cited therein.

Periodic solutions of a difference equations have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution.

The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}} \quad (1.1)$$

was studied by Çinar in [5].

Elsayed [11] has obtained the solution of the following system of the difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, \quad y_{n+1} = \frac{y_{n-k}}{x_n y_n}. \quad (1.2)$$

The behavior of the positive solution of the following system:

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1} x_n}. \quad (1.3)$$

has been studied by Kurbanli et al. [22].

Özban [24] has investigated the positive solution of the system of rational difference equations as

$$x_{n+1} = \frac{1}{y_{n-k}}, \quad y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}. \quad (1.4)$$

Özban [25] has investigated the solution of the following system:

$$x_{n+1} = \frac{a}{y_{n-3}}, \quad y_{n+1} = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}. \quad (1.5)$$

In [26] Yalcinkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations:

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}. \quad (1.6)$$

Also, Yalcinkaya [27] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations as

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}. \quad (1.7)$$

Yang et al. [28] has investigated the positive solution of the system following:

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}}. \quad (1.8)$$

Similar nonlinear systems of rational difference equations were investigated [26–41].

In this paper, we investigate the behavior of the solutions of the difference equations systems as

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-5} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5} x_{n-2}}, \quad (1.9)$$

where the initial conditions x_i, y_i for $i = -5, -4, -3, -2, -1, 0$ are real numbers.

2. The First System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(1 + x_{n-2}y_{n-5})$

In this section, we investigate the solution of the system of two difference equations as

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2}y_{n-5}}, \quad (2.1)$$

where the initial conditions are arbitrary real numbers with $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq 1$, $\neq 1/2$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq \pm 1$.

The following theorem is devoted to the form of the solutions of system (2.1).

Theorem 2.1. Suppose that $\{x_n, y_n\}$ are solutions of system (2.1). Also, assume that the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary real numbers and let $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$. Then for $n = 0, 1, 2, \dots$, one has

$$\begin{aligned} x_{12n-5} &= \frac{(-1)^n f (-1 + 2pf)^n}{(-1 + pf)^{2n}}, & x_{12n-4} &= \frac{(-1)^n e (-1 + 2eh)^n}{(-1 + eh)^{2n}}, \\ x_{12n-3} &= \frac{(-1)^n d (-1 + 2dg)^n}{(-1 + dg)^{2n}}, & x_{12n-2} &= c(1 + sc)^n (1 - sc)^n, \\ x_{12n-1} &= b(1 + br)^n (1 - br)^n, & x_{12n} &= a(1 + aq)^n (1 - aq)^n, \\ x_{12n+1} &= \frac{(-1)^n f (-1 + 2pf)^n}{(-1 + pf)^{2n+1}}, & x_{12n+2} &= \frac{(-1)^n e (-1 + 2eh)^n}{(-1 + eh)^{2n+1}}, \\ x_{12n+3} &= \frac{(-1)^n d (-1 + 2dg)^n}{(-1 + dg)^{2n+1}}, & x_{12n+4} &= -c(1 + sc)^{n+1} (1 - sc)^n, \\ x_{12n+5} &= -b(1 + br)^{n+1} (1 - br)^n, & x_{12n+6} &= -a(1 + aq)^{n+1} (1 - aq)^n, \\ y_{12n-5} &= \frac{s}{(1 + sc)^n (1 - sc)^n}, & y_{12n-4} &= \frac{r}{(1 + br)^n (1 - br)^n}, \\ y_{12n-3} &= \frac{q}{(1 + aq)^n (1 - aq)^n}, & y_{12n-2} &= \frac{(-1)^n p (-1 + pf)^{2n}}{(-1 + 2pf)^n}, \\ y_{12n-1} &= \frac{(-1)^n h (-1 + eh)^{2n}}{(-1 + 2eh)^n}, & y_{12n} &= \frac{(-1)^n g (-1 + dg)^{2n}}{(-1 + 2dg)^n}, \end{aligned}$$

$$\begin{aligned}
y_{12n+1} &= \frac{s}{(1+sc)^{n+1}(1-sc)^n}, & y_{12n+2} &= \frac{r}{(1+br)^{n+1}(1-br)^n}, \\
y_{12n+3} &= \frac{q}{(1+aq)^{n+1}(1-aq)^n}, & y_{12n+4} &= \frac{(-1)^n p(-1+pf)^{2n+1}}{(-1+2pf)^{n+1}}, \\
y_{12n+5} &= \frac{(-1)^n h(-1+eh)^{2n+1}}{(-1+2eh)^{n+1}}, & y_{12n+6} &= \frac{(-1)^n g(-1+dg)^{2n+1}}{(-1+2dg)^{n+1}}.
\end{aligned} \tag{2.2}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, that is,

$$\begin{aligned}
x_{12n-17} &= \frac{(-1)^{n-1} f(-1+2pf)^{n-1}}{(-1+pf)^{2n-2}}, & x_{12n-16} &= \frac{(-1)^{n-1} e(-1+2eh)^{n-1}}{(-1+eh)^{2n-2}}, \\
x_{12n-15} &= \frac{(-1)^{n-1} d(-1+2dg)^{n-1}}{(-1+dg)^{2n-2}}, & x_{12n-14} &= c(1+sc)^{n-1}(1-sc)^{n-1}, \\
x_{12n-13} &= b(1+br)^{n-1}(1-br)^{n-1}, & x_{12n-12} &= a(1+aq)^{n-1}(1-aq)^{n-1}, \\
x_{12n-11} &= \frac{(-1)^{n-1} f(-1+2pf)^{n-1}}{(-1+pf)^{2n-1}}, & x_{12n-10} &= \frac{(-1)^{n-1} e(-1+2eh)^{n-1}}{(-1+eh)^{2n-1}}, \\
x_{12n-9} &= \frac{(-1)^{n-1} d(-1+2dg)^{n-1}}{(-1+dg)^{2n-1}}, & x_{12n-8} &= -c(1+sc)^n(1-sc)^{n-1}, \\
x_{12n-7} &= -b(1+br)^n(1-br)^{n-1}, & x_{12n-6} &= -a(1+aq)^n(1-aq)^{n-1}, \\
y_{12n-17} &= \frac{s}{(1+sc)^{n-1}(1-sc)^{n-1}}, & y_{12n-16} &= \frac{r}{(1+br)^{n-1}(1-br)^{n-1}}, \\
y_{12n-15} &= \frac{q}{(1+aq)^{n-1}(1-aq)^{n-1}}, & y_{12n-14} &= \frac{(-1)^{n-1} p(-1+pf)^{2n-2}}{(-1+2pf)^{n-1}}, \\
y_{12n-13} &= \frac{(-1)^{n-1} h(-1+eh)^{2n-2}}{(-1+2eh)^{n-1}}, & y_{12n-12} &= \frac{(-1)^{n-1} g(-1+dg)^{2n-2}}{(-1+2dg)^{n-1}}, \\
y_{12n-11} &= \frac{s}{(1+sc)^n(1-sc)^{n-1}}, & y_{12n-10} &= \frac{r}{(1+br)^n(1-br)^{n-1}}, \\
y_{12n-9} &= \frac{q}{(1+aq)^n(1-aq)^{n-1}}, & y_{12n-8} &= \frac{(-1)^{n-1} p(-1+pf)^{2n-1}}{(-1+2pf)^n}, \\
y_{12n-7} &= \frac{(-1)^{n-1} h(-1+eh)^{2n-1}}{(-1+2eh)^n}, & y_{12n-6} &= \frac{(-1)^{n-1} g(-1+dg)^{2n-1}}{(-1+2dg)^n}.
\end{aligned} \tag{2.3}$$

Now, it follows from (2.1) that

$$\begin{aligned}
x_{12n-5} &= \frac{x_{12n-11}}{-1 + y_{12n-8}x_{12n-11}} \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}/(-1+pf)^{2n-1}}{\left(-1 + \left((-1)^{n-1}p(-1+pf)^{2n-1}/(-1+2pf)^n\right)\left((-1)^{n-1}f(-1+2pf)^{n-1}/(-1+pf)^{2n-1}\right)\right)} \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}}{(-1+pf)^{2n-1}(-1+pf/(-1+2pf))} \left(\frac{-1+2pf}{-1+2pf}\right) \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}(-1+2pf)}{(-1+pf)^{2n-1}(1-2pf+pf)} = \frac{(-1)^nf(-1+2pf)^n}{(-1+pf)^{2n}}, \\
y_{12n-5} &= \frac{y_{12n-11}}{1 + x_{12n-8}y_{12n-11}} \\
&= \frac{s/(1+sc)^n(1-sc)^{n-1}}{\left(1 - c(1+sc)^n(1-sc)^{n-1}\left(s/(1+sc)^n(1-sc)^{n-1}\right)\right)} \\
&= \frac{s}{(1+sc)^n(1-sc)^{n-1}(1-sc)} = \frac{s}{(1+sc)^n(1-sc)^n}, \\
x_{12n-4} &= \frac{x_{12n-10}}{-1 + y_{12n-7}x_{12n-10}} \\
&= \frac{(-1)^{n-1}e(-1+2eh)^{n-1}/(-1+eh)^{2n-1}}{\left(-1 + \left((-1)^{n-1}h(-1+eh)^{2n-1}/(-1+2eh)^n\right)\left((-1)^{n-1}e(-1+2eh)^{n-1}/(-1+eh)^{2n-1}\right)\right)} \\
&= \frac{(-1)^{n-1}e(-1+2eh)^{n-1}}{(-1+eh)^{2n-1}(-1+he/(-1+2eh))} \left(\frac{-1+2eh}{-1+2eh}\right) \\
&= \frac{(-1)^{n-1}e(-1+2eh)^n}{(-1+eh)^{2n-1}(1-2eh+he)} = \frac{(-1)^ne(-1+2eh)^n}{(-1+eh)^{2n}}, \\
y_{12n-4} &= \frac{y_{12n-10}}{1 + x_{12n-7}y_{12n-10}} = \frac{r/(1+br)^n(1-br)^{n-1}}{\left(1 - b(1+br)^n(1-br)^{n-1}\left(r/(1+br)^n(1-br)^{n-1}\right)\right)} \\
&= \frac{r}{(1+br)^n(1-br)^{n-1}(1-br)} = \frac{r}{(1+br)^n(1-br)^n}.
\end{aligned} \tag{2.4}$$

Similarly, we can prove the other relations. $\square$

Lemma 2.2. Let $\{x_n, y_n\}$ be a positive solution of system (2.1), then $\{y_n\}$ is bounded and converges to zero.

Figure 1

Proof. It follows from (2.1) that

$$y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2}y_{n-5}} \leq y_{n-5}. \quad (2.5)$$

Then, the subsequences $\{y_{6n-5}\}_{n=0}^{\infty}$, $\{y_{6n-4}\}_{n=0}^{\infty}$, $\{y_{6n-3}\}_{n=0}^{\infty}$, $\{y_{6n-2}\}_{n=0}^{\infty}$, $\{y_{6n-1}\}_{n=0}^{\infty}$, and $\{y_{6n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0\}$. $\square$

Example 2.3. We consider interesting numerical example for the difference system (2.1) with the initial conditions, where $x_{-5} = 0.8$, $x_{-4} = 0.1$, $x_{-3} = -1.6$, $x_{-2} = 0.3$, $x_{-1} = 0.1$, $x_0 = -0.7$, $y_{-5} = 1.7$, $y_{-4} = 0.3$, $y_{-3} = 0.4$, $y_{-2} = -0.2$, $y_{-1} = 0.5$, and $y_0 = 0.6$ (see Figure 1).

3. The Second System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(-1 + x_{n-2}y_{n-5})$

In this section, we study the solution of the following system of the difference equations:

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{-1 + x_{n-2}y_{n-5}}, \quad (3.1)$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers such that $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0$, $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq 1$.

Theorem 3.1. Assume that $\{x_n, y_n\}$ are solutions of system (3.1). Then for $n = 0, 1, 2, \dots$, one has

$$\begin{aligned}
 x_{6n-5} &= \frac{f}{(-1+pf)^n}, & x_{6n-4} &= \frac{e}{(-1+eh)^n}, & x_{6n-3} &= \frac{d}{(-1+dg)^n}, \\
 x_{6n-2} &= c(-1+sc)^n, & x_{6n-1} &= b(-1+br)^n, & x_{6n} &= a(-1+aq)^n, \\
 y_{6n-5} &= \frac{s}{(-1+sc)^n}, & y_{6n-4} &= \frac{r}{(-1+br)^n}, & y_{6n-3} &= \frac{q}{(-1+aq)^n}, \\
 y_{6n-2} &= p(-1+pf)^n, & y_{6n-1} &= h(-1+eh)^n, & y_{6n} &= g(-1+dg)^n.
 \end{aligned} \tag{3.2}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$, that is,

$$\begin{aligned}
 x_{6n-11} &= \frac{f}{(-1+pf)^{n-1}}, & x_{6n-10} &= \frac{e}{(-1+eh)^{n-1}}, & x_{6n-9} &= \frac{d}{(-1+dg)^{n-1}}, \\
 x_{6n-8} &= c(-1+sc)^{n-1}, & x_{6n-7} &= b(-1+br)^{n-1}, & x_{6n-6} &= a(-1+aq)^{n-1}, \\
 y_{6n-11} &= \frac{s}{(-1+sc)^{n-1}}, & y_{6n-10} &= \frac{r}{(-1+br)^{n-1}}, & y_{6n-9} &= \frac{q}{(-1+aq)^{n-1}}, \\
 y_{6n-8} &= p(-1+pf)^{n-1}, & y_{6n-7} &= h(-1+eh)^{n-1}, & y_{6n-6} &= g(-1+dg)^{n-1}.
 \end{aligned} \tag{3.3}$$

Now, it follows from (3.1) that

$$\begin{aligned}
 x_{6n-5} &= \frac{x_{6n-11}}{-1+y_{6n-8}x_{6n-11}} = \frac{f/(-1+pf)^{n-1}}{\left(-1+p(-1+pf)^{n-1} \times \left(f/(-1+pf)^{n-1}\right)\right)} \\
 &= \frac{f/(-1+pf)^{n-1}}{(-1+pf)^{n-1}} = \frac{f}{(-1+pf)^n}, \\
 y_{6n-5} &= \frac{y_{6n-11}}{-1+x_{6n-8}y_{6n-11}} = \frac{(s/(-1+sc)^{n-1})}{\left(-1+c(-1+sc)^{n-1} \left(s/(-1+sc)^{n-1}\right)\right)} \\
 &= \frac{s/(-1+sc)^{n-1}}{(-1+cs)^{n-1}} = \frac{s}{(-1+sc)^n}.
 \end{aligned} \tag{3.4}$$

Also, we see from (3.1) that

$$\begin{aligned}
 x_{6n-2} &= \frac{x_{6n-8}}{-1 + y_{6n-5}x_{6n-8}} = \frac{c(-1 + sc)^{n-1}}{\left(-1 + (s/(-1 + sc)^n)c(-1 + sc)^{n-1}\right)} \\
 &= \frac{c(-1 + sc)^{n-1}}{(-1 + sc/(-1 + sc))} \left(\frac{(-1 + sc)}{(-1 + sc)}\right) = \frac{c(-1 + sc)^n}{1 - sc + sc} = c(-1 + sc)^n, \\
 y_{6n-2} &= \frac{y_{6n-8}}{-1 + x_{6n-5}y_{6n-8}} = \frac{p(-1 + pf)^{n-1}}{\left(-1 + p(-1 + pf)^{n-1}(f/(-1 + pf)^n)\right)} \\
 &= \frac{p(-1 + pf)^{n-1}}{(-1 + pf/(-1 + pf))} \left(\frac{-1 + pf}{-1 + pf}\right) = \frac{p(-1 + pf)^n}{(1 - pf + pf)} = p(-1 + pf)^n.
 \end{aligned} \tag{3.5}$$

Similarly, we can prove the other relations. $\square$

Lemma 3.2. *The solutions of system (3.1) has unboundedness solutions except in the following case.*

Theorem 3.3. *System (3.1) has a periodic solution of period six if and only if $pf = eh = dg = br = aq = sc = 2$ and it will take the form $\{x_n\} = \{f, e, d, c, b, a, f, e, \dots\}$, $\{y_n\} = \{s, r, q, p, h, g, s, r, \dots\}$.*

Proof. First suppose that there exists a prime period-six solution

$$\{x_n\} = \{f, e, d, c, b, a, f, e, \dots\}, \quad \{y_n\} = \{s, r, q, p, h, g, s, r, \dots\}, \tag{3.6}$$

of system (3.1). We see from the form of the solution of system (3.1) that

$$\begin{aligned}
 f &= \frac{f}{(-1 + pf)^n}, & e &= \frac{e}{(-1 + eh)^n}, & d &= \frac{d}{(-1 + dg)^n}, \\
 c &= c(-1 + sc)^n, & b &= b(-1 + br)^n, & a &= a(-1 + aq)^n, \\
 s &= \frac{s}{(-1 + sc)^n}, & r &= \frac{r}{(-1 + br)^n}, & q &= \frac{q}{(-1 + aq)^n}, \\
 p &= p(-1 + pf)^n, & h &= h(-1 + eh)^n, & g &= g(-1 + dg)^n.
 \end{aligned} \tag{3.7}$$

Then, we get

$$-1 + pf = -1 + eh = -1 + dg = -1 + sc = -1 + br = -1 + aq = 1. \tag{3.8}$$

Thus,

$$pf = eh = dg = br = aq = sc = 2. \tag{3.9}$$

Figure 2

Second, assume that $pf = eh = dg = br = aq = sc = 2$. Then, we see from the form of the solution of system (3.1) that

$$\begin{aligned}
 x_{6n-5} &= f, & x_{6n-4} &= e, & x_{6n-3} &= d, \\
 x_{6n-2} &= c, & x_{6n-1} &= b, & x_{6n} &= a, \\
 y_{6n-5} &= s, & y_{6n-4} &= r, & y_{6n-3} &= q, \\
 y_{6n-2} &= p, & y_{6n-1} &= h, & y_{6n} &= g.
 \end{aligned} \tag{3.10}$$

Thus, we have a periodic solution of period six and the proof is complete. $\square$

Example 3.4. Figure 2 shows the behavior of the solution of the difference system (3.1) with the initial conditions, where $x_{-5} = 0.18$, $x_{-4} = -0.41$, $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = -0.21$, $x_0 = .7$, $y_{-5} = -0.17$, $y_{-4} = 1.3$, $y_{-3} = .14$, $y_{-2} = 0.2$, $y_{-1} = -0.15$, and $y_0 = 0.16$.

Example 3.5. If we consider the difference equation system (3.1) with the initial conditions, where $x_{-5} = 5$, $x_{-4} = -2$, $x_{-3} = .1$, $x_{-2} = -7$, $x_{-1} = 4$, $x_0 = -3$, $y_{-5} = -2/7$, $y_{-4} = 0.5$, $y_{-3} = -2/3$, $y_{-2} = 0.4$, $y_{-1} = -1$, and $y_0 = 20$, then we get the shape of Figure 3.

Figure 3

4. The Third System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(1 - x_{n-2}y_{n-5})$

In this section, we obtain the form of the solution of the system of two difference equations as

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{1 - x_{n-2}y_{n-5}}, \quad (4.1)$$

where the initial conditions are arbitrary real numbers such that $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq \pm 1$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq 1/2, 1$.

Theorem 4.1. Suppose that $\{x_n, y_n\}$ are solutions of system (4.1). Then

$$\begin{aligned} x_{12n-5} &= \frac{(-1)^n f}{(-1 + pf)^n (1 + pf)^n}, & x_{12n-4} &= \frac{(-1)^n e}{(-1 + eh)^n (1 + eh)^n}, \\ x_{12n-3} &= \frac{(-1)^n d}{(-1 + dg)^n (1 + dg)^n}, & x_{12n-2} &= \frac{(-1)^n c(1 - sc)^{2n}}{(-1 + 2sc)^n}, \end{aligned}$$

$$\begin{aligned}
x_{12n-1} &= \frac{(-1)^n b(1-br)^{2n}}{(-1+2br)^n}, & x_{12n} &= \frac{(-1)^n a(1-aq)^{2n}}{(-1+2aq)^n}, \\
x_{12n+1} &= \frac{(-1)^n f}{(-1+pf)^{n+1}(1+pf)^n}, & x_{12n+2} &= \frac{(-1)^n e}{(-1+eh)^{n+1}(1+eh)^n}, \\
x_{12n+3} &= \frac{(-1)^n d}{(-1+dg)^{n+1}(1+dg)^n}, & x_{12n+4} &= \frac{(-1)^n c(1-sc)^{2n+1}}{(-1+2sc)^{n+1}}, \\
x_{12n+5} &= \frac{(-1)^n b(1-br)^{2n+1}}{(-1+2br)^{n+1}}, & x_{12n+6} &= \frac{(-1)^n a(1-aq)^{2n+1}}{(-1+2aq)^{n+1}}, \\
y_{12n-5} &= \frac{(-1)^n s(-1+2sc)^n}{(1-sc)^{2n}}, & y_{12n-4} &= \frac{(-1)^n r(-1+2br)^n}{(1-br)^{2n}}, \\
y_{12n-3} &= \frac{(-1)^n q(-1+2aq)^n}{(1-aq)^{2n}}, & y_{12n-2} &= (-1)^n p(-1+pf)^n(1+pf)^n, \\
y_{12n-1} &= (-1)^n h(-1+eh)^n(1+eh)^n, & y_{12n} &= (-1)^n g(-1+dg)^n(1+dg)^n, \\
y_{12n+1} &= \frac{(-1)^n s(-1+2sc)^n}{(1-sc)^{2n+1}}, & y_{12n+2} &= \frac{(-1)^n r(-1+2br)^n}{(1-br)^{2n+1}}, \\
y_{12n+3} &= \frac{(-1)^n q(-1+2aq)^n}{(1-aq)^{2n+1}}, & y_{12n+4} &= (-1)^{n+1} p(-1+pf)^{n+1}(1+pf)^n, \\
y_{12n+5} &= (-1)^{n+1} h(-1+eh)^{n+1}(1+eh)^n, & y_{12n+6} &= (-1)^{n+1} g(-1+dg)^{n+1}(1+dg)^n,
\end{aligned} \tag{4.2}$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$.

Proof. As the proof of Theorem 2.1, and so it will be omitted. $\square$

Example 4.2. Figure 4 shows the behavior of the solutions of the system (4.1) with the initial conditions $x_{-5} = 0.05$, $x_{-4} = -0.42$, $x_{-3} = 0.11$, $x_{-2} = 0.07$, $x_{-1} = -0.4$, $x_0 = -3$, $y_{-5} = -1.7$, $y_{-4} = 0.12$, $y_{-3} = -1.2$, $y_{-2} = 0.2$, $y_{-1} = -1$, and $y_0 = 0.13$.

5. The Fourth System: $x_{n+1} = x_{n-5}/(-1+y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(-1-x_{n-2}y_{n-5})$

We get, in this section, the solution of the following system of the difference equations:

$$x_{n+1} = \frac{x_{n-5}}{-1+y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{-1-x_{n-2}y_{n-5}}, \tag{5.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Figure 4

Theorem 5.1. Let $\{x_n, y_n\}_{n=-5}^{+\infty}$ be solutions of system (5.1). Then for $n = 0, 1, 2, \dots$ one has

$$\begin{aligned}
 x_{6n-5} &= f \prod_{i=0}^{n-1} \frac{(1 - (2i)pf)}{(-1 + (2i+1)pf)}, & x_{6n-4} &= e \prod_{i=0}^{n-1} \frac{(1 - (2i)eh)}{(-1 + (2i+1)eh)}, \\
 x_{6n-3} &= d \prod_{i=0}^{n-1} \frac{(1 - (2i)dg)}{(-1 + (2i+1)dg)}, & x_{6n-2} &= c \prod_{i=0}^{n-1} \frac{(-1 - (2i+1)sc)}{(1 + (2i+2)sc)}, \\
 x_{6n-1} &= b \prod_{i=0}^{n-1} \frac{(-1 - (2i+1)br)}{(1 + (2i+2)br)}, & x_{6n} &= a \prod_{i=0}^{n-1} \frac{(-1 - (2i+1)aq)}{(1 + (2i+2)aq)}, \\
 y_{6n-5} &= s \prod_{i=0}^{n-1} \frac{(1 + (2i)sc)}{(-1 - (2i+1)sc)}, & y_{6n-4} &= r \prod_{i=0}^{n-1} \frac{(1 + (2i)br)}{(-1 - (2i+1)br)}, \\
 y_{6n-3} &= q \prod_{i=0}^{n-1} \frac{(1 + (2i)aq)}{(-1 - (2i+1)aq)}, & y_{6n-2} &= p \prod_{i=0}^{n-1} \frac{(-1 + (2i+1)pf)}{(1 - (2i+2)pf)}, \\
 y_{6n-1} &= h \prod_{i=0}^{n-1} \frac{(-1 + (2i+1)eh)}{(1 - (2i+2)eh)}, & y_{6n} &= g \prod_{i=0}^{n-1} \frac{(-1 + (2i+1)dg)}{(1 - (2i+2)dg)},
 \end{aligned} \tag{5.2}$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$, that is,

$$\begin{aligned}
 x_{6n-11} &= f \prod_{i=0}^{n-2} \frac{(1 - (2i)pf)}{(-1 + (2i+1)pf)}, & x_{6n-10} &= e \prod_{i=0}^{n-2} \frac{(1 - (2i)eh)}{(-1 + (2i+1)eh)}, \\
 x_{6n-9} &= d \prod_{i=0}^{n-2} \frac{(1 - (2i)dg)}{(-1 + (2i+1)dg)}, & x_{6n-8} &= c \prod_{i=0}^{n-2} \frac{(-1 - (2i+1)sc)}{(1 + (2i+2)sc)}, \\
 x_{6n-7} &= b \prod_{i=0}^{n-2} \frac{(-1 - (2i+1)br)}{(1 + (2i+2)br)}, & x_{6n-6} &= a \prod_{i=0}^{n-2} \frac{(-1 - (2i+1)aq)}{(1 + (2i+2)aq)}, \\
 y_{6n-11} &= s \prod_{i=0}^{n-2} \frac{(1 + (2i)sc)}{(-1 - (2i+1)sc)}, & y_{6n-10} &= r \prod_{i=0}^{n-2} \frac{(1 + (2i)br)}{(-1 - (2i+1)br)}, \\
 y_{6n-9} &= q \prod_{i=0}^{n-2} \frac{(1 + (2i)aq)}{(-1 - (2i+1)aq)}, & y_{6n-8} &= p \prod_{i=0}^{n-2} \frac{(-1 + (2i+1)pf)}{(1 - (2i+2)pf)}, \\
 y_{6n-7} &= h \prod_{i=0}^{n-2} \frac{(-1 + (2i+1)eh)}{(1 - (2i+2)eh)}, & y_{6n-6} &= g \prod_{i=0}^{n-2} \frac{(-1 + (2i+1)dg)}{(1 - (2i+2)dg)}.
 \end{aligned} \tag{5.3}$$

It follows from (3.1) that

$$\begin{aligned}
 &x_{6n-5} \\
 &= \frac{x_{6n-11}}{-1 + y_{6n-8}x_{6n-11}} \\
 &= \frac{f \prod_{i=0}^{n-2} ((1 - (2i)pf) / (-1 + (2i+1)pf))}{\left(-1 + p \prod_{i=0}^{n-2} ((-1 + (2i+1)pf) / (1 - (2i+2)pf)) f \prod_{i=0}^{n-2} ((1 - (2i)pf) / (-1 + (2i+1)pf)) \right)} \\
 &= \frac{f \prod_{i=0}^{n-2} ((1 - (2i)pf) / (-1 + (2i+1)pf))}{\left(-1 + p f \prod_{i=0}^{n-2} ((1 - (2i)pf) / (1 - (2i+2)pf)) \right)} = \frac{f \prod_{i=0}^{n-2} ((1 - (2i)pf) / (-1 + (2i+1)pf))}{(-1 + pf / (1 - (2n-2)pf))} \\
 &= f \prod_{i=0}^{n-2} \frac{(1 - (2i)pf)}{(-1 + (2i+1)pf)} \frac{(1 - (2n-2)pf)}{(-1 + (2n-2)pf + pf)} = f \prod_{i=0}^{n-1} \frac{(1 - (2i)pf)}{(-1 + (2i+1)pf)},
 \end{aligned}$$

$$\begin{aligned}
& y_{6n-5} \\
&= \frac{y_{6n-11}}{-1 - x_{6n-8}y_{6n-11}} \\
&= \frac{\prod_{i=0}^{n-2} ((1 + (2i)sc) / (-1 - (2i + 1)sc))}{\left(-1 - c \prod_{i=0}^{n-2} ((-1 - (2i+1)sc) / (1 + (2i+2)sc)) s \prod_{i=0}^{n-2} ((1 + (2i)sc) / (-1 - (2i + 1)sc)) \right)} \\
&= \frac{\prod_{i=0}^{n-2} ((1 + (2i)sc) / (-1 - (2i + 1)sc))}{\left(-1 - sc \prod_{i=0}^{n-2} ((1 + (2i)sc) / (1 + (2i + 2)sc)) \right)} = \frac{\prod_{i=0}^{n-2} ((1 + (2i)sc) / (-1 - (2i + 1)sc))}{(-1 - sc / (1 + (2n - 2)sc))} \\
&= \prod_{i=0}^{n-2} \frac{(1 + (2i)sc)}{(-1 - (2i + 1)sc)} \frac{(1 + (2n - 2)sc)}{(-1 - (2n - 2)sc - sc)}.
\end{aligned} \tag{5.4}$$

Then, we see that

$$y_{6n-5} = s \prod_{i=0}^{n-1} \frac{(1 + (2i)sc)}{(-1 - (2i + 1)sc)}. \tag{5.5}$$

Similarly, we can prove the other relations. This completes the proof. $\square$

Lemma 5.2. *If $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 are arbitrary real numbers and $\{x_n, y_n\}$ are solutions of system (5.1), then the following statements are true.*

- (i) *If $x_{-5} = 0, y_{-2} \neq 0$, then we have $x_{6n-5} = 0$ and $y_{6n-2} = (-1)^n y_{-2}$.*
- (ii) *If $x_{-4} = 0, y_{-1} \neq 0$, then we have $x_{6n-4} = 0$ and $y_{6n-1} = (-1)^n y_{-1}$.*
- (iii) *If $x_{-3} = 0, y_0 \neq 0$, then we have $x_{6n-3} = 0$ and $y_{6n} = (-1)^n y_0$.*
- (iv) *If $x_{-2} = 0, y_{-5} \neq 0$, then we have $x_{6n-2} = 0$ and $y_{6n-5} = (-1)^n y_{-5}$.*
- (v) *If $x_{-1} = 0, y_{-4} \neq 0$, then we have $x_{6n-1} = 0$ and $y_{6n-4} = (-1)^n y_{-4}$.*
- (vi) *If $x_0 = 0, y_{-3} \neq 0$, then we have $x_{6n} = 0$ and $y_{6n-3} = (-1)^n y_{-3}$.*
- (vii) *If $y_{-5} = 0, x_{-2} \neq 0$, then we have $y_{6n-5} = 0$ and $x_{6n-2} = (-1)^n x_{-2}$.*
- (viii) *If $y_{-4} = 0, x_{-1} \neq 0$, then we have $y_{6n-4} = 0$ and $x_{6n-1} = (-1)^n x_{-1}$.*
- (ix) *If $y_{-3} = 0, x_0 \neq 0$, then we have $y_{6n-3} = 0$ and $x_{6n} = (-1)^n x_0$.*
- (x) *If $y_{-2} = 0, x_{-5} \neq 0$, then we have $y_{6n-2} = 0$ and $x_{6n-5} = (-1)^n x_{-5}$.*
- (xi) *If $y_{-1} = 0, x_{-4} \neq 0$, then we have $y_{6n-1} = 0$ and $x_{6n-4} = (-1)^n x_{-4}$.*
- (xii) *If $y_0 = 0, x_{-3} \neq 0$, then we have $y_{6n} = 0$ and $x_{6n-3} = (-1)^n x_{-3}$.*

Proof. The proof follows from the form of the solution of system (5.1). $\square$

Example 5.3. If we take the system of difference equations (5.1) with the initial conditions $x_{-5} = 0.05, x_{-4} = -0.42, x_{-3} = .101, x_{-2} = 0.07, x_{-1} = -0.4, x_0 = -3, y_{-5} = -0.7, y_{-4} = .12, y_{-3} = -1.2, y_{-2} = 0.2, y_{-1} = -0.11$, and $y_0 = 0.13$, we get the following shape of the solution, see Figure 5.

Figure 5

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks the DSR technical and financial support.

References

- [1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.
- [2] R. P. Agarwal and E. M. Elsayed, "On the solution of fourth-order rational recursive sequence," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 4, pp. 525–545, 2010.
- [3] N. Battaloglu, C. Cinar, and I. Yalcinkaya, "The dynamics of the difference equation $x_{n+1} = (\alpha x_{n-m})/(\beta + \gamma x_{n-(k+1)}^p)$," *Ars Combinatoria*, vol. 97, pp. 281–288, 2010.
- [4] E. Camouzis and G. Papaschinopoulos, "Global asymptotic behavior of positive solutions on the system of rational difference equations $x_{n+1} = 1 + 1/y_{n-k}$, $y_{n+1} = y_n/x_{n-m}y_{n-m-k}$," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 733–737, 2004.
- [5] C. Çinar, "On the positive solutions of the difference equation system $x_{n+1} = 1/y_n$, $y_{n+1} = y_n/x_{n-1}y_{n-1}$," *Applied Mathematics and Computation*, vol. 158, no. 2, pp. 303–305, 2004.
- [6] C. Cinar, I. Yalçinkaya, and R. Karatas, "On the positive solutions of the difference equation system $x_{n+1} = m/y_n$, $y_{n+1} = py_n/x_{n-1}y_{n-1}$," *Journal of Institute of Mathematics and Computer Science*, vol. 18, pp. 135–136, 2005.
- [7] D. Clark and M. R. S. Kulenović, "A coupled system of rational difference equations," *Computers & Mathematics with Applications*, vol. 43, no. 6-7, pp. 849–867, 2002.
- [8] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, "Global behavior of the solutions of difference equation," *Advances in Difference Equations*, vol. 2011, 28 pages, 2011.

- [9] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, "Some properties and expressions of solutions for a class of nonlinear difference equation," *Utilitas Mathematica*, vol. 87, pp. 93–110, 2012.
- [10] E. M. Elabbasy and E. M. Elsayed, "Global attractivity and periodic nature of a difference equation," *World Applied Sciences Journal*, vol. 12, no. 1, pp. 39–47, 2011.
- [11] E. M. Elsayed, "On the solutions of a rational system of difference equations," *Polytechnica Posnaniensis*, no. 45, pp. 25–36, 2010.
- [12] E. M. Elsayed, "Dynamics of recursive sequence of order two," *Kyungpook Mathematical Journal*, vol. 50, no. 4, pp. 483–497, 2010.
- [13] E. M. M. Elsayed, "Behavior of a rational recursive sequences," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 56, no. 1, pp. 27–42, 2011.
- [14] E. M. Elsayed, "Solution of a recursive sequence of order ten," *General Mathematics*, vol. 19, no. 1, pp. 145–162, 2011.
- [15] E. M. Elsayed, "Solution and attractivity for a rational recursive sequence," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 982309, 17 pages, 2011.
- [16] E. M. Elsayed, "On the solution of some difference equations," *European Journal of Pure and Applied Mathematics*, vol. 4, no. 3, pp. 287–303, 2011.
- [17] E. M. Elsayed, "On the dynamics of a higher-order rational recursive sequence," *Communications in Mathematical Analysis*, vol. 12, no. 1, pp. 117–133, 2012.
- [18] E. M. Elsayed, "Solutions of rational difference system of order two," *Mathematical and Computer Modelling*, vol. 55, pp. 378–384, 2012.
- [19] E. M. Elsayed, M. M. El-Dessoky, and A. Alotaibi, "On the solutions of a general system of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 892571, 12 pages, 2012.
- [20] M. E. Erdogan, C. Cinar, and I. Yalcinkaya, "On the dynamics of the recursive sequence $x_{n+1} = (bx_{n-1}) / (A + Bx_n^p x_{n-2}^q)$," *Computers & Mathematics with Applications*, vol. 61, no. 3, pp. 533–537, 2011.
- [21] E. A. Grove, G. Ladas, L. C. McGrath, and C. T. Teixeira, "Existence and behavior of solutions of a rational system," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 1–25, 2001.
- [22] A. S. Kurbanlı, C. Çinar, and I. Yalcinkaya, "On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = x_{n-1} / (y_n x_{n-1} - 1)$, $y_{n+1} = y_{n-1} / (x_n y_{n-1} - 1)$," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 1261–1267, 2011.
- [23] A. S. Kurbanlı, "On the behavior of solutions of the system of rational difference equations: $x_{n+1} = x_{n-1} / (y_n x_{n-1} - 1)$, $y_{n+1} = y_{n-1} / (x_n y_{n-1} - 1)$, and $z_{n+1} = z_{n-1} / (y_n z_{n-1} - 1)$," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 932362, 12 pages, 2011.
- [24] A. Y. Özban, "On the positive solutions of the system of rational difference equations $x_{n+1} = 1 / y_{n-k}$, $y_{n+1} = y_n / x_{n-m} y_{n-m-k}$," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 26–32, 2006.
- [25] A. Y. Özban, "On the system of rational difference equations $x_{n+1} = a / y_{n-3}$, $y_{n+1} = b y_{n-3} / x_{n-q} y_{n-q}$," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 833–837, 2007.
- [26] I. Yalcinkaya, "On the global asymptotic stability of a second-order system of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 860152, 12 pages, 2008.
- [27] I. Yalcinkaya, "On the global asymptotic behavior of a system of two nonlinear difference equations," *Ars Combinatoria*, vol. 95, pp. 151–159, 2010.
- [28] X. Yang, Y. Liu, and S. Bai, "On the system of high order rational difference equations $x_n = a / y_{n-p}$, $y_n = b y_{n-p} / x_{n-q} y_{n-q}$," *Applied Mathematics and Computation*, vol. 171, no. 2, pp. 853–856, 2005.
- [29] C. J. Schinas, "Invariants for difference equations and systems of difference equations of rational form," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 164–179, 1997.
- [30] Y. Zhang, X. Yang, D. J. Evans, and C. Zhu, "On the nonlinear difference equation system $x_{n+1} = A + y_{n-m} / x_n$, $y_{n+1} = A + x_{n-m} / y_n$," *Computers & Mathematics with Applications*, vol. 53, no. 10, pp. 1561–1566, 2007.
- [31] D. Simsek, B. Demir, and C. Cinar, "On the solutions of the system of difference equations $x_{n+1} = \max\{A / x_n, y_n / x_n\}$, $y_{n+1} = \max\{A / y_n, x_n / y_n\}$," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 325296, 11 pages, 2009.
- [32] S. Stević, "On a system of difference equations with period two coefficients," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4317–4324, 2011.
- [33] S. Stević, "On a system of difference equations," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3372–3378, 2011.
- [34] S. Stević, "On some solvable systems of difference equations," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5010–5018, 2012.

- [35] N. Touafek and E. M. Elsayed, "On the solutions of systems of rational difference equations," *Mathematical and Computer Modelling*, vol. 55, pp. 1987–1997, 2012.
- [36] I. Yalcinkaya, C. Cinar, and D. Simsek, "Global asymptotic stability of a system of difference equations," *Applicable Analysis*, vol. 87, no. 6, pp. 677–687, 2008.
- [37] I. Yalcinkaya and C. Çinar, "Global asymptotic stability of a system of two nonlinear difference equations," *Fasciculi Mathematici*, no. 43, pp. 171–180, 2010.
- [38] I. Yalçinkaya, C. Çinar, and M. Atalay, "On the solutions of systems of difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 143943, 9 pages, 2008.
- [39] X. Yang, "On the system of rational difference equations $x_n = A + y_{n-1}/x_{n-p}y_{n-q}$, $y_n = A + x_{n-1}/x_{n-r}y_{n-s}$," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 1, pp. 305–311, 2005.
- [40] E. M. E. Zayed and M. A. El-Moneam, "On the rational recursive sequence $x_{n+1} = ax_n - (bx_n/cx_n - dx_{n-k})$," *Communications on Applied Nonlinear Analysis*, vol. 15, no. 2, pp. 47–57, 2008.
- [41] Y. Zhang, X. Yang, G. M. Megson, and D. J. Evans, "On the system of rational difference equations $x_{n+1} = a/y_{n-3}$, $y_{n+1} = by_{n-3}/x_{n-q}y_{n-q}$," *Applied Mathematics and Computation*, vol. 176, no. 2, pp. 403–408, 2006.