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Research Article

Positive Solution of a Nonlinear Fractional Differential Equation Involving Caputo Derivative

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This paper is concerned with a nonlinear fractional differential equation involving Caputo derivative. By constructing the upper and lower control functions of the nonlinear term without any monotone requirement and applying the method of upper and lower solutions and the Schauder fixed point theorem, the existence and uniqueness of positive solution for the initial value problem are investigated. Moreover, the existence of maximal and minimal solutions is also obtained.

1. Introduction

Fractional differential equation can be extensively applied to various disciplines such as physics, mechanics, chemistry, and engineering, see [1–3]. Hence, in recent years, fractional differential equations have been of great interest and there have been many results on existence and uniqueness of the solution of FDE, see [4–8]. Especially, Diethelm and Ford [9] have gained existence, uniqueness, and structural stability of solution of the type of fractional differential equation

$$D^q\left(y - T_{m-1}[y]\right)(x) = f\left(x, y(x)\right), \quad y^{(k)}(0) = y_0^{(k)},\ k = 0, 1, \ldots, m-1, \tag{1.1}$$

where $q > 0$ is a real number, D^q denotes the Riemann-Liouville differential operator of order q, and $T_{m-1}[y]$ is the Taylor polynomial of order $(m-1)$ for the function $y(x)$ at

$x_0 = 0$. Recently, Daftardar-Gejji and Jafari [10] have discussed the existence, uniqueness, and stability of solution of the system of nonlinear fractional differential equation

$$D_*^\alpha y(t) = f\big(t, y(t)\big), \quad 0 < t < 1, \qquad y^{(k)}(0) = c_k, \quad 0 \le k \le m, \quad 0 \le c_k, \tag{1.2}$$

where $m < \alpha \le m + 1$ and D_*^α denotes Caputo fractional derivative (see Definition 2.3). Delbosco and Rodino [11] have proved existence and uniqueness theorems for the nonlinear fractional equation

$$D^\delta u = f(t, u), \quad 0 < t < 1, \ u(0) = 0, \tag{1.3}$$

where $0 < \delta < 1$, D^δ is the Riemann-Liouville fractional derivative. Zhang [12] used the method of the upper and lower solution and cone fixed point theorem to obtain the existence and uniqueness of positive solution to (1.3). Yao [13] considered the existence of positive solution to (1.3) controlled by the power function employing Krasnosel'skii fixed point theorem of cone expansion-compression type. The existence of the local and global solution for (1.3) was obtained by Lakshmikantham and Vatsala [14] utilizing classical differential equation theorem.

More recently, Zhang [15] shows the existence of positive solutions to the singular boundary value problem for fractional differential equation

$$\begin{gathered} D_{0+}^\alpha u(t) + g(t) f\left(u, u', \ldots, u^{(n-2)}\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{gathered} \tag{1.4}$$

where D_{0+}^α is the Riemann-Liouville fractional derivative of order $n - 1 < \alpha \le 2$, $n \ge 2$.

However, in the previous works, the nonlinear term has to satisfy the monotone or others control conditions. In fact, the fractional differential equations with nonmonotone function can respond better to impersonal law, so it is very important to weaken monotone condition. Considering this, in this paper, we mainly investigate the fractional differential Equation (1.2) without any monotone requirement on nonlinear term by constructing upper and lower control function and exploiting the method of upper and lower solutions and the Schauder fixed point theorem. The existence and uniqueness of positive solution for (1.2) are obtained. Some properties concerning the maximal and minimal solutions are also given. This work is motivated by the above references and my previous work [16, 17]. Other related results on the fractional differential equations can be found in [18–24].

This paper is organized as follow. In Section 2, we recall briefly some notions of the fractional calculus and the theory of the operators for integration and differentiation of fractional order. Section 3 is devoted to the study of the existence and uniqueness of positive solution for (1.2) utilizing the upper and lower solution method and the Schauder fixed point theorem. The existence of maximal and minimal solutions for (1.2) is given in Section 4.

2. Preliminaries and Notations

First, we give some basic definitions and theorems which are basically used throughout this paper. $C[0,1]$ denotes the space of continuous functions defined on $[0,1]$ and $C^n[0,1]$

denotes the class of all real valued functions defined on $[0,1]$ which have continuous nth order derivative.

Definition 2.1. Let $f(x) \in C[0,1]$ and $\alpha > 0$, then the expression

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad 0 < x < 1 \tag{2.1}$$

is called the (left-sided) Riemann-Liouville integral of order α.

Definition 2.2. Let $n-1 < \alpha \leq n$, $n \in N$, then the expression

$$D_0^\alpha f(x) = \frac{d^n}{dx^n}\left[I_0^{n-\alpha} f(x)\right], \quad 0 < x < 1 \tag{2.2}$$

is called the (left-sided) Riemann-Liouville derivative of $f(x)$ of order α whenever the expression on the right-hand side is defined.

Definition 2.3. Let $f(x) \in C^n[0,1]$ and $n-1 < \alpha \leq n$, $n \in N$, then the expression

$$D_{*0}^\alpha f(x) = I_0^{n-\alpha} f^{(n)}(x) \tag{2.3}$$

is called the (left-sided) Caputo derivative of $f(x)$ of order α.

In further discussion we will denote D_0^α, I_0^α, and D_{*0}^α as D^α, I^α, and D_*^α, respectively.

Lemma 2.4 (see [25, 26]). *Let $f(x) \in C^n[0,1]$ and $n-1 < \alpha \leq n$, $n \in N$, then we one has*

$$I^\mu I^\nu f(x) = I^{\mu+\nu} f(x), \quad \mu, \nu \geq 0, \tag{2.4}$$

$$D^\alpha I^\alpha f(x) = f(x), \tag{2.5}$$

$$I^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0+). \tag{2.6}$$

Lemma 2.5 (see, [10]). *If the function $f(t, y(t))$ is $C^1[0,1]$, then the initial value problem* (1.2) *is equivalent to the Volterra integral equations*

$$y(t) = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^\alpha f(t, y(t)), \quad 0 < t < 1, \ m < \alpha \leq m+1. \tag{2.7}$$

Proof. Suppose $y(t)$ satisfies the initial value problem (1.2), then applying I^{α} to both sides of (1.2) and using Lemma 2.4 (2.7) follows. Conversely, suppose $y(t)$ satisfies (2.7). Then observe that $D^{(m+1)}y(t)$ exists and is integrable, because

$$\begin{aligned} y^{(m+1)}(t) &= D^{m+1}\left(\sum_{k=0}^{m}\frac{t^k}{k!}\,c_k + I^{\alpha}f(t,y(t))\right) \\ &= D^{m+1}I^{\alpha}f(t,y(t)) = DD^{m}I^{m}I^{\alpha-m}f(t,y(t)) \\ &= DI^{\alpha-m}f(t,y(t)) = D^{m+1-\alpha}f(t,y(t)), \end{aligned} \tag{2.8}$$

which exists and is integrable as $f(t,y(t))$ is $C^1[0,1]$. Thus $I^{m+1-\alpha}y^{(m+1)}(t) = D_*^{\alpha}y(t)$ exists.

Applying D_*^{α} on both sides of (2.7), one has

$$\begin{aligned} D_*^{\alpha}y(t) &= I^{m+1-\alpha}D^{m+1}\left(\sum_{k=0}^{m}\frac{t^k}{k!}\,c_k + I^{\alpha}f(t,y(t))\right) \\ &= I^{m+1-\alpha}D^{m+1-\alpha}f(t,y(t)) = f(t,y(t)), \end{aligned} \tag{2.9}$$

as $f(t,y(t))$ is continuous and $0 < m+1-\alpha < 1$. Hence $y(t)$ satisfies (1.2). Moreover, from (2.4), $y^{(k)}(0) = c_k$, $0 \le k \le m$ hold. □

Let $X = C[0,1]$ be the Banach space endowed with the infinity norm and K a nonempty closed subset of X defined as $K = \{y(t) \in X \mid 0 < y(t) \le l, 0 < t \le 1, y^{(k)}(0) = c_k, 1 \le k \le m, 0 \le c_k\}$. The positive solution which we consider in this paper is a function such that $y(t) \in K$.

According to Lemma 2.5, (1.2) is equivalent to the fractional integral Equation (2.7). The integral equation (2.7) is also equivalent to fixed point equation $Ty(t) = y(t)$, $y(t) \in C[0,1]$, where operator $T : K \to K$ is defined as

$$Ty(t) = \sum_{k=0}^{m}\frac{t^k}{k!}\,c_k + I^{\alpha}f(t,y(t)), \tag{2.10}$$

then we have the following lemma.

Lemma 2.6. *Let $f : [0,1]\times[0,l] \to R^{+}$ a given continuous function. Then the operator $T : K \to K$ is completely continuous.*

Proof. Let $M \subset K$ be bounded, that is, there exists a positive constant l^* such that $\|y\|_{\infty} \le l^*$ for any $y(t) \in M$. Since $f(t,y(t))$ is a given continuous function, we have

$$\max_{0\le t\le 1} f(t,y(t)) \le \max_{(t,y)\in D} f(t,y), \quad \text{for any } y(t) \in M, \tag{2.11}$$

where $D = \{(t,y) \mid 0 \le t \le 1,\ 0 \le y \le l^*\}$.

Let $L = \max_{(t,y)\in D} f(t,y)$, then for any $y(t) \in M$, we have

$$\begin{aligned}
\left|Ty(t)\right| &= \left|\sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^{\alpha} f\left(t, y(t)\right)\right| \\
&= \left|\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s)\right) ds\right| \\
&\leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f\left(s, y(s)\right)\right| ds \\
&\leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \ \leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{L}{\Gamma(1+\alpha)} t^{\alpha}.
\end{aligned} \tag{2.12}$$

Thus,

$$\|Ty\| \leq \sum_{k=0}^{m} \frac{c_k}{k!} \ + \frac{L}{\Gamma(1+\alpha)}. \tag{2.13}$$

Hence $T : K \to K$ is uniformly bounded.

Now, we prove that $T : K \to K$ is continuous. Since $f(t, y(t))$ is continuous function in a compact set $[0,1] \times [0,l]$, then it is uniformly continuous there. Thus given $\varepsilon > 0$, we can find $\mu > 0$ such that $\|f(t,y) - f(t,z)\| < \varepsilon^*$ whenever $\|y - z\| < \mu$, where $\varepsilon^* = \varepsilon\Gamma(\alpha+1)$. Then

$$\begin{aligned}
\left|Ty(t) - Tz(t)\right| &\leq \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f\left(s, y(s)\right) - f\left(s, z(s)\right)\right] ds\right| \\
&\leq \varepsilon^* \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds\right| \\
&= \varepsilon^* \frac{t^{\alpha}}{\Gamma(\alpha+1)} < \varepsilon,
\end{aligned} \tag{2.14}$$

proving the continuity of the operators $T : K \to K$.

Now, we will prove that the operator $T : K \to K$ is equicontinuous. For each $y(t) \in M$, any $\varepsilon > 0, t_1, t_2 \in [0,1]$ and $t_1 < t_2$. Let $\delta = \min\{(\varepsilon/2)(\sum_{k=1}^{m}(c_k/(k-1)!\,)^{-1}, (\varepsilon\,\Gamma(1+\alpha)/4L)^{1/\alpha}\}$, then when $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}
\left|Ty(t_1) - Ty(t_2)\right| &= \left|\sum_{k=0}^{m}\frac{t_1^k}{k!}c_k + I^\alpha f\left(t_1, y(t_1)\right) - \sum_{k=0}^{m}\frac{t_2^k}{k!}c_k - I^\alpha f\left(t_2, y(t_2)\right)\right| \\
&\le \left|\sum_{k=0}^{m}\frac{t_1^k}{k!}c_k - \sum_{k=0}^{m}\frac{t_2^k}{k!}c_k\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s))ds\right| \\
&\le (t_2-t_1)\sum_{k=1}^{m}\frac{c_k}{(k-1)!} + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right|\left|f(s,u(s))\right|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\
&\le (t_2-t_1)\sum_{k=1}^{m}\frac{c_k}{(k-1)!} + \frac{L}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)ds \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&= (t_2-t_1)\sum_{k=1}^{m}\frac{c_k}{(k-1)!} \\
&\quad + \frac{L}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_1-s)^{\alpha-1}ds - \int_0^{t_1}(t_2-s)^{\alpha-1}ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right) \\
&= (t_2-t_1)\sum_{k=1}^{m}\frac{c_k}{(k-1)!} + \frac{L}{\Gamma(1+\alpha)}\left[t_1^\alpha + (t_2-t_1)^\alpha - t_2^\alpha + (t_2-t_1)^\alpha\right] \\
&\le (t_2-t_1)\sum_{k=1}^{m}\frac{c_k}{(k-1)!} + \frac{2L}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha < \delta\sum_{k=1}^{m}\frac{c_k}{(k-1)!} + \frac{2L}{\Gamma(1+\alpha)}\delta^\alpha = \varepsilon.
\end{aligned} \tag{2.15}$$

The Arzela-Ascoli Theorem implies that T is completely continuous. The proof is therefore completed. □

Lemma 2.7. *If the operator $A : X \to X$ is the contraction mapping, where X is the Banach space, then A has a unique fixed point in X.*

Let $f : [0,1] \times [0,l) \to R^+$ be a given function. Take $a, b \in R^+$, and $a < b < l$. For any $y \in [a,b]$ one defines the upper-control function $H(t,y) = \sup_{a\le\eta\le y} f(t,\eta)$, and lower-control function $h(t,y) = \inf_{y\le\eta\le b} f(t,\eta)$, obviously $H(t,y), h(t,y)$ is monotonous nondecreasing on y and $h(t,y) \le f(t,y) \le H(t,y)$.

Definition 2.8. Let $\widetilde{y}(t), \widehat{y}(t) \in K$, $b \geq \widetilde{y}(t) \geq \widehat{y}(t) \geq a$, and satisfy

$$\begin{gathered} D_*^\alpha \widetilde{y}(t) \geq f(t, \widetilde{y}(t)), \quad \widetilde{y}^{(k)}(0) \geq c_k, \quad m < \alpha \leq m+1, \quad 1 \leq k \leq m, \quad c_k \geq 0, \\ \left(\text{or } \widetilde{y}(t) \geq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^\alpha f(t, \widetilde{y}(t)),\ m < \alpha \leq m+1 \right), \\ D_*^\alpha \widehat{y}(t) \leq f(t, \widehat{y}(t)), \quad \widehat{y}^{(k)}(0) \leq c_k, \quad m < \alpha \leq m+1, \quad 1 \leq k \leq m, \quad c_k \geq 0, \\ \left(\text{or } \widehat{y}(t) \leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^\alpha f(t, \widehat{y}(t)),\ m < \alpha \leq m+1 \right), \end{gathered} \tag{2.16}$$

then the functions $\widetilde{y}(t), \widehat{y}(t)$ are called a pair of order upper and lower solutions for (1.2).

3. Existence and Uniqueness of Positive Solution

Now, we give and prove the main results of this paper.

Theorem 3.1. *Assume $f : [0,1] \times [0,l) \to [0,+\infty)$ is continuous, and $\widetilde{y}(t), \widehat{y}(t)$ are a pair of order upper and lower solutions of* (1.2)*, then the boundary value problem* (1.2) *exists one solution $y(t) \in C[0,1]$; moreover,*

$$\widetilde{y}(t) \geq y(t) \geq \widehat{y}(t), \quad t \in [0,1]. \tag{3.1}$$

Proof. Let

$$S = \{ z(t) \mid z(t) \in K,\ \widehat{y}(t) \leq z(t) \leq \widetilde{y}(t),\ t \in [0,1] \}, \tag{3.2}$$

endowed with the norm $\|z\| = \max_{t\in[0,1]} z(t)$, then we have $\|z\| \leq b$. Hence S is a convex, bounded, and closed subset of the Banach space X. According to Lemma 2.6, the operator $T : K \to K$ is completely continuous. Then we need only to prove $T : S \to S$.

For any $z(t) \in S$, we have $\widetilde{y}(t) \geq z(t) \geq \widehat{y}(t)$, then

$$\begin{aligned} Tz(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^\alpha f(t, z(t)) \\ &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) ds \\ &\leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(s, z(s)) ds \\ &\leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(s, \widetilde{y}(s)) ds \\ &\leq \widetilde{y}(t), \end{aligned}$$

$$\begin{aligned}
Tz(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^{\alpha} f(t, z(t)) \\
&= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s)) ds \\
&\geq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, z(s)) ds \\
&\geq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, \widehat{y}(s)) ds \\
&\geq \widehat{y}(t).
\end{aligned} \tag{3.3}$$

Hence $\widetilde{y}(t) \geq Tz(t) \geq \widehat{y}(t), 1 > t > 0$, that is, $T : S \to S$. According to Schauder fixed point theorem, the operator T exists at least one fixed point $y(t) \in S$, $0 < t < 1$. Therefore the boundary value problem (1.2) exists at least one solution $y(t) \in C[0,1]$, and $\widetilde{y}(t) \geq y(t) \geq \widehat{y}(t)$, $t \in [0,1]$. □

Corollary 3.2. *Assume $f : [0,1] \times [0,l) \to [0,+\infty)$ is continuous, and there exist $p_2 > p_1 \geq 0$, such that*

$$p_1 \leq f(t,s) \leq p_2, \quad (t,s) \in [0,1] \times [0,l), \tag{3.4}$$

then the boundary value problem (1.2) *exists at least one positive solution $y(t) \in C[0,1]$, moreover*

$$\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_1\, t^{\alpha}}{\Gamma(\alpha+1)} \leq y(t) \leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_2\, t^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.5}$$

Proof. By assumption (3.4) and the definition of control function, we have

$$p_1 \leq h(t,s) \leq H(t,s) \leq p_2, \quad (t,s) \in [0,1] \times [a,b]. \tag{3.6}$$

Now, we consider the equation

$$D_*^{\alpha} w(t) = p_2, \quad w^{(k)}(0) = c_k, \quad m < \alpha \leq m+1,\ 1 \leq k \leq m,\ c_k \geq 0. \tag{3.7}$$

Obviously, (3.7) has a positive solution

$$w(t) = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_2 ds = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_2 t^\alpha}{\Gamma(\alpha+1)}, \quad t \in [0,1],$$

$$\begin{aligned} w(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_2 t^\alpha}{\Gamma(\alpha+1)} \\ &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_2 ds \\ &\geq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(t, w(t)) ds, \end{aligned} \tag{3.8}$$

namely, $w(t)$ is a upper solution of (1.2). In the similar way, we obtain $v(t) = \sum_{k=0}^{m}((t^k/k!)c_k) + p_1 t^\alpha/\Gamma(\alpha+1)$ is the lower solution of (1.2). An application of Theorem 3.1 now yields that the boundary value problem (1.2) exists at least one positive solution $y(t) \in C[0,1]$, moreover

$$\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_1 t^\alpha}{\Gamma(\alpha+1)} \leq y(t) \leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{p_2 t^\alpha}{\Gamma(\alpha+1)}. \tag{3.9}$$

□

Corollary 3.3. *Assume $f : [0,1] \times [0,+\infty) \to [c,+\infty)$ is continuous, where $c > 0$, moreover*

$$c < \lim_{y \to +\infty} f(t,y) < +\infty, \quad t \in [0,1], \tag{3.10}$$

then the boundary value problem (1.2) *has at least one positive solution $u(t) \in C[0,1]$.*

Proof. By assumption (3.9), there are positive constants N, R, such that $f(t,y) \leq N$ whenever $u > R$. Let $M = \max_{0\leq t\leq 1, 0\leq y\leq R} f(t,y)$, then $f(t,y) \leq N + M$, $0 \leq y < +\infty$. By the definition of control function, one has $H(t,y) \leq N + M$, $0 \leq t \leq 1$, $0 \leq y < +\infty$.

Now, we consider the equation

$$D_*^\alpha w(t) = N + M, \quad w^{(k)}(0) = c_k, \quad m < \alpha \leq m+1, \quad 1 \leq k \leq m, \quad c_k \geq 0. \tag{3.11}$$

Obviously, (3.11) has a positive solution

$$w(t) = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(N+M)ds = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{(N+M)t^\alpha}{\Gamma(\alpha+1)}, \quad t \in [0,1],$$

$$\begin{aligned} w(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{(N+M)t^\alpha}{\Gamma(\alpha+1)} \\ &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(N+M)ds \\ &\geq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H(t, w(t))ds, \end{aligned} \tag{3.12}$$

namely, $w(t)$ is the upper solution of (1.2). In the similar way, we obtain $v(t) = \sum_{k=0}^{m}(t^k/k!)c_k + ct^\alpha/\Gamma(\alpha+1)$ is the lower solution of (1.2). Therefore, the boundary value problem of (1.2) has at least one positive solution $y(t) \in C[0,1]$, what is more, we have

$$\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{ct^\alpha}{\Gamma(\alpha+1)} \leq y(t) \leq \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{(N+M)t^\alpha}{\Gamma(\alpha+1)}. \tag{3.13}$$

□

Corollary 3.4. *Assume $f : [0,1] \times [0,+\infty) \to [c,+\infty)$ is continuous, where $c > 0$, moreover*

$$c < \lim_{u \to +\infty} \max_{0 \leq t \leq 1} \frac{f(t,y)}{y} = M < \Gamma(\alpha+1), \tag{3.14}$$

then the boundary value problem (1.2) *exists at least at one positive solution $u(t) \in C[0,1]$.*

Proof. According to $c < \lim_{y \to +\infty} \max_{0 \leq t \leq 1}(f(t,y)/y) = M < +\infty$, there exists $D > 0$, such that for any $y(t) \in X$, we have

$$f(t, y(t)) \leq My(t) + D. \tag{3.15}$$

By the definition of control function, we have

$$H(t, y(t)) \leq My(t) + D. \tag{3.16}$$

We now consider the equation

$$D_*^\alpha w(t) = My(t) + D, \quad w^{(k)}(0) = c_k, \quad m < \alpha \leq m+1, \quad 1 \leq k \leq m, \quad c_k \geq 0. \tag{3.17}$$

According to Lemma 2.5, (3.17) is equivalent to the integral equation

$$\begin{aligned} y(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^{\alpha}\big(My(t) + D\big) \\ &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(My(s) + D\big) ds, \quad m < \alpha \le m+1. \end{aligned} \tag{3.18}$$

Let $A : K \to K$ be an operator as follows:

$$\begin{aligned} A(y)(t) &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + I^{\alpha}\big(My(t) + D\big) \\ &= \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(My(s) + D\big) ds, \quad m < \alpha \le m+1. \end{aligned} \tag{3.19}$$

by Lemma 2.6, the operator A is completely continuous.

Let

$$B_R = \left\{ u(t) \in K \mid \left\| u - \frac{Dt^{\alpha}}{\Gamma(\alpha+1)} \right\| \le R < +\infty \right\}, \tag{3.20}$$

where $R > 0$ and satisfies that $\sum_{k=0}^{m}(c_k/k!) + (M/\Gamma(\alpha+1))[(D/\Gamma(\alpha+1)) + R] - R \le 0$, then B_R is convex, bounded, and closed subset of the Banach space $C[0,1]$. For any $y(t) \in B_R$, we have

$$\|u\| \le \frac{D}{\Gamma(\alpha+1)} + R, \tag{3.21}$$

then

$$\begin{aligned} \left\| Ay(t) - \frac{Dt^{\alpha}}{\Gamma(\alpha+1)} \right\| &= \left\| \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(My(s) + D\big) ds - \frac{Dt^{\alpha}}{\Gamma(\alpha+1)} \right\| \\ &= \left\| \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \right\| \\ &\le \sum_{k=0}^{m} \frac{t^k}{k!} c_k + M\|y(t)\| \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\le \sum_{k=0}^{m} \frac{c_k}{k!} + \frac{M}{\Gamma(\alpha+1)} \left(R + \frac{D}{\Gamma(\alpha+1)} \right) \le R, \end{aligned} \tag{3.22}$$

thus

$$\left\| Ay(t) - \frac{Dt^{\alpha}}{\Gamma(\alpha+1)} \right\| \le R. \tag{3.23}$$

Hence, the Schauder fixed theorem assures that the operator A has at least one fixed point and then (3.17) has at least one positive solution $y^*(t)$, therefore we have

$$y^*(t) = \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(My^*(s) + D\right) ds, \quad m < \alpha \le m+1. \tag{3.24}$$

Combining condition (3.16), we have

$$y^*(t) \ge \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H\left(s, y^*(s)\right) ds, \quad m < \alpha \le m+1. \tag{3.25}$$

Obviously, $y^*(t)$ is the upper solution of initial value problem (1.2), and $v(t) = \sum_{k=0}^{m}(c_k/k!) + ct^\alpha/\Gamma(\alpha+1)$ is the lower solution. By Theorem 3.1, system (1.2) has at least one positive solution $u(t) \in C[0,1]$. □

Corollary 3.5. *Assume $f : [0,1] \times [0,+\infty) \to [c,+\infty)$ is continuous and there exists $d > 0$, $e > 0$, such that*

$$\max\left\{f(t,l) : (t,l) \in [0,1] \times [0,d]\right\} \le e, \tag{3.26}$$

then the boundary value problem (1.2) *has at least one positive solution $y(t) \in C[0,1]$, moreover*

$$\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{c\ t^\alpha}{\Gamma(\alpha+1)} \le y(t) \le \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{et^\alpha}{\Gamma(\alpha+1)}. \tag{3.27}$$

Proof. By the definition of control function, we have

$$c \le h(t,l) \le H(t,l) \le e, \quad (t,l) \in [0,1] \times [0,d]. \tag{3.28}$$

By Corollary 3.2, the boundary value problem (1.2) has at least one positive solution $y(t) \in C[0,1]$, moreover

$$\sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{ct^\alpha}{\Gamma(\alpha+1)} \le y(t) \le \sum_{k=0}^{m} \frac{t^k}{k!} c_k + \frac{et^\alpha}{\Gamma(\alpha+1)}. \tag{3.29}$$

□

Theorem 3.6. *Let the conditions in Theorem 3.1 hold. Moreover for any $y_1(t), y_2(t) \in X$, $0 < t < 1$, there exists $l > 0$, such that*

$$\left|f(t,y_1) - f(t,y_2)\right| \le l\left|y_1 - y_2\right|, \tag{3.30}$$

then when $l/\Gamma(\alpha+1) < 1$, the boundary value problem (1.2) *has a unique positive solution $y(t) \in S$.*

Proof. According to Theorem 3.1, if the conditions in Theorem 3.1 hold, then the boundary value problem (1.2) has at least one positive solution in S. Hence we need only to prove that the operator T defined in (2.10) is the contraction mapping in X. In fact, for any $y_1(t), y_2(t) \in X$, by assumption (3.30), we have

$$\begin{aligned}
\left|Ty_1(t) - Ty_2(t)\right| &= \left|\sum_{k=0}^{m}\frac{t^k}{k!}c_k + I^\alpha f\left(t, y_1(t)\right) - \sum_{k=0}^{m}\frac{t^k}{k!}c_k - I^\alpha f\left(t, y_2(t)\right)\right| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y_1(s)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y_2(s)\right)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y_1(s)\right) - f\left(s, y_2(s)\right)\right|ds \\
&\le \frac{lt^\alpha}{\Gamma(\alpha+1)}\left|y_1 - y_2\right| < \frac{l}{\Gamma(\alpha+1)}\left|y_1 - y_2\right|.
\end{aligned} \tag{3.31}$$

Thus, when $l/\Gamma(\alpha+1) < 1$, the operator T is the contraction mapping. Then by Lemma 2.7, the boundary value problem (1.2) has a unique positive solution $y(t) \in S$. □

4. Maximal and Minimal Solutions Theorem

In this section, we consider the existence of maximal and minimal solutions for (1.2).

Definition 4.1. Let $m(t)$ be a solution of (1.2) in $[0,1]$, then $m(t)$ is said to be a maximal solution of (1.2), if for every solution $y(t)$ of (1.2) existing on $[0,1]$ the inequality $y(t) \le m(t)$, $t \in [0,1]$ holds. A minimal solution may be defined similarly by reversing the last inequality.

Theorem 4.2. *Let $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ be a given continuous and monotone nondecreasing with respect to the second variable. Assume that there exist two positive constants λ, μ $(\mu > \lambda)$ such that*

$$\lambda \le f(t, y) \le \mu. \tag{4.1}$$

Then there exist maximal solution $\varphi(t)$ and minimal solution $\eta(t)$ of (1.2) *on* $[0,1]$, *moreover*

$$\sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} \le \eta(t) \le \varphi(t) \le \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\mu t^\alpha}{\Gamma(\alpha+1)}, \quad 0 \le t \le 1. \tag{4.2}$$

Proof. It is easy to know that $\sum_{k=0}^{m}((t^k/k!)c_k) + ((\mu t^\alpha)/\Gamma(\alpha+1))$ and $\sum_{k=0}^{m}((t^k/k!)c_k) + ((\lambda t^\alpha)/\Gamma(\alpha+1))$ are the upper and lower solutions of (1.2), respectively. Then by using $\overline{y}^{(0)} = \sum_{k=0}^{m}((t^k/k!)c_k) + \mu t^\alpha/\Gamma(\alpha+1)$, $\underline{y}^{(0)} = \sum_{k=0}^{m}((t^k/k!)c_k) + \lambda t^\alpha/\Gamma(\alpha+1)$ as a pair of

coupled initial iterations we construct two sequences $\{\overline{y}^{(m)}\}$, $\{\underline{y}^{(m)}\}$from the following linear iteration process:

$$\begin{aligned}\overline{y}^{(m)}(t) &= \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \overline{y}^{(m-1)}(t)\right)ds,\\ \underline{y}^{(m)}(t) &= \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \underline{y}^{(m-1)}(t)\right)ds.\end{aligned} \tag{4.3}$$

It is easy to show from the monotone property of $f(t,y)$ and condition (4.1) that the sequences $\{\overline{y}^{(m)}\}$, $\{\underline{y}^{(m)}\}$ possess the following monotone property:

$$\begin{aligned}\sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} &= \underline{y}^{(0)} \le \underline{y}^{(m)} \le \underline{y}^{(m+1)} \le \overline{y}^{(m+1)} \le \overline{y}^{(m)} \le \overline{y}^{(0)}\\ &= \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\mu t^\alpha}{\Gamma(\alpha+1)} \quad (m = 1, 2, \ldots).\end{aligned} \tag{4.4}$$

The above property implies that

$$\lim_{m\to\infty} \overline{y}(t)^{(m)} = \varphi(t), \qquad \lim_{m\to\infty} \underline{y}(t)^{(m)} = \eta(t) \tag{4.5}$$

exist and satisfy the relation

$$\sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} \le \eta(t) \le \varphi(t) \le \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{\mu t^\alpha}{\Gamma(\alpha+1)}, \qquad 0 \le t \le 1. \tag{4.6}$$

Letting $m \to \infty$ in (4.3) shows that $\varphi(t)$ and $\eta(t)$ satisfy the equations

$$\begin{aligned}\varphi(t) &= \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \varphi(t)\right)ds,\\ \eta(t) &= \sum_{k=0}^{m}\frac{t^k}{k!}c_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \eta(t)\right)ds.\end{aligned} \tag{4.7}$$

It is easy to verify that the limits $\varphi(t)$ and $\eta(t)$ are maximal and minimal solutions of (1.2) in $S^* = \{\psi(t) \mid \psi(t) \in K, \sum_{k=0}^{m}((t^k/k!)c_k) + \lambda t^\alpha/\Gamma(\alpha+1) \le \psi(t) \le \sum_{k=0}^{m}((t^k/k!)c_k) + \mu t^\alpha/\Gamma(\alpha+1), st \in [0,1], \|\psi(t)\| = \max_{0\le t\le 1}\psi(t)\}$, respectively, furthermore, if $\varphi(t) = \eta(t)$ $(\equiv \zeta(t))$ then $\zeta(t)$ is the unique solution in S^*, and hence the proof is completed. □

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