Research Article
Some Identities on Bernoulli and Euler Numbers

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Recently, Kim introduced the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \). By using the equations of the fermionic and bosonic \( p \)-adic integral on \( \mathbb{Z}_p \), we give some interesting identities on Bernoulli and Euler numbers.

1. Introduction/Preliminaries

Let \( p \) be a fixed odd prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). The \( p \)-adic absolute value \(| \cdot |_p \) is normally defined by \(| p |_p = 1/p \).

Let \( \text{UD}(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \) and \( \text{C}(\mathbb{Z}_p) \) the space of continuous function on \( \mathbb{Z}_p \). For \( f \in \text{C}(\mathbb{Z}_p) \), the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see [1]}). \tag{1.1}
\]

The following fermionic \( p \)-adic integral equation on \( \mathbb{Z}_p \) is well known (see [1–3]):

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.2}
\]

where \( f_1(x) = f(x + 1) \).
From (1.1) and (1.2), we can derive the generating function of Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

(1.3)

where $E_n(x)$ is the $n$th ordinary Euler polynomial (see [1–4]). In the special case, $x = 0$, $E_n(0) = E_n$ is called the $n$th ordinary Euler number.

By (1.3), we get Witt’s formula for the $n$th Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = E_n(x), \quad \text{for } n \in \mathbb{Z}_+.$$  

(1.4)

Thus, by (1.4), we have

$$E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l,$$

(1.5)

with the usual convention about replacing $E^n$ by $E_n$ (see [5, 6]). From (1.3), we note that

$$(E + 1)^n + E_n = 2\delta_{0,n},$$

(1.6)

where $\delta_{k,n}$ is the Kronecker symbol (see [3]). By (1.2) and (1.4), we get

$$\int_{\mathbb{Z}_p} (x + y + 1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = 2x^n.$$  

(1.7)

Thus, by (1.4) and (1.7), we have

$$E_n(x + 1) + E_n(x) = 2x^n, \quad \text{for } n \in \mathbb{Z}_+.$$  

(1.8)

Equation (1.8) is equivalent to

$$x^n = E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x).$$

(1.9)

From (1.6), we can derive the following equation:

$$E_n(2) = 2 - E_n(1) = 2 + E_n - 2\delta_{0,n}, \quad \text{for } n \in \mathbb{Z}_+.$$  

(1.10)

For $f \in \text{UD}(\mathbb{Z}_p)$, the bosonic $p$-adic integral on $\mathbb{Z}_p$ is defined by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [4]).}$$

(1.11)
From (1.11), we can easily derive the following $I_1$-integral equation:

$$I_1(f) = I(f) + f'(0), \quad \text{(see [4, 7, 8])}, \quad (1.12)$$

where $f_1(x) = f(x + 1)$ and $f'(0) = df(x)/dx|_{x=0}$.

It is well known that the Bernoulli polynomial can be represented by the bosonic $p$-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.13)$$

where $B_n(x)$ is called the $n$th Bernoulli polynomial (see [4, 7–13]). In the special case, $x = 0$, $B_0(0) = B_0$ is called the $n$th Bernoulli number. By the definition of Bernoulli numbers and polynomials, we get

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l. \quad (1.14)$$

Thus, by (1.13) and (1.14), we see that

$$B_0 = 1, \quad (B + 1)^n - B_n = \delta_{1,n}. \quad (1.15)$$

with the usual convention about replacing $B^n$ by $B_n$ (see [1–22]).

By (1.11), we easily get

$$\int_{\mathbb{Z}_p} (1-x+y)^n d\mu_1(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y). \quad (1.16)$$

From (1.13), (1.14), and (1.16), we have

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{for } n \in \mathbb{Z}_+. \quad (1.17)$$

By (1.15), we get

$$B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n}. \quad (1.18)$$

Thus, by (1.17) and (1.18), we have

$$(-1)^n B_n(-1) = B_n(2) = n + B_n + \delta_{1,n}, \quad \text{(see [4])}. \quad (1.19)$$

From (1.12) and (1.13), we get

$$\int_{\mathbb{Z}_p} (x + 1 + y)^{n+1} d\mu_1(y) - \int_{\mathbb{Z}_p} (x + y)^{n+1} d\mu_1(y) = (n + 1)x^n. \quad (1.20)$$
Thus, by (1.13) and (1.20), we have
\[ B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n \quad \text{for } n \in \mathbb{Z}_+. \] (1.21)

Equation (1.21) is equivalent to the following equation:
\[ x^n = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} B_l(x) \quad \text{for } n \in \mathbb{Z}_+. \] (1.22)

In this paper we derive some interesting and new identities for the Bernoulli and Euler numbers from the \( p \)-adic integral equations on \( \mathbb{Z}_p \).

2. Some Identities on Bernoulli and Euler Numbers

From (1.1), we note that
\[
\int_{\mathbb{Z}_p} (1-x+y)^n \, d\mu_{-1}(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_{-1}(y). \tag{2.1}
\]

By (1.14) and (2.1), we get
\[ E_n(1-x) = (-1)^n E_n(x), \quad \text{where } n \in \mathbb{Z}_+. \] (2.2)

In the special case, \( x = -1 \), we have
\[ E_n(2) = (-1)^n E_n(-1) = 2 + E_n - 2\delta_{0,n}. \] (2.3)

Let us consider the following fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) as follows:
\[
\int_{\mathbb{Z}_p} x^n \, d\mu_{-1}(x) = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} \int_{\mathbb{Z}_p} B_l(x) \, d\mu_{-1}(x)
\]
\[ = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} \sum_{k=0}^{l} \binom{l}{k} B_{l-k} \int_{\mathbb{Z}_p} x^k \, d\mu_{-1}(x) \tag{2.4}
\]
\[ = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k. \]

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+ \), one has
\[ E_n = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k. \] (2.5)
It is known that $B_n(x) = (-1)^n B_n(1 - x)$. If we take the fermionic $p$-adic integral on both sides of (1.22), then we have

$$
\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} \int_{\mathbb{Z}_p} B_l(x) d\mu_{-1}(x)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \int_{\mathbb{Z}_p} B_l(1-x) d\mu_{-1}(x)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} B_{l-k} \int_{\mathbb{Z}_p} (1-x)^k d\mu_{-1}(x)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} B_{l-k} (-1)^k E_k (-1). \quad (2.6)
$$

From (2.2) and (2.6), we note that

$$
\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k (2)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} B_{l-k} (2 + E_k - 2\delta_{0,k}) \quad (2.7)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \left( 2B_l(1) + \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k - 2B_l \right)
$$

$$
= \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \left( \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k + 2\delta_{1,l} \right). \quad (2.8)
$$

Therefore, by (1.4) and (2.7), we obtain the following theorem.

**Theorem 2.2.** For $n \in \mathbb{Z}_+$, one has

$$
E_n = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \left( \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k + 2\delta_{1,l} \right). \quad (2.8)
$$

**Corollary 2.3.** For $n \in \mathbb{N}$, one has

$$
2 + E_n = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \left( \sum_{k=0}^{l} \binom{l}{k} B_{l-k} E_k \right). \quad (2.9)
$$
Let us take the bosonic $p$-adic integral on both sides of (1.9) as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = \int_{\mathbb{Z}_p} \left( E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x) \right) d\mu_1(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_1(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} \int_{\mathbb{Z}_p} x^k d\mu_1(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k. \quad (2.10)$$

Thus, by (1.14) and (2.10), we obtain the following theorem.

**Theorem 2.4.** For $n \in \mathbb{Z}_+$, one has

$$B_n = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k. \quad (2.11)$$

On the other hand, by (2.2) and (2.10), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = (-1)^n \int_{\mathbb{Z}_p} E_n(1-x) d\mu_1(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} E_l(1-x) d\mu_1(x)$$

$$= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} (-1)^l \int_{\mathbb{Z}_p} (1-x)^l d\mu_1(x)$$

$$+ \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} (-1)^k \int_{\mathbb{Z}_p} (1-x)^k d\mu_1(x)$$

$$= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} (-1)^l B_l(-1) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} (-1)^k B_k(-1)$$

$$= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_l(2) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k(2)$$

$$= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} (I - B_l + O_{l,l}) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} (I + B_k + O_{k,k})$$
\[ (-1)^n n E_{n-1}(1) + (-1)^n \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l} B_l + (-1)^n n E_{n-1} + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l E_{l-1}(1) \]
\[ + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l E_{l-1} \]
\[ = (-1)^n n(2 + E_{n-1} - 2\delta_{0,n-1}) + (-1)^n \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l} B_l + (-1)^n n E_{n-1} \]
\[ + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l (2 + E_{l-1} - \delta_{0,l-1}) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k \]
\[ + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l E_{l-1}, \]
(2.12)

where \( n \in \mathbb{N} \) with \( n \geq 2 \). Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.5.** For \( n \in \mathbb{N} \) with \( n \geq 2 \), one has

\[ B_{2n-1} = -\frac{2n-1}{2} - (2n-1)E_{2n-2}(-1) - \sum_{l=0}^{2n-2} \binom{2n-1}{l} E_{2n-1-l} B_l \]
\[ + \frac{1}{2} \sum_{l=0}^{2n-2} \binom{2n-1}{l} (-1)^l \sum_{k=0}^{l} \binom{l}{k} E_{l-k} B_k. \]
(2.13)

By (1.9) and (1.22), we get

\[
\int_{Z_p} \int_{Z_p} x^m y^n d\mu_{-1}(x) d\mu_1(y) \\
= \int_{Z_p} \left( \frac{1}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} B_k(x) \right) \left( E_n(y) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(y) \right) d\mu_{-1}(x) d\mu_1(y) \\
= \frac{1}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} \int_{Z_p} B_k(x) E_n(y) d\mu_{-1}(x) d\mu_1(y) \\
+ \frac{1}{2(m+1)} \sum_{k=0}^{m+1} \sum_{l=0}^{n-1} \binom{m+1}{k} \binom{n}{l} \int_{Z_p} B_k(x) E_l(y) d\mu_{-1}(x) d\mu_1(y) \\
= \frac{1}{m+1} \sum_{k=0}^{m} \sum_{l=0}^{n-1} \binom{m+1}{k} \binom{n}{l} B_k \delta_0 \delta_{n-p} B_p E_l \\
+ \frac{1}{2(m+1)} \sum_{k=0}^{m+1} \sum_{l=0}^{n-1} \sum_{s=0}^{l} \binom{m+1}{k} \binom{n}{l} \binom{k}{s} \binom{l}{p} B_k \delta_0 \delta_{n-p} E_s B_p. 
\]
(2.14)

Therefore, by (1.4), (1.14), and (2.14), we obtain the following theorem.
Theorem 2.6. For \( m \in \mathbb{Z}_+ \) and \( n \in \mathbb{N} \), one has

\[
E_m B_n = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{n} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}
\]

\[+ \frac{1}{2(m+1)} \sum_{k=0}^{m} \sum_{i=0}^{n-1} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}. \]  

(2.15)

It is easy to show that

\[
\int_{\mathbb{Z}_p} x^{m+n} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \left( \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k(x) \right) \left( E_n(x) + \frac{1}{2} \sum_{i=0}^{n-1} \binom{n}{i} E_i(x) \right) d\mu_{-1}(x)
\]

\[= \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{n} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}
\]

\[+ \frac{1}{2(m+1)} \sum_{k=0}^{m} \sum_{i=0}^{n-1} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}. \]

(2.16)

Therefore, by (2.16), we obtain the following corollary.

Corollary 2.7. For \( m \in \mathbb{Z}_+ \) and \( n \in \mathbb{N} \), one has

\[
E_{m+n} = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{i=0}^{n} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}
\]

\[+ \frac{1}{2(m+1)} \sum_{k=0}^{m} \sum_{i=0}^{n-1} \sum_{j=0}^{l} \binom{m+1}{k} \binom{n}{i} \binom{l}{j} B_{k-i} E_{n-j} E_{i+j}. \]  

(2.17)

For \( f \in C(\mathbb{Z}_p) \), \( p \)-adic analogue of Bernstein operator of order \( n \) for \( f \) is given by

\[
B_n(f | x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x),
\]

(2.18)

where \( B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) for \( n, k \in \mathbb{Z}_+ \) is called the Bernstein polynomial of degree \( n \) (see [8]). From the definition of \( B_{k,n}(x) \), we note that \( B_{n-k,n}(1-x) = B_{k,n}(x) \).
Let us take the fermionic $p$-adic integral on $\mathbb{Z}_p$ for the product of $x^m$ and $B_{k,n}(x)$ as follows:

\[
\int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu_{-1}(x) = \frac{1}{m+1} \sum_{l=0}^{m} \binom{m+1}{l} \int_{\mathbb{Z}_p} B_l(x) B_{k,n}(x) d\mu_{-1}(x)
\]

\[
= \frac{n}{m+1} \sum_{l=0}^{m} \binom{m+1}{l} \binom{l}{j} B_{l-j} \int_{\mathbb{Z}_p} x^{j+k}(1-x)^{n-k} d\mu_{-1}(x)
\]

\[
= \frac{n}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{n-k} (-1)^j B_{l-j} \binom{m+1}{l} \binom{l}{j} \binom{n-k}{i} \int_{\mathbb{Z}_p} x^{j+k+l} d\mu_{-1}(x)
\]

\[
= \frac{n}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{n-k} (-1)^j \binom{m+1}{l} \binom{l}{j} \binom{n-k}{i} B_{l-j} E_{j+k+l}.
\]  

(2.19)

From (2.18), we note that

\[
\int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} x^{m+k}(1-x)^{n-k} d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{m+k+j} d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{m+k+j}.
\]  

(2.20)

Therefore, by (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.8.** For $m, n, k \in \mathbb{Z}_+$, one has

\[
\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{m+k+j} = \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{n-k} (-1)^j \binom{m+1}{l} \binom{l}{j} \binom{n-k}{i} B_{l-j} E_{j+k+l}.
\]  

(2.21)

In particular,

\[
(m+1) E_{m+n} = \sum_{l=0}^{m} \sum_{j=0}^{l} \binom{m+1}{l} \binom{l}{j} B_{l-j} E_{j+n}.
\]  

(2.22)
By (1.17) and the symmetric property of \( B_{k,n}(x) \), we get

\[
\int_{Z_p} x^m B_{k,n}(x) d\mu_{-1}(x) = \int_{Z_p} x^m B_{n-k,n}(1-x) d\mu_{-1}(x)
\]

\[
= \frac{1}{m+1} \sum_{l=0}^{m} (-1)^l \binom{m+1}{l} \int_{Z_p} B_l(1-x)B_{n-k,n}(1-x) d\mu_{-1}(x)
\]

\[
= \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} \int_{Z_p} (1-x)^{i+j+n-k} d\mu_{-1}(x).
\]

(2.23)

From (1.4) and (2.2), we note that

\[
\int_{Z_p} (1-x)^n d\mu_{-1}(x) = (-1)^n E_n(-1) = E_n(2) = 2 + E_n - 2\delta_{0,n}.
\]

(2.24)

By (2.23) and (2.24), we see that

\[
\int_{Z_p} x^m B_{k,n}(x) d\mu_{-1}(x) = \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} (2 + E_{i+j+n-k} - 2\delta_{0,i+j+n-k}).
\]

(2.25)

From (2.20) and (2.25), we have

\[
\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{m+k+j}
\]

\[
= \frac{2\delta_{0,k}}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} - \frac{2}{m+1} \sum_{l=0}^{m} \sum_{i=0}^{l} (m+1) \binom{m+1}{l} B_l \delta_{k,n}
\]

\[
+ \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} E_{i+j+n-k}
\]

(2.26)

\[
= \frac{2\delta_{0,k}}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} - \frac{2}{m+1} \sum_{l=0}^{m+1} (B_{m+1}(2) + (-1)^m B_{m+1}) \delta_{k,n}
\]

\[
+ \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{k}{i} B_{l-j} E_{i+j+n-k}.
\]

Therefore, by (1.19) and (2.26), we obtain the following theorem.
Theorem 2.9. For \( m, n, k \in \mathbb{N} \) with \( n \geq k \), one has

\[
\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{m+j} = \frac{1}{m+1} \sum_{l=0}^{m} \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^{i+l} \binom{m+1}{l} \binom{l}{j} \binom{n}{i} B_{l-j} E_{i+j, n-k} \tag{2.27}
\]

\[
- \frac{2}{m+1} (B_{m+1} + m + 1 + (-1)^m B_{m+1}).
\]

In particular,

\[
(2m + 2) (E_{2m+n+1} + 2) = \sum_{l=0}^{2m+1} \sum_{j=0}^{n} (-1)^{i+l} \binom{2m+2}{l} \binom{n}{i} B_{l-j} E_{i+j}. \tag{2.28}
\]

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References


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