Research Article

Some Identities on Bernoulli and Hermite Polynomials Associated with Jacobi Polynomials

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We investigate some identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space $\mathbf{P}_n$.

1. Introduction

For $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are defined as

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2} \right) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^n \frac{\binom{n}{k} (1+\alpha+\beta+n)_k}{(\alpha+1)_k} \left(\frac{x-1}{2} \right)^k, \end{aligned} \quad (1.1)$$

(see [1–4]), where $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$.

From (1.1), we note that

$$P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(\alpha+1+n)}{n! \Gamma(\alpha+\beta+n+1)} \sum_{k=0}^n \frac{\binom{n}{k} \Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{x-1}{2} \right)^k. \quad (1.2)$$

By (1.2), we see that $P_n^{(\alpha, \beta)}(x)$ is polynomial of degree n with real coefficients. It is not difficult to show that the leading coefficient of $P_n^{(\alpha, \beta)}(x)$ is $2^{-n} \binom{\alpha + \beta + 2n}{n}$. From (1.2), we have $P_n^{(\alpha, \beta)}(1) = \binom{\alpha + n}{n}$.

By (1.1), we get

$$\begin{aligned} \left(\frac{d}{dx}\right)^k P_n^{(\alpha, \beta)}(x) &= 2^{-k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} P_{n-k}^{(\alpha+k, \beta+k)}(x) \\ &= \frac{1}{2^k} (n + \alpha + \beta + k)(n + \alpha + \beta + k - 1) \cdots (n + \alpha + \beta + 1) P_{n-k}^{(\alpha+k, \beta+k)}(x), \end{aligned} \quad (1.3)$$

where k is a positive integer (see [1–4]).

The Rodrigues' formula for $P_n^{(\alpha, \beta)}(x)$ is given by

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^k \left\{ (1-x)^{n+\alpha} (1+x)^{n+\beta} \right\}. \quad (1.4)$$

It is easy to show that $u = P_n^{(\alpha, \beta)}(x)$ is a solution of the following differential equation:

$$(1-x^2)u'' + \{\beta - \alpha - (\alpha + \beta + 2)x\}u' + n(n + \alpha + \beta + 1)u = 0. \quad (1.5)$$

As is well known, the generating function of $P_n^{(\alpha, \beta)}(x)$ is given by

$$F(x, t) = \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) t^n = \frac{2^{\alpha+\beta}}{R(1-t+R)^\alpha (1+t+R)^\beta}, \quad (1.6)$$

where $R = \sqrt{1 - 2xt + t^2}$, (see [1–4]).

From (1.3), (1.4), and (1.6), we can derive the following identity:

$$\begin{aligned} \int_{-1}^1 P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx \\ = \frac{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1) \Gamma(n + 1)} \delta_{n,m}, \end{aligned} \quad (1.7)$$

where $\delta_{n,m}$ is the Kronecker symbol.

Let $\mathbf{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \leq n\}$. Then $\mathbf{P}_n$ is an inner product space with respect to the inner product $\langle q_1(x), q_2(x) \rangle = \int_{-1}^1 (1-x)^\alpha (1+x)^\beta q_1(x) q_2(x) dx$, where $q_1(x), q_2(x) \in \mathbf{P}_n$. From (1.7), we note that $\{P_0^{(\alpha, \beta)}(x), P_1^{(\alpha, \beta)}(x), \dots, P_n^{(\alpha, \beta)}(x)\}$ is an orthogonal basis for $\mathbf{P}_n$.

The so-called Euler polynomials $E_n(x)$ may be defined by means of

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.8)$$

(see [5–22]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the Euler numbers.

The Bernoulli polynomials are also defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.9)$$

(see [11–21]), with the usual convention about replacing $B^n(x)$ by $B_n(x)$. From (1.8) and (1.9), we note that

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k, \quad E_n(x) = \sum_{k=0}^n \binom{n}{k} E_{n-k} x^k. \quad (1.10)$$

For $n \in \mathbb{Z}_+$, we have

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \quad \frac{dE_n(x)}{dx} = nE_{n-1}(x) \quad (1.11)$$

(see [23–29]) By the definition of Bernoulli and Euler polynomials, we get

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad E_0 = 1, \quad E_n(1) + E_n = 2\delta_{0,n}. \quad (1.12)$$

In this paper we give some interesting identities on the Bernoulli and the Hermite polynomials arising from the orthogonality of Jacobi polynomials in the inner product space $\mathbf{P}_n$.

2. Bernoulli, Euler and Jacobi Polynomials

From (1.4), we have

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}. \quad (2.1)$$

By (2.1), we have

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) t^n = \frac{1}{2\pi i} \oint \frac{(1 + ((x+1)/2)z)^{n+\alpha} (1 + ((x-1)/2)z)^{n+\beta}}{z^{n+1}} dz, \quad (2.2)$$

where we assume $x \neq \pm 1$ and circle around 0 is taken so small that $-2(x \pm 1)^{-1}$ lie neither on it nor in its interior. It is not so difficult to show that $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$.

For $q(x) \in \mathbf{P}_n$, let

$$q(x) = \sum_{k=0}^n C_k P_k^{(\alpha, \beta)}(x), \quad (C_k \in \mathbb{R}). \quad (2.3)$$

From (1.7), we note that

$$\begin{aligned}
 \langle q(x), P_k^{(\alpha, \beta)}(x) \rangle &= C_k \langle P_k^{(\alpha, \beta)}(x), P_k^{(\alpha, \beta)}(x) \rangle \\
 &= C_k \int_{-1}^1 (1-x)^\alpha (1+x)^\beta \left(P_k^{(\alpha, \beta)}(x) \right)^2 dx \\
 &= C_k \frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1) \Gamma(\alpha+\beta+k+1) k!}.
 \end{aligned} \tag{2.4}$$

Thus, by (2.4), we get

$$C_k = \frac{(2k+\alpha+\beta+1) \Gamma(\alpha+\beta+k+1) k!}{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)} \int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_k^{(\alpha, \beta)}(x) q(x) dx. \tag{2.5}$$

Therefore, by (1.7), (2.3), and (2.5), we obtain the following proposition.

Proposition 2.1. *For $q(x) \in \mathbf{P}_n(n \in \mathbb{N})$, one has*

$$q(x) = \sum_{k=0}^n C_k P_k^{(\alpha, \beta)}(x), \tag{2.6}$$

where

$$C_k = \frac{(-1)^k (2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k} \Gamma(\alpha+k+1) \Gamma(\beta+k+1)} \int_{-1}^1 \left(\frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta} \right) q(x) dx. \tag{2.7}$$

Let us take $q(x) = x^n \in \mathbf{P}_n$. First, we consider the following integral:

$$\begin{aligned}
 &\int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} q(x) dx \\
 &= \int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} x^n dx \\
 &= (-n) \int_{-1}^1 \left(\frac{d}{dx} \right)^{k-1} \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} x^{n-1} dx \\
 &= \dots \\
 &= (-1)^k \frac{n!}{(n-k)!} \int_{-1}^1 (1-x)^{k+\alpha} (1+x)^{k+\beta} x^{n-k} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \int_0^1 y^{k+\beta} (1-y)^{k+\alpha} (2y-1)^{n-k} dy \\
&= \frac{(-1)^k n!}{(n-k)!} 2^{2k+\alpha+\beta+1} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} B(k+l+\beta+1, k+\alpha+1) \\
&= \frac{(-1)^k n!}{(n-k)!} 2^{2k+\alpha+\beta+1} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1) \Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)}.
\end{aligned} \tag{2.8}$$

From (2.5) and (16), we have

$$\begin{aligned}
C_k &= \frac{(-1)^k (2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k} \Gamma(\alpha+k+1) \Gamma(\beta+k+1)} \\
&\quad \times \int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} x^n dx \\
&= \frac{(-1)^k (2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1+k} \Gamma(\alpha+k+1) \Gamma(\beta+k+1)} \cdot \frac{(-1)^k n! 2^{2k+\alpha+\beta+1}}{(n-k)!} \\
&\quad \times \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l (-1)^{n-k-l} \frac{\Gamma(k+l+\beta+1) \Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+l+2)} \\
&= \frac{(2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1) n! 2^k}{\Gamma(\beta+k+1) (n-k)!} \sum_{l=0}^{n-k} \frac{(-1)^{n-k-l} \binom{n-k}{l} 2^l \Gamma(k+l+\beta+1)}{\Gamma(2k+\alpha+\beta+l+2)}.
\end{aligned} \tag{2.9}$$

By Proposition 2.1, we get

$$\begin{aligned}
x^n &= n! \sum_{k=0}^n \sum_{l=0}^{n-k} \left(\frac{(2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1) (n-k)!} 2^k \right) \\
&\quad \times \left(\frac{(-1)^{n-k-l} \binom{n-k}{l} 2^l \Gamma(k+l+\beta+1)}{\Gamma(2k+\alpha+\beta+l+2)} \right) P_k^{(\alpha, \beta)}(x).
\end{aligned} \tag{2.10}$$

From (1.9), we have

$$e^{xt} = \frac{1}{t} \frac{t}{e^t - 1} e^{xt} (e^t - 1) = \sum_{n=0}^{\infty} \left(\frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1} \right) \frac{t^n}{n!}. \tag{2.11}$$

By (2.11), we get

$$x^n = \frac{B_{n+1}(x+1) - B_{n+1}(x)}{n+1}, \quad (n \in \mathbb{Z}_+). \tag{2.12}$$

Therefore, by (2.10) and (2.12), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} & \frac{1}{(n+1)!} \{B_{n+1}(x+1) - B_{n+1}(x)\} \\ &= \sum_{k=0}^n \left(\sum_{l=0}^{n-k} \frac{(-1)^{n-k-l} 2^{k+l} (2k+\alpha+\beta+1) \binom{n-k}{l}}{\Gamma(k+\beta+1) \Gamma(2k+\alpha+\beta+l+2) (n-k)!} \right. \\ & \quad \left. \times \Gamma(k+\alpha+\beta+1) \Gamma(k+l+\beta+1) \right) P_k^{(\alpha, \beta)}(x). \end{aligned} \quad (2.13)$$

Let us take $q(x) = B_n(x) \in \mathbf{P}_n$. Then we evaluate the following integral:

$$\begin{aligned} & \int_{-1}^1 \left(\frac{d}{dx} \right)^k \{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \} B_n(x) dx \\ &= \sum_{l=k}^n \binom{n}{l} B_{n-l} \frac{(-1)^k l!}{(l-k)!} 2^{2k+\alpha+\beta+1} \int_0^1 y^{k+\beta} (1-y)^{k+\alpha} (2y-1)^{l-k} dy \\ &= \sum_{l=k}^n \binom{n}{l} B_{n-l} \frac{(-1)^k l!}{(l-k)!} 2^{2k+\alpha+\beta+1} \sum_{m=0}^{l-k} \binom{l-k}{m} 2^m (-1)^{l-k-m} \\ & \quad \times \frac{\Gamma(k+m+\beta+1) \Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+m+2)} \\ &= \sum_{l=k}^n \sum_{m=0}^{l-k} \frac{\binom{n}{l} B_{n-l} (-1)^{l-m} l! 2^{2k+\alpha+\beta+1} \binom{l-k}{m} 2^m \Gamma(k+m+\beta+1) \Gamma(k+\alpha+1)}{(l-k)! \Gamma(2k+\alpha+\beta+m+2)}. \end{aligned} \quad (2.14)$$

Finding (2.5) and (21), we have

$$\begin{aligned} C_k &= \frac{(-1)^k (2k+\alpha+\beta+1) \Gamma(\alpha+\beta+k+1)}{2^{\alpha+\beta+k+1} \Gamma(\alpha+k+1) \Gamma(\beta+k+1)} \\ & \quad \times \int_{-1}^1 \left(\frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta} \right) B_n(x) dx \\ &= \sum_{l=k}^n \sum_{m=0}^{l-k} \frac{2^{k+m} \binom{n}{l} B_{n-l} (-1)^{l-m-k} l! (2k+\alpha+\beta+1) \binom{l-k}{m}}{\Gamma(\beta+k+1) (l-k)! \Gamma(2k+\alpha+\beta+m+2)} \\ & \quad \times \Gamma(k+m+\beta+1) \Gamma(k+\alpha+\beta+1). \end{aligned} \quad (2.15)$$

Theorem 2.3. For $n \in \mathbb{Z}_+$, one has

$$B_n(x) = \sum_{k=0}^n \left(\sum_{l=k}^n \sum_{m=0}^{l-k} \frac{2^{k+m} \binom{n}{l} B_{n-l} (-1)^{l-m-k} l! (2k + \alpha + \beta + 1) \binom{l-k}{m}}{\Gamma(\beta + k + 1) (l-k)! \Gamma(2k + \alpha + \beta + m + 2)} \right. \\ \left. \times \Gamma(k + m + \beta + 1) \Gamma(k + \alpha + \beta + 1) \right) P_k^{(\alpha, \beta)}(x). \quad (2.16)$$

Let $q(x) = P_n^{(\alpha, \beta)}(x) \in \mathbf{P}_n$. From Proposition 2.1, we firstly evaluate the following integral:

$$\int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} P_n^{(\alpha, \beta)}(x) dx \\ = (-1)^k \frac{1}{2^k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \int_{-1}^1 (1-x)^{k+\alpha} (1+x)^{k+\beta} P_{n-k}^{(\alpha+k, \beta+k)}(x) dx. \quad (2.17)$$

By (2.1) and (2.17), we get

$$\int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1-x)^{k+\alpha} (1+x)^{k+\beta} \right\} P_n^{(\alpha, \beta)}(x) dx \\ = \frac{(-1)^k}{2^k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} \\ \times \int_{-1}^1 (1-x)^{k+\alpha} (1+x)^{k+\beta} \left(\frac{x-1}{2} \right)^l \left(\frac{x+1}{2} \right)^{n-k-l} dx \\ = \frac{(-1)^k}{2^k} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} (-1)^l 2^{2k+\alpha+\beta+1} \\ \times \int_0^1 (1-y)^{k+\alpha+l} y^{n+\beta-l} dy \\ = (-1)^k 2^{\alpha+\beta+k+1} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} (-1)^l \\ \times B(k + \alpha + l + 1, n + \beta - l + 1) \\ = (-1)^k 2^{\alpha+\beta+k+1} \frac{\Gamma(n + \alpha + \beta + k + 1)}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} (-1)^l$$

$$\begin{aligned}
& \times \frac{\Gamma(\alpha + k + l + 1)\Gamma(n + \beta - l + 1)}{\Gamma(\alpha + \beta + k + n + 2)} \\
& = (-1)^k 2^{\alpha + \beta + k + 1} \frac{1}{\Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} (-1)^l \\
& \quad \times \frac{\Gamma(\alpha + k + l + 1)\Gamma(n + \beta - l + 1)}{(\alpha + \beta + k + n + 1)}.
\end{aligned} \tag{2.18}$$

It is easy to show that

$$\begin{aligned}
\frac{\Gamma(n + \beta - l + 1)}{\Gamma(\beta + k + 1)} &= \frac{(n + \beta - l) \cdots \beta \Gamma(\beta)}{(\beta + k) \cdots \beta \Gamma(\beta)} = (n + \beta - l) \cdots (\beta + k + 1) \\
&= \binom{n + \beta - l}{n - k - l} (n - k - l)!.
\end{aligned} \tag{2.19}$$

From (2.5), (2.18), and (2.19), we can derive the following equation:

$$\begin{aligned}
C_k &= \frac{(-1)^k (2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1)}{2^{\alpha + \beta + k + 1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\
& \quad \times \int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1 - x)^{k + \alpha} (1 + x)^{k + \beta} \right\} P_n^{(\alpha, \beta)}(x) dx \\
&= \frac{(2k + \alpha + \beta + 1) \Gamma(\alpha + \beta + k + 1)}{\Gamma(\beta + k + 1) \Gamma(n + \alpha + \beta + 1)} \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} \binom{\alpha + k + l}{l} \\
& \quad \times l! (-1)^l \frac{\Gamma(n + \beta - l + 1)}{(\alpha + \beta + k + n + 1)} \\
&= (2k + \alpha + \beta + 1) \Gamma(\alpha + \beta + k + 1) \sum_{l=0}^{n-k} \binom{n + \alpha}{n - k - l} \binom{n + \beta}{l} \binom{n + \beta - l}{n - k - l} \\
& \quad \times \frac{(n - k - l)!!!}{\alpha + \beta + k + n + 1} (-1)^l.
\end{aligned} \tag{2.20}$$

Therefore, by Proposition 2.1, we obtain the following theorem.

Theorem 2.4. For $(n \in \mathbb{Z}_+)$, one has

$$\begin{aligned} \frac{\Gamma(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x)}{\Gamma(\alpha + \beta + 1)} &= \sum_{k=0}^n \left\{ \sum_{l=0}^{n-k} (2k + \alpha + \beta + 1) \binom{\alpha + \beta + k}{k} \binom{n + \alpha}{n - k - l} \right. \\ &\quad \times \binom{n + \beta}{l} \binom{\alpha + k + l}{l} \binom{n + \beta - l}{n - k - l} \frac{(-1)^l (n - k - l)! k! l!}{\alpha + \beta + n + k + 1} \left. \right\} P_k^{(\alpha, \beta)}(x). \end{aligned} \quad (2.21)$$

Let $H_n(x)$ be the Hermite polynomial with

$$H_n(x) = q(x) = \sum_{k=0}^n C_k P_k^{(\alpha, \beta)}(x), \quad (2.22)$$

where

$$\begin{aligned} C_k &= \frac{(-1)^k (2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1)}{2^{\alpha + \beta + k + 1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\ &\quad \times \int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1 - x)^{k + \alpha} (1 + x)^{k + \beta} \right\} H_n(x) dx. \end{aligned} \quad (2.23)$$

Integrating by parts, one has

$$\begin{aligned} &\int_{-1}^1 \left(\frac{d}{dx} \right)^k \left\{ (1 - x)^{k + \alpha} (1 + x)^{k + \beta} \right\} H_n(x) dx \\ &= \frac{2^k (-1)^k n!}{(n - k)!} \int_{-1}^1 (1 - x)^{k + \alpha} (1 + x)^{k + \beta} H_{n-k}(x) dx \\ &= \frac{2^k (-1)^k n!}{(n - k)!} \sum_{l=0}^{n-k} \binom{n - k}{l} H_{n-k-l} 2^l \int_{-1}^1 (1 - x)^{k + \alpha} (1 + x)^{k + \beta} x^l dx \\ &= \frac{2^k (-1)^k n!}{(n - k)!} \sum_{l=0}^{n-k} \binom{n - k}{l} H_{n-k-l} 2^{2k + \alpha + \beta + l + 1} \sum_{m=0}^l \binom{l}{m} (-1)^{l-m} 2^m \\ &\quad \times \int_0^1 (1 - y)^{k + \alpha} y^{k + \beta + m} dy \\ &= \frac{2^k (-1)^k n!}{(n - k)!} \sum_{l=0}^{n-k} \sum_{m=0}^l \binom{n - k}{l} \binom{l}{m} H_{n-k-l} (-1)^{l-m} 2^{2k + \alpha + \beta + m + l + 1} \\ &\quad \times \frac{\Gamma(k + \alpha + 1) \Gamma(\beta + k + m + 1)}{\Gamma(2k + \alpha + \beta + m + 2)}. \end{aligned} \quad (2.24)$$

By (2.23) and (29), we get

$$C_k = \sum_{l=0}^{n-k} \sum_{m=0}^l \frac{\binom{n-k}{l} \binom{l}{m} H_{n-k-l} (-1)^{l-m} (2k + \alpha + \beta + 1) \binom{\alpha + \beta + k}{k} k!}{(\alpha + \beta + 1) \binom{2k + \alpha + \beta + m + 1}{m + 2k} (m + 2k)! (n - k)!} \times 2^{2k+m+l} n! \binom{\beta + k + m}{m} m!. \quad (2.25)$$

Therefore, by (2.22) and (2.25), we obtain the following theorem.

Theorem 2.5. For $n \in \mathbb{Z}_+$, one has

$$\frac{(\alpha + \beta + 1) H_n(x)}{n!} = \sum_{k=0}^n \left\{ \sum_{l=0}^{n-k} \sum_{m=0}^l \frac{\binom{n-k}{l} \binom{l}{m} H_{n-k-l} (-1)^{l-m} (2k + \alpha + \beta + 1)}{\binom{2k + \alpha + \beta + m + 1}{m + 2k} (m + 2k)! (n - k)!} \times \binom{\alpha + \beta + k}{k} k! 2^{2k+m+l} \binom{\beta + k + m}{m} m! \right\} P_k^{(\alpha, \beta)}(x), \quad (2.26)$$

where H_n is the n th Hermite number.

Remark 2.6. By the same method as Theorem 2.3, we get

$$\frac{1}{2n!} \{E_n(x+1) + E_n(x)\} = \sum_{k=0}^n \left(\sum_{l=0}^{n-k} \frac{2^{k+l} (2k + \alpha + \beta + 1) \binom{n-k}{l} (-1)^{n-k-l}}{\Gamma(k + \beta + 1) \Gamma(2k + \alpha + \beta + l + 2) (n - k)!} \times \Gamma(k + \alpha + \beta + 1) \Gamma(k + l + \beta + 1) \right) P_k^{(\alpha, \beta)}(x). \quad (2.27)$$

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