Research Article

Stability of a Bi-Additive Functional Equation in Banach Modules Over a $C^\star$-Algebra

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We solve the bi-additive functional equation $f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w)$ and prove that every biadditive Borel function is bilinear. And we investigate the stability of a biadditive functional equation in Banach modules over a unital $C^\star$-algebra.

1. Introduction

In 1940, Ulam proposed the stability problem (see [1]).

Let G_1 be a group, and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [2] in the case of Banach space. Thereafter, many authors investigated solutions or stability of various functional equations (see [3–21]).

Let X and Y be real or complex vector spaces. In 1989, Aczél and Dhombres [22] proved that a mapping $g : X \to Y$ satisfies the quadratic functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y) \tag{1.1}$$

if and only if there exists a symmetric bi-additive mapping $S : X \times X \to Y$ such that $g(x) = S(x, x)$, where

$$S(x, y) := \frac{1}{4}[g(x + y) - g(x - y)] \tag{1.2}$$

for all $x, y \in X$. For a mapping $f : X \times X \to Y$, consider the bi-additive functional equation:

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w). \tag{1.3}$$

For a mapping $g : X \to Y$ satisfying (1.1), the Aczél's bi-additive mapping $S : X \times X \to Y$ given by (1.2) is a solution of (1.3).

In this paper, we find out the general solution of the bi-additive functional equation (1.3) and investigate the linearity of bi-additive Borel functions. And we investigate the stability of (1.3) in Banach modules over a unital $C^\star$-algebra.

2. Solution of the bi-additive Functional Equation (1.3)

The general solution of the bi-additive functional equation (1.3) is as follows.

Theorem 2.1. *A mapping $f : X \times X \to Y$ satisfies* (1.3) *if and only if the mapping f is bi-additive.*

Proof. Assume that the mapping f satisfies (1.3). Letting $x = y = z = w = 0$ in (1.3), we gain $f(0, 0) = 0$. Putting $w = z$ in (1.3), we get

$$f(x + y, 0) + f(x - y, 2z) = 2f(x, z) - 2f(y, z) \tag{2.1}$$

for all $x, y, z \in X$. Setting $y = x$ in (2.1), we have

$$f(x, 0) = -f(0, z) \tag{2.2}$$

for all $x, z \in X$. Taking $z = 0$ (resp., $x = 0$) in the above equation, we obtain

$$f(x, 0) = 0 \ \left(\text{resp.,} \ f(0, z) = 0\right) \tag{2.3}$$

for all $x \in X$ (resp., for all $z \in X$). Letting $x = w = 0$ in (1.3) and using (2.3), we gain

$$f(-y, z) = -f(y, z) \tag{2.4}$$

for all $y, z \in X$. Putting $y = 0$ in (2.1) and using (2.3), we get

$$f(x, 2z) = 2f(x, z) \tag{2.5}$$

for all $x, z \in X$. Replacing y by $-y$ in (2.1) and using (2.3), (2.4), and (2.5) and the above equation, we see that $f(x + y, z) = f(x, z) + f(y, z)$ for all $x, y, z \in X$.

On the other hand, letting $y = x$ in (1.3) and using (2.3), we gain

$$f(2x, z - w) = 2f(x, z) - 2f(x, w) \tag{2.6}$$

for all $x, z, w \in X$. Putting $y = z = 0$ in (1.3) and using (2.3), we get

$$f(x, -w) = -f(x, w) \tag{2.7}$$

for all $x, w \in X$. Setting $w = 0$ in (2.6) and using (2.3), we have

$$f(2x, z) = 2f(x, z) \tag{2.8}$$

for all $x, z \in X$. Replacing w by $-w$ in (2.6) and using (2.7) and (2.8), we obtain that $f(x, z + w) = f(x, z) + f(x, w)$ for all $x, z, w \in X$.

The converse is trivial. □

The bi-additive functional equation (1.3) is related to the quadratic functional equation (1.1).

If $f : X \times X \rightarrow Y$ is a mapping satisfying (1.3) and $g : X \rightarrow Y$ is the mapping given by $g(x) := f(x, x)$ for all $x \in X$, then one can easily obtain that g satisfies (1.1).

Let $a \in \mathbb{R}$ and $g : X \rightarrow Y$ be a mapping satisfying (1.1). If $f : X \times X \rightarrow Y$ is the mapping given by $f(x, y) := (a/4)[g(x + y) - g(x - y)]$ for all $x, y \in X$, then one can easily prove that f satisfies (1.3). Furthermore, $g(x) = f(x, x)$ holds for all $x \in X$ if $a = 1$.

The following is a result on bi-additive Borel functions.

Theorem 2.2. *Let $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a bi-additive Borel function; then it is bilinear, that is, it satisfies $\psi(s, t) = st\psi(1, 1)$ for all $s, t \in \mathbb{R}$.*

Proof. Since the function ψ is bi-additive, we gain

$$\psi(pu, qv) = pq\psi(u, v) \tag{2.9}$$

for all $p, q \in \mathbb{Q}$ and all $u, v \in \mathbb{R}$. Letting $p = v = 1$ in equality (2.9), we get

$$\psi(u, q) = q\psi(u, 1) \tag{2.10}$$

for all $q \in \mathbb{Q}$ and all $u \in \mathbb{R}$. Putting $u = v = 1$ in equality (2.9) again, we have

$$\psi(p, q) = pq\psi(1, 1) \tag{2.11}$$

for all $p, q \in \mathbb{Q}$. Note that the function $v \rightarrow \psi(u, v)$ is measurable for each fixed $u \in \mathbb{R}$ (see [23, Proposition 2.34]). Since the function $v \rightarrow \psi(u, v)$ is additive for each fixed $u \in \mathbb{R}$, by [24], it is continuous for each fixed $u \in \mathbb{R}$. By the same reasoning, the function $u \rightarrow \psi(u, v)$ is also continuous for each fixed $v \in \mathbb{R}$. Let $s, t \in \mathbb{R}$ be fixed. Since ψ is measurable, by [25, Theorem 7.14.26], for every $m \in \mathbb{N}$ there is a closed set $F_m \subset [s, s + 1]$ such that $\mu([s, s + 1] \setminus F_m) < 1/m$ and $\psi|_{F_m \times \mathbb{R}}$ is continuous. Since $\mu(F_m) \rightarrow 1$, one can choose $u_m \in F_m$ satisfying $u_m \rightarrow s$.

Take a sequence $\{q_n\}$ in $\mathbb{Q}$ converging to t. For each fixed $m \in \mathbb{N}$, take a sequence $\{p_n\}$ in $\mathbb{Q}$ converging to u_m. By equalities (2.10) and (2.11), we see that

$$\psi(u_m, t) = \psi\left(u_m, \lim_{n \to \infty} q_n\right) = \lim_{n \to \infty} \psi(u_m, q_n) = \lim_{n \to \infty} q_n \psi(u_m, 1) = t\psi(u_m, 1)$$

$$= t\psi\left(\lim_{n \to \infty} p_n, 1\right) = t \lim_{n \to \infty} \psi(p_n, 1) = t \lim_{n \to \infty} p_n \psi(1, 1) = t u_m \psi(1, 1) \tag{2.12}$$

for all $m \in \mathbb{N}$. Hence we obtain that

$$\psi(s, t) = \psi\left(\lim_{m \to \infty} u_m, t\right) = \lim_{m \to \infty} \psi(u_m, t) = \lim_{m \to \infty} t u_m \psi(1, 1) = st\psi(1, 1), \tag{2.13}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

3. Stability of the bi-additive Functional Equation (1.3)

From now on, let X be a normed space, Y a complete normed space, and $r \neq 2$ a nonnegative real number. In this section, we investigate the stability of the bi-additive functional equation (1.3).

Lemma 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\left\| f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w) \right\|$$

$$\leq \begin{cases} 4\varepsilon, & (r = 0), \\ \varepsilon(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r), & (0 < r \neq 2) \end{cases} \tag{3.1}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $F : X \times X \to Y$ satisfying (1.3) such that

$$\left\| f(x, y) - F(x, y) \right\| \leq \begin{cases} 2\varepsilon + \left\| f(0, 0) \right\|, & (r = 0), \\[2mm] \dfrac{3\varepsilon}{4 - 2^r}(\|x\|^r + \|y\|^r), & (0 < r < 2), \\[2mm] \dfrac{3\varepsilon}{2^r - 4}(\|x\|^r + \|y\|^r), & (r > 2) \end{cases} \tag{3.2}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \begin{cases} \lim_{j \to \infty} \dfrac{1}{4^j} f(2^j x, 2^j y), & (0 \leq r < 2), \\[2mm] \lim_{j \to \infty} 4^j f\left(\dfrac{x}{2^j}, \dfrac{y}{2^j}\right), & (r > 2) \end{cases} \tag{3.3}$$

for all $x, y \in X$.

Proof. Consider the case $r \in (0, 2)$. Letting $y = x$ and $w = -z$ in (3.1), we gain

$$\|f(2x, 2z) + f(0, 0) - 2f(x, z) + 2f(x, -z)\| \le 2\varepsilon(\|x\|^r + \|z\|^r) \tag{3.4}$$

for all $x, z \in X$. Putting $x = z = 0$ in (3.4), we get $f(0, 0) = 0$. Putting $x = z = 0$ in (3.1), we get

$$\|f(y, -w) + f(-y, w) + 2f(y, w)\| \le \varepsilon(\|y\|^r + \|w\|^r) \tag{3.5}$$

for all $y, w \in X$. Replacing y by x and w by z in the above inequality, we have

$$\|f(x, -z) + f(-x, z) + 2f(x, z)\| \le \varepsilon(\|x\|^r + \|z\|^r) \tag{3.6}$$

for all $x, z \in X$. Setting $y = -x$ and $w = z$ in (3.1), we obtain

$$\|f(2x, 2z) - 2f(x, z) + 2f(-x, z)\| \le 2\varepsilon(\|x\|^r + \|z\|^r) \tag{3.7}$$

for all $x, z \in X$. By (3.4) and (3.6), we gain

$$\|f(2x, 2z) - 4f(x, z) + f(x, -z) - f(-x, z)\| \le 3\varepsilon(\|x\|^r + \|z\|^r) \tag{3.8}$$

for all $x, z \in X$. By (3.4) and (3.7), we get

$$\|f(x, -z) - f(-x, z)\| \le 2\varepsilon(\|x\|^r + \|z\|^r) \tag{3.9}$$

for all $x, z \in X$. By (3.4), (3.6), and (3.7), we have

$$\|f(2x, 2z) - 4f(x, z)\| \le 3\varepsilon(\|x\|^r + \|z\|^r) \tag{3.10}$$

for all $x, z \in X$. Replacing x by $2^j x$ and z by $2^j z$ and dividing 4^{j+1}, we obtain that

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j z\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x, 2^{j+1} z\right) \right\| \le \frac{3\varepsilon \cdot 2^{rj}}{4^{j+1}} (\|x\|^r + \|z\|^r) \tag{3.11}$$

for all $x, z \in X$ and all $j = 0, 1, 2, \ldots$. For given integers l, m $(0 \le l < m)$, we obtain that

$$\left\| \frac{1}{4^l} f\left(2^l x, 2^l z\right) - \frac{1}{4^m} f(2^m x, 2^m z) \right\| \le \sum_{j=l}^{m-1} \frac{3\varepsilon \cdot 2^{rj}}{4^{j+1}} (\|x\|^r + \|z\|^r) \tag{3.12}$$

for all $x, z \in X$. By (3.12), the sequence $\{(1/4^j)f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x, y) := \lim_{j \to \infty}(1/4^j)f(2^j x, 2^j y)$ for all $x, y \in X$. By (3.1), we have

$$
\left\| \frac{1}{4^j} f\left(2^j (x + y), 2^j (z - w)\right) + \frac{1}{4^j} f\left(2^j (x - y), 2^j (z + w)\right) - \frac{2}{4^j} f\left(2^j x, 2^j z\right) + \frac{2}{4^j} f\left(2^j y, 2^j w\right) \right\|
$$

$$
\leq \varepsilon \frac{2^{rj}}{4^j} \left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r \right)
$$

(3.13)

for all $x, y, z, w \in X$ and all $j = 0, 1, 2, \dots$. Letting $j \to \infty$ in the above inequality, we see that F satisfies (1.3). Setting $l = 0$ and taking $m \to \infty$ in (3.12), one can obtain inequality (3.2). If $G : X \times X \to Y$ is another mapping satisfying (1.3) and (3.2), by Theorem 2.1, we obtain that

$$
\begin{aligned}
\|F(x, y) - G(x, y)\| &= \frac{1}{4^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\
&\leq \frac{1}{4^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\
&\leq \frac{6\varepsilon \cdot 2^{n(r-2)}}{4 - 2^r} \left(\|x\|^r + \|y\|^r \right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty
\end{aligned}
$$

(3.14)

for all $x, y \in X$. Hence the mapping F is the unique bi-additive mapping satisfying (1.3), as desired.

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. $\square$

From now on, let A be a unital $C^\star$-algebra with a norm $|\cdot|$, and let $_A\mathcal{M}$ and $_A\mathcal{N}$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Put $A_1 := \{a \in A \mid |a| = 1\}$.

A bi-additive mapping $F : {_A\mathcal{M}} \times {_A\mathcal{M}} \to {_A\mathcal{N}}$ satisfying (1.3) is called A-*quadratic* if $F(ax, ay) = a^2 F(x, y)$ for all $a \in A$ and all $x, y \in {_A\mathcal{M}}$.

Theorem 3.2. *Let* $f : {_A\mathcal{M}} \times {_A\mathcal{M}} \to {_A\mathcal{N}}$ *be a mapping such that*

$$
\left\| f\left(ax + ay, az - aw\right) + f\left(ax - ay, az + aw\right) - 2a^2 f(x, z) + 2a^2 f(y, w) \right\|
$$

$$
\leq \begin{cases} 4\varepsilon, & (r = 0), \\ \varepsilon\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right), & (0 < r \neq 2) \end{cases}
$$

(3.15)

for all $a \in A_1$ *and all* $x, y, z, w \in {_A\mathcal{M}}$. *If* $f(tx, ty)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x, y \in {_A\mathcal{M}}$, *then there exists a unique bi-additive A-quadratic mapping* $F : {_A\mathcal{M}} \times {_A\mathcal{M}} \to {_A\mathcal{N}}$ *satisfying* (1.3) *and inequality* (3.2).

Proof. Consider the case $r \in (0, 2)$. By Lemma 3.1, it follows from the inequality of the statement for $a = 1$ that there exists a unique bi-additive mapping $F : {_A\mathcal{M}} \times {_A\mathcal{M}} \to {_A\mathcal{N}}$ satisfying (1.3) and inequality (3.2). Let $x_0, y_0 \in {_A\mathcal{M}}$ be fixed. And let $L : {_A\mathcal{N}} \to \mathbb{R}$ be any

real continuous linear functional, that is, L is an arbitrary real functional element of the dual space of $_A\mathcal{N}$ restricted to the scalar field $\mathbb{R}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \to \mathbb{R}$ defined by $\psi_n(t) := (1/4^n)L[f(2^n t x_0, 2^n t y_0)]$ for all $t \in \mathbb{R}$. By the assumption that $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(t) = (1/4^n)L[f(2^n t x_0, 2^n t y_0)] = L[(1/4^n)f(2^n t x_0, 2^n t y_0)]$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. By the proof of Lemma 3.1, the sequence $\psi_n(t)$ is a Cauchy sequence for all $t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(t) := \lim_{n \to \infty} \psi_n(t)$ for all $t \in \mathbb{R}$. Note that $\psi(t) = L[F(tx_0, ty_0)]$ for all $t \in \mathbb{R}$. Since F is bi-additive, we get

$$
\begin{aligned}
\psi(s+t) + \psi(s-t) &= L\big(F\big[(s+t)x_0, (s+t)y_0\big]\big) + L\big(F\big[(s-t)x_0, (s-t)y_0\big]\big) \\
&= L\big(F\big[(s+t)x_0, (s+t)y_0\big] + F\big[(s-t)x_0, (s-t)y_0\big]\big) \\
&= L\big[F(sx_0 + tx_0, sy_0 + ty_0) + F(sx_0 - tx_0, sy_0 - ty_0)\big] \\
&= L\big[2F(sx_0, sy_0) + 2F(tx_0, ty_0)\big] \\
&= 2L\big[F(sx_0, sy_0)\big] + 2L\big[F(tx_0, ty_0)\big] = 2\psi(s) + 2\psi(t)
\end{aligned}
\tag{3.16}
$$

for all $s, t \in \mathbb{R}$. Since ψ is the pointwise limit of continuous functions, it is a Borel function. Thus the function ψ as a measurable quadratic function is continuous (see [26]) so has the form $\psi(t) = t^2\psi(1)$ for all $t \in \mathbb{R}$. Hence we have

$$
L\big[F(tx_0, ty_0)\big] = \psi(t) = t^2\psi(1) = t^2 L\big[F(x_0, y_0)\big] = L\big[t^2 F(x_0, y_0)\big]
\tag{3.17}
$$

for all $t \in \mathbb{R}$. Since L is any continuous linear functional, the bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(tx_0, ty_0) = t^2 F(x_0, y_0)$ for all $t \in \mathbb{R}$. Therefore we obtain

$$
F(tx, ty) = t^2 F(x, y)
\tag{3.18}
$$

for all $t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in inequality (3.15), we gain

$$
\left\| f\big(2^j a x, 2^j a z\big) - a^2 f\big(2^j x, 2^j z\big) + a^2 f(0,0) \right\| \leq 2^{rj-1}\varepsilon\big(\|x\|^r + \|z\|^r\big)
\tag{3.19}
$$

for all $a \in A_1$ and all $x, z \in {}_A\mathcal{M}$. Note that there is a constant $K > 0$ such that the condition

$$
\|av\| \leq K|a|\|v\|
\tag{3.20}
$$

for each $a \in A$ and each $v \in {}_A\mathcal{N}$ (see [27, Definition 12]). For all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$, we get

$$
\frac{1}{4^j}\left\| f\big(2^j a x, 2^j a y\big) - a^2 f\big(2^j x, 2^j y\big) \right\| \leq 2^{(r-2)j-1}\varepsilon\big(\|x\|^r + \|z\|^r\big) + \frac{K|a|^2}{4^j}\|f(0,0)\| \longrightarrow 0
\tag{3.21}
$$

as $j \to \infty$. Hence we have

$$F(ax, ay) = \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j ax, 2^j ay\right) = a^2 \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j x, 2^j y\right) = a^2 F(x, y) \qquad (3.22)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$. Since $F(ax, ay) = a^2 F(x, y)$ for each $a \in A_1$, by (3.18), we obtain

$$F(ax, ay) = F\left(|a| \frac{a}{|a|} x, |a| \frac{a}{|a|} y\right) = |a|^2 F\left(\frac{a}{|a|} x, \frac{a}{|a|} y\right) = a^2 F(x, y) \qquad (3.23)$$

for all nonzero $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By (3.18), we get $F(0x, 0y) = 0^2 F(x, y)$ for all $x, y \in {}_A\mathcal{M}$. Therefore the bi-additive mapping F is the unique A-quadratic mapping satisfying the inequality (3.2).

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. □

We obtain the Hyers-Ulam stability of (1.3) as a corollary of Theorem 3.2.

Corollary 3.3. *Let E be a complex normed space and $f : E \times E \to \mathbb{C}$ a function such that*

$$\left\| f\left(\lambda x + \lambda y, \lambda z - \lambda w\right) + f\left(\lambda x - \lambda y, \lambda z + \lambda w\right) - 2\lambda^2 f(x, z) + 2\lambda^2 f(y, w) \right\| \leq \varepsilon \qquad (3.24)$$

for all $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z, w \in E$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in E$, then there exists a unique bi-additive $\mathbb{C}$-quadratic mapping $F : E \times E \to \mathbb{C}$ satisfying (1.3) such that $\|f(x, y) - F(x, y)\| \leq \varepsilon/2 + \|f(0, 0)\|$ for all $x, y \in E$.

Put $A_{in} := \{a \in A \mid a \text{ is invertible in } A\}$, $A_{sa} := \{a \in A \mid a^\star = a\}$, $A^+ := \{a \in A_{sa} \mid Sp(a) \subset [0, \infty)\}$, and $A_1^+ := A_1 \cap A^+$.

A unital $C^\star$-algebra A is said to have *real rank* 0 (see [28]) if the invertible self-adjoint elements are dense in A_{sa}.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 := (a + a^\star)/2$ and $a_2 := (a - a^\star)/2i$ are self-adjoint elements, furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+, and a_2^- are positive elements (see [27, Lemma 38.8]).

Theorem 3.4. *Let A be of real rank 0, and let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\left\| f(ax + ay, bz - bw) + f(ax - ay, bz + bw) - 2abf(x, z) + 2ab(y, w) \right\|$$
$$\leq \begin{cases} 4\varepsilon, & (r = 0), \\ \varepsilon\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right), & (0 < r \neq 2) \end{cases} \qquad (3.25)$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y, z, w \in {}_A\mathcal{M}$. For each fixed $x, y \in {}_A\mathcal{M}$, let the sequence $\{(1/4^j) f(2^j ax, 2^j by)\}$ converge uniformly on $A_1 \times A_1$. If $f(ax, by)$ is continuous in $(a, b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique bi-additive A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) and inequality (3.2) such that $F(ax, by) = abF(x, y)$ for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$.

Proof. Consider the case $r \in (0,2)$. By Lemma 3.1, there exists a unique bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$ satisfying (1.3) and inequality (3.2) on ${}_A\mathcal{M} \times {}_A\mathcal{M}$. Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let L be an arbitrary real functional element of the dual space of ${}_A\mathcal{N}$ restricted to the scalar field $\mathbb{R}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\psi_n(s,t) := (1/4^n)L[f(2^n s x_0, 2^n t y_0)]$ for all $s, t \in \mathbb{R}$. By the assumption that $f(ax, by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(s,t) = (1/4^n)L[f(2^n s x_0, 2^n t y_0)] = L[(1/4^n)f(2^n s x_0, 2^n t y_0)]$ for all $n \in \mathbb{N}$ and all $s, t \in \mathbb{R}$. By the proof of Lemma 3.1, the sequence $\psi_n(s,t)$ is a Cauchy sequence for all $s, t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by $\psi(s,t) := \lim_{n\to\infty}\psi_n(s,t)$ for all $s, t \in \mathbb{R}$. Note that $\psi(s,t) = L[F(s x_0, t y_0)]$ for all $s, t \in \mathbb{R}$. Since the mapping F is bi-additive, we have

$$
\begin{aligned}
\psi(s_1 &+ s_2, t_1 - t_2) + \psi(s_1 - s_2, t_1 + t_2) \\
&= L\big(F\big[(s_1+s_2)x_0, (t_1-t_2)y_0\big]\big) + L\big(F\big[(s_1-s_2)x_0, (t_1+t_2)y_0\big]\big) \\
&= L\big(F\big[(s_1+s_2)x_0, (t_1-t_2)y_0\big] + F\big[(s_1-s_2)x_0, (t_1+t_2)y_0\big]\big) \\
&= L\big[F\big(s_1 x_0 + s_2 x_0, t_1 y_0 - t_2 y_0\big) + F\big(s_1 x_0 - s_2 x_0, t_1 y_0 + t_2 y_0\big)\big] \\
&= L\big[2F\big(s_1 x_0, t_1 y_0\big) - 2F\big(s_2 x_0, t_2 y_0\big)\big] = 2L\big[F\big(s_1 x_0, t_1 y_0\big)\big] - 2L\big[F\big(s_2 x_0, t_2 y_0\big)\big] \\
&= 2\psi(s_1, t_1) - 2\psi(s_2, t_2)
\end{aligned}
\tag{3.26}
$$

for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$. Since ψ is the pointwise limit of continuous functions, it is a Borel function. By Theorem 2.2, we gain $\psi(s,t) = st\psi(1,1)$ for all $s, t \in \mathbb{R}$. Hence we get

$$
L\big[F(s x_0, t y_0)\big] = \psi(s,t) = st\psi(1,1) = stL\big[F(x_0, y_0)\big] = L\big[stF(x_0, y_0)\big]
\tag{3.27}
$$

for all $s, t \in \mathbb{R}$. Since L is any continuous linear functional, the bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$ satisfies $F(s x_0, t y_0) = stF(x_0, y_0)$ for all $s, t \in \mathbb{R}$. Therefore we obtain

$$
F(sx, ty) = stF(x, y)
\tag{3.28}
$$

for all $s, t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in inequality (3.25), we get

$$
\left\| f\left(2^j ax, 2^j bz\right) - abf\left(2^j x, 2^j z\right) + abf(0,0) \right\| \le 2^{rj-1}\varepsilon\big(\|x\|^r + \|z\|^r\big)
\tag{3.29}
$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, z \in {}_A\mathcal{M}$. By inequality (3.20) and the above inequality, for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, z \in {}_A\mathcal{M}$, we have

$$
\begin{aligned}
\frac{1}{4^j}&\left\| f\left(2^j ax, 2^j bz\right) - abf\left(2^j x, 2^j z\right) \right\| \\
&\le 2^{(r-2)j-1}\varepsilon\big(\|x\|^r + \|z\|^r\big) + \frac{K|a||b|}{4^j}\|f(0,0)\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.
\end{aligned}
\tag{3.30}
$$

Hence we obtain that

$$
F(ax, by) = \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j ax, 2^j by\right) = ab \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j x, 2^j y\right) = abF(x, y) \tag{3.31}
$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$. Let $c, d \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists two sequences $\{c_j\}$ and $\{d_j\}$ in $A_{in} \cap A_{sa}$ such that $c_j \to c$ and $d_j \to d$ as $j \to \infty$. Put $p_j := (1/|c_j|)c_j$ and $q_j := (1/|d_j|)d_j$ for all $j \in \mathbb{N}$. Then $p_j \to c$ and $q_j \to d$ as $j \to \infty$. Set $a_j := \sqrt{p_j^\star p_j}$ and $b_j := \sqrt{q_j^\star q_j}$ for all $j \in \mathbb{N}$. Then $a_j \to c$ and $b_j \to d$ as $j \to \infty$ and $a_j, b_j \in A_1^+ \cap A_{in}$. Since $\{(1/4^j)f(2^j ax, 2^j by)\}$ is uniformly converges on $A_1 \times A_1$ for each $x, y \in {}_A\mathcal{M}$ and $f(ax, by)$ is continuous in $a, b \in A_1$ for each $x, y \in {}_A\mathcal{M}$, we see that $F(ax, by)$ is also continuous in $a, b \in A_1$ for each $x, y \in {}_A\mathcal{M}$. In fact, we gain

$$
\lim_{(a,b) \to (c,d)} F(ax, by) = \lim_{(a,b) \to (c,d)} \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j ax, 2^j by\right)
$$
$$
= \lim_{j \to \infty} \lim_{(a,b) \to (c,d)} \frac{1}{4^j} f\left(2^j ax, 2^j by\right) = \lim_{j \to \infty} \frac{1}{4^j} f\left(2^j cx, 2^j dy\right) = F(cx, dy) \tag{3.32}
$$

for all $x, y \in {}_A\mathcal{M}$. Thus we get

$$
\lim_{j \to \infty} F(a_j x, b_j y) = F\left(\lim_{j \to \infty} a_j x, \lim_{j \to \infty} b_j y\right) = F(cx, dy) \tag{3.33}
$$

for all $x, y \in {}_A\mathcal{M}$. By equality (3.31), we have

$$
\left\| F(a_j x, b_j y) - cdF(x, y) \right\| = \left\| a_j b_j F(x, y) - cdF(x, y) \right\|
$$
$$
\longrightarrow \left\| cdF(x, y) - cdF(x, y) \right\| = 0 \tag{3.34}
$$

as $j \to \infty$ for all $x, y \in {}_A\mathcal{M}$. By equality (3.33) and the above convergence, we see that

$$
\left\| F(cx, dy) - cdF(x, y) \right\| \le \left\| F(cx, dy) - F(a_j x, b_j y) \right\| + \left\| F(a_j x, b_j y) - cdF(x, y) \right\| \longrightarrow 0 \tag{3.35}
$$

as $j \to \infty$ for all $x, y \in {}_A\mathcal{M}$. By equality (3.31) and the above convergence, we obtain

$$
F(ax, by) = abF(x, y) \tag{3.36}
$$

for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A \mathcal{M}$. Since the mapping F is bi-additive, we see that

$$
\begin{aligned}
F(ax, ay) &= F\left(a_1^+ x - a_1^- x + i a_2^+ x - i a_2^- x, a_1^+ y - a_1^- y + i a_2^+ y - i a_2^- y\right) \\
&= F\left(a_1^+ x, a_1^+ y\right) - F\left(a_1^+ x, a_1^- y\right) + F\left(a_1^+ x, i a_2^+ y\right) - F\left(a_1^+ x, i a_2^- y\right) \\
&\quad - F\left(a_1^- x, a_1^+ y\right) + F\left(a_1^- x, a_1^- y\right) - F\left(a_1^- x, i a_2^+ y\right) + F\left(a_1^- x, i a_2^- y\right) \\
&\quad + F\left(i a_2^+ x, a_1^+ y\right) - F\left(i a_2^+ x, a_1^- y\right) + F\left(i a_2^+ x, i a_2^+ y\right) - F\left(i a_2^+ x, i a_2^- y\right) \\
&\quad - F\left(i a_2^- x, a_1^+ y\right) + F\left(i a_2^- x, a_1^- y\right) - F\left(i a_2^- x, i a_2^+ y\right) + F\left(i a_2^- x, i a_2^- y\right)
\end{aligned} \tag{3.37}
$$

for all $a \in A$ and all $x, y \in {}_A \mathcal{M}$. By (3.28) and equality (3.36), we have

$$
F(px, qy) = F\left(|p| \frac{p}{|p|} x, |q| \frac{q}{|q|} y\right) = |p||q| F\left(\frac{p}{|p|} x, \frac{q}{|q|} y\right) = pq F(x, y) \tag{3.38}
$$

for all $p, q \in \{a_1^+, a_1^-, a_2^+, a_2^-\}$ and all $x, y \in_A \mathcal{M}$. Note that $a_1^+ a_1^- = a_1^- a_1^+ = a_2^+ a_2^- = a_2^- a_2^+ = 0$. Hence we obtain that

$$
\begin{aligned}
F(ax, ay) &= \left(a_2^+\right)^2 F(x, y) + i a_1^+ a_2^+ F(x, y) - i a_1^+ a_2^- F(x, y) + \left(a_1^-\right)^2 F(x, y) \\
&\quad - i a_1^- a_2^+ F(x, y) + i a_1^- a_2^- F(x, y) + i a_2^+ a_1^+ F(x, y) - i a_2^+ a_1^- F(x, y) \\
&\quad - \left(a_2^+\right)^2 F(x, y) - i a_2^- a_1^+ F(x, y) + i a_2^- a_1^- F(x, y) - \left(a_2^-\right)^2 F(x, y) \\
&= \Big[\left(a_1^+\right)^2 + i a_1^+ a_2^+ - i a_1^+ a_2^- + \left(a_1^-\right)^2 - i a_1^- a_2^+ + i a_1^- a_2^- \\
&\quad + i a_2^+ a_1^+ - i a_2^+ a_1^- - \left(a_2^+\right)^2 - i a_2^- a_1^+ + i a_2^- a_1^- - \left(a_2^-\right)^2\Big] F(x, y) \\
&= \left(a_1^+ - a_1^- + i a_2^+ - i a_2^-\right)^2 F(x, y) = a^2 F(x, y)
\end{aligned} \tag{3.39}
$$

for all $a \in A$ and all $x, y \in {}_A \mathcal{M}$.

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. $\qquad\square$

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, vol. 8 of *Interscience Tracts in Pure and Applied Mathematics*, Interscience Publishers, New York, NY, USA, 1968.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] J.-H. Bae and W.-G. Park, "On stability of a functional equation with n variables," *Nonlinear Analysis*, vol. 64, no. 4, pp. 856–868, 2006.

[4] J.-H. Bae and W.-G. Park, "On a cubic equation and a Jensen-quadratic equation," *Abstract and Applied Analysis*, vol. 2007, Article ID 45179, 10 pages, 2007.

[5] M. E. Gordji, A. Ebadian, and N. Ghobadipour, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 51, 10 pages, 2010.

[6] M. E. Gordji and A. Fazeli, "Stability and superstability of $*$-bihomomorphisms on C^*-ternary algebras," *Journal of Concrete and Applicable Mathematics*, vol. 10, pp. 245–258, 2012.

[7] M. E. Gordji, R. Khodabakhsh, and H. Khodaei, "On approximate n-ary derivations," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 3, pp. 485–500, 2011.

[8] K.-W. Jun and H.-M. Kim, "Remarks on the stability of additive functional equation," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 4, pp. 679–687, 2001.

[9] K.-W. Jun and H.-M. Kim, "On the Hyers-Ulam stability of a generalized quadratic and additive functional equation," *Bulletin of the Korean Mathematical Society*, vol. 42, no. 1, pp. 133–148, 2005.

[10] S.-M. Jung, "On the Hyers-Ulam-Rassias stability of the equation $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$," *Bulletin of the Korean Mathematical Society*, vol. 33, no. 4, pp. 513–519, 1996.

[11] S.-M. Jung, T.-S. Kim, and K.-S. Lee, "A fixed point approach to the stability of quadratic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 3, pp. 531–541, 2006.

[12] M. B. Moghimi, A. Najati, and C. Park, "A fixed point approach to the stability of a quadratic functional equation in C^*-algebras," *Advances in Difference Equations*, vol. 2009, Article ID 256165, 10 pages, 2009.

[13] A. Najati and C. Park, "On the stability of an n-dimensional functional equation originating from quadratic forms," *Taiwanese Journal of Mathematics*, vol. 12, no. 7, pp. 1609–1624, 2008.

[14] A. Najati, C. Park, and J. R. Lee, "Homomorphisms and derivations in C^*-ternary algebras," *Abstract and Applied Analysis*, vol. 2009, Article ID 612392, 16 pages, 2009.

[15] A. Najati and T. M. Rassias, "Stability of a mixed functional equation in several variables on Banach modules," *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1755–1767, 2010.

[16] C. Park and J. S. An, "Isomorphisms in quasi-Banach algebras," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 1, pp. 111–118, 2008.

[17] C. Park, S.-K. Hong, and M.-J. Kim, "Jensen type quadratic-quadratic mapping in Banach spaces," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 4, pp. 703–709, 2006.

[18] W.-G. Park and J.-H. Bae, "On a Cauchy-Jensen functional equation and its stability," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 634–643, 2006.

[19] W.-G. Park and J.-H. Bae, "A multidimensional functional equation having quadratic forms as solutions," *Journal of Inequalities and Applications*, vol. 2007, Article ID 24716, 8 pages, 2007.

[20] W.-G. Park and J.-H. Bae, "A functional equation originating from elliptic curves," *Abstract and Applied Analysis*, vol. 2008, Article ID 135237, 10 pages, 2008.

[21] W.-G. Park and J.-H. Bae, "Stability of a 2-dimensional functional equation in a class of vector variable functions," *Journal of Inequalities and Applications*, vol. 2010, Article ID 167042, 12 pages, 2010.

[22] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[23] G. B. Folland, *Real Analysis*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1999.

[24] M. Fréchet, "Pri la Funkcia Ekvacio $f(x + y) = f(x) + f(y)$," *L'Enseignement Mathématique*, vol. 15, pp. 390–393, 1913.

[25] V. I. Bogachev, *Measure Theory*, vol. 2, Springer, Berlin, Germany, 2007.

[26] S. Kurepa, "On the quadratic functional," *Académie Serbe des Sciences, Publications de l'Institut Mathématique*, vol. 13, pp. 57–72, 1959.

[27] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, Germany, 1973.

[28] K. R. Davidson, C^*-*Algebras by Example*, vol. 6 of *Fields Institute Monographs*, American Mathematical Society, Providence, RI, USA, 1996.