Review Article

Incomplete Bivariate Fibonacci and Lucas p -Polynomials

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We define the incomplete bivariate Fibonacci and Lucas p -polynomials. In the case $x = 1, y = 1$, we obtain the incomplete Fibonacci and Lucas p -numbers. If $x = 2, y = 1$, we have the incomplete Pell and Pell-Lucas p -numbers. On choosing $x = 1, y = 2$, we get the incomplete generalized Jacobsthal number and besides for $p = 1$ the incomplete generalized Jacobsthal-Lucas numbers. In the case $x = 1, y = 1, p = 1$, we have the incomplete Fibonacci and Lucas numbers. If $x = 1, y = 1, p = 1, k = \lfloor (n-1)/(p+1) \rfloor$, we obtain the Fibonacci and Lucas numbers. Also generating function and properties of the incomplete bivariate Fibonacci and Lucas p -polynomials are given.

1. Introduction

Djordjević introduced incomplete generalized Fibonacci and Lucas numbers using explicit formulas of generalized Fibonacci and Lucas numbers in [1]. In [2] incomplete Fibonacci and Lucas numbers are given as follows:

$$\begin{aligned} F_n(k) &= \sum_{j=0}^k \binom{n-1-j}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \\ L_n(k) &= \sum_{j=0}^k \frac{n}{n-j} \binom{n-j}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \end{aligned} \tag{1.1}$$

where $n = 1, 2, 3, \dots$. Note that for the case $k = \lfloor (n-1)/2 \rfloor$ incomplete Fibonacci numbers are reduced to Fibonacci numbers and for the case $k = \lfloor n/2 \rfloor$ incomplete Lucas numbers are

reduced to Lucas numbers in [2]. Also the authors considered the generating functions of the incomplete Fibonacci and Lucas numbers in [3]. In [4] Djordjević and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers.

The generalized Fibonacci and Lucas p -numbers were studied in [5, 6]. Incomplete Fibonacci and Lucas p -numbers are defined by

$$\begin{aligned} F_p^k(n) &= \sum_{j=0}^k \binom{n-jp-1}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor, \\ L_p^k(n) &= \sum_{j=0}^k \frac{n}{n-jp} \binom{n-jp}{j}, \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor, \end{aligned} \quad (1.2)$$

for $n \geq 1$ in [7]. In [8] the authors introduced incomplete Pell and Pell-Lucas p -numbers.

The generalized bivariate Fibonacci p -polynomials $F_{p,n}(x, y)$ and generalized bivariate Lucas p -polynomials $L_{p,n}(x, y)$ are defined the recursion for $p \geq 1$

$$F_{p,n}(x, y) = xF_{p,n-1}(x, y) + yF_{p,n-p-1}(x, y), \quad n > p, \quad (1.3)$$

with

$$F_{p,0}(x, y) = 0, \quad F_{p,n}(x, y) = x^{n-1} \quad \text{for } n = 1, 2, \dots, p, \quad (1.4)$$

and

$$L_{p,n}(x, y) = xL_{p,n-1}(x, y) + yL_{p,n-p-1}(x, y), \quad n > p, \quad (1.5)$$

with

$$L_{p,0}(x, y) = p + 1, \quad L_{p,n}(x, y) = x^n \quad \text{for } n = 1, 2, \dots, p \quad (1.6)$$

in [5]. When $x = y = 1$, $F_{p,n}(1, 1) = F_p(n)$. In [5], the authors obtained some relations for these polynomials sequences. In addition, in [5], the explicit formula of bivariate Fibonacci p -polynomials is

$$F_{p,n}(x, y) = \sum_{j=0}^{\lfloor (n-1)/(p+1) \rfloor} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^j, \quad n \geq 0, \quad p \geq 1, \quad (1.7)$$

and the explicit formula of bivariate Lucas p -polynomials is

$$L_{p,n}(x, y) = \sum_{j=0}^{\lfloor n/(p+1) \rfloor} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j, \quad n \geq 0, \quad p \geq 1. \quad (1.8)$$

In this paper, we defined incomplete bivariate Fibonacci and Lucas p -polynomials. We generalize incomplete Fibonacci and Lucas numbers, incomplete generalized Fibonacci numbers, incomplete generalized Jacobsthal numbers, incomplete Fibonacci and Lucas p -numbers, incomplete Pell and Pell-Lucas p -numbers.

2. Incomplete Bivariate Fibonacci and Lucas p -Polynomials

Definition 2.1. For $p \geq 1, n \geq 1$, incomplete bivariate Fibonacci p -polynomials are defined as

$$F_{p,n}^k(x, y) = \sum_{j=0}^k \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^j, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor. \quad (2.1)$$

For $x = 1, y = 1, F_{p,n}^k(x, y) = F_p^k(n)$, we get incomplete Fibonacci p -numbers [7].

If $x = 2, y = 1, F_{p,n}^k(x, y) = P_p^k(n)$, we obtained incomplete Pell p -numbers [8].

On choosing $x = 1, y = 2, F_{p,n}^k(x, y) = J_{n,p+1}^k$, we have incomplete generalized Jacobsthal numbers [4].

If $x = 1, y = 1, p = 1, F_{p,n}^k(x, y) = F_n(k)$, we get incomplete Fibonacci numbers [2].

For $x = 1, y = 1, p = 1, k = \lfloor (n-1)/(p+1) \rfloor, F_{p,n}^k(x, y) = F_n$, we obtained Fibonacci numbers [9].

Definition 2.2. For $p \geq 1, n \geq 1$, incomplete bivariate Lucas p -polynomials are defined as

$$L_{p,n}^k(x, y) = \sum_{j=0}^k \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j, \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \quad (2.2)$$

If $x = 1, y = 1, L_{p,n}^k(x, y) = L_p^k(n)$, we obtained incomplete Lucas p -numbers [7].

For $x = 2, y = 1, L_{p,n}^k(x, y) = Q_p^k(n)$, we have incomplete Pell-Lucas p -numbers [8].

On choosing $x = 1, y = 2, p = 1, L_{p,n}^k(x, y) = j_{n,p+1}^k$, we get incomplete generalized Jacobsthal-Lucas numbers [4].

If $x = 1, y = 1, p = 1, L_{p,n}^k(x, y) = L_n(k)$, we obtained incomplete Lucas numbers [2].

For $x = 1, y = 1, p = 1, k = \lfloor n/(p+1) \rfloor, L_{p,n}^k(x, y) = L_n$, we have Lucas numbers [9].

Proposition 2.3. The incomplete bivariate Fibonacci p -polynomials satisfy the following recurrence relation:

$$F_{p,n}^{k+1}(x, y) = xF_{p,n-1}^{k+1}(x, y) + yF_{p,n-p-1}^k(x, y), \quad 0 \leq k \leq \frac{n-p-3}{p+1}. \quad (2.3)$$

Proof. Using (2.1), we obtain

$$\begin{aligned}
& xF_{p,n-1}^{k+1}(x, y) + yF_{p,n-p-1}^k(x, y) \\
&= x \sum_{j=0}^{k+1} \binom{n-jp-2}{j} x^{n-j(p+1)-2} y^j + y \sum_{j=0}^k \binom{n-p-pj-2}{j} x^{n-p-j(p+1)-2} y^j \\
&= \sum_{j=0}^{k+1} \binom{n-jp-2}{j} x^{n-j(p+1)-1} y^j + \sum_{j=0}^k \binom{n-p-pj-2}{j} x^{n-p-j(p+1)-2} y^{j+1} \\
&= \sum_{j=0}^{k+1} \binom{n-jp-2}{j} x^{n-j(p+1)-1} y^j + \sum_{j=1}^{k+1} \binom{n-pj-2}{j-1} x^{n-j(p+1)-1} y^j \\
&= \sum_{j=0}^{k+1} \left[\binom{n-jp-2}{j} + \binom{n-pj-2}{j-1} \right] x^{n-j(p+1)-1} y^j - \binom{n-2}{-1} x^{n-1} \\
&= \sum_{j=0}^{k+1} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^j - 0 \\
&= F_{p,n}^{k+1}(x, y).
\end{aligned} \tag{2.4}$$

□

Taking $x = y = 1$ in (2.3), we could obtain a formula for incomplete Fibonacci p -numbers (see [7, Proposition 3]). Taking $x = y = p = 1$ in (2.3), we could obtain a formula for incomplete Fibonacci numbers (see [2, Proposition 1]).

Proposition 2.4. *The nonhomogeneous recurrence relation of incomplete bivariate Fibonacci p -polynomials is*

$$F_{p,n}^k(x, y) = xF_{p,n-1}^k(x, y) + yF_{p,n-p-1}^k(x, y) - \binom{n-p(k+1)-2}{k} x^{n-p(k+1)-k-2} y^{k+1}. \tag{2.5}$$

Proof. It is easy to obtain from (2.1) and (2.3). □

Proposition 2.5. *For $0 \leq k \leq (n-h-p-1)/(p+1)$, one has*

$$\sum_{j=0}^h \binom{h}{j} y^{h-j} x^j F_{p,n+p(j-1)}^{k+j}(x, y) = F_{p,n+(p+1)h-p}^{k+h}(x, y). \tag{2.6}$$

Proof. Equation (2.6) clearly holds for $h = 0$. Suppose that the equation holds for $h > 0$. We show that the equation holds for $(h + 1)$. We have

$$\begin{aligned}
& \sum_{j=0}^{h+1} \binom{h+1}{j} y^{h+1-j} x^j F_{p,n+p(j-1)}^{k+j}(x, y) \\
&= \sum_{j=0}^{h+1} \left[\binom{h}{j} + \binom{h}{j-1} \right] y^{h+1-j} x^j F_{p,n+p(j-1)}^{k+j}(x, y) \\
&= \sum_{j=0}^{h+1} \binom{h}{j} y^{h+1-j} x^j F_{p,n+p(j-1)}^{k+j}(x, y) + \sum_{j=0}^{h+1} \binom{h}{j-1} y^{h+1-j} x^j F_{p,n+p(j-1)}^{k+j}(x, y) \\
&= y F_{p,n+(p+1)h-p}^{k+h}(x, y) + \binom{h}{h+1} x^{h+1} F_{p,n+ph}^{k+h+1}(x, y) \\
&\quad + \sum_{j=-1}^h \binom{h}{j} y^{h-j} x^{j+1} F_{p,n+pj}^{k+j+1}(x, y) \tag{2.7} \\
&= y F_{p,n+(p+1)h-p}^{k+h}(x, y) + x \sum_{j=0}^h \binom{h}{j} y^{h-j} x^j F_{p,n+pj}^{k+j+1}(x, y) \\
&\quad + \binom{h}{-1} y^{h+1} F_{p,n-p}^k(x, y) \\
&= y F_{p,n+(p+1)h-p}^{k+h}(x, y) + x F_{p,n+(p+1)h}^{k+h+1}(x, y) \\
&= F_{p,n+(p+1)h+1}^{k+h+1}(x, y).
\end{aligned}$$

□

Proposition 2.6. For $n \geq k(p+1) + p + 2$,

$$\sum_{j=0}^{h-1} \frac{y}{x^j} F_{p,n-p+j}^k(x, y) = \frac{1}{x^{h-1}} F_{p,n+h}^{k+1}(x, y) - x F_{p,n}^{k+1}(x, y). \tag{2.8}$$

Proof. Equation (2.8) can be easily proved by using (2.3) and induction on h . □

We have the following proposition in which the relationship between the incomplete bivariate Fibonacci and Lucas p -polynomials is preserved as found in [5] before.

Proposition 2.7. One has

$$L_{p,n}^k(x, y) = F_{p,n+1}^k(x, y) + py F_{p,n-p}^{k-1}(x, y), \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \tag{2.9}$$

Proof. By (2.1), rewrite the right-hand side of (2.9) as

$$\begin{aligned}
F_{p,n+1}^k(x, y) + pyF_{p,n-p}^{k-1}(x, y) &= \sum_{j=0}^k \binom{n-jp}{j} x^{n-j(p+1)} y^j + py \sum_{j=0}^{k-1} \binom{n-p-jp-1}{j} x^{n-p-j(p+1)-1} y^j \\
&= \sum_{j=0}^k \binom{n-jp}{j} x^{n-j(p+1)} y^j + py \sum_{j=1}^k \binom{n-jp-1}{j-1} x^{n-j(p+1)} y^{j-1} \\
&= \sum_{j=0}^k \left[\binom{n-jp}{j} + \binom{n-jp-1}{j-1} \right] x^{n-j(p+1)} y^j - \binom{n-1}{-1} x^n \\
&= \sum_{j=0}^k \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j \\
&= L_{p,n}^k(x, y).
\end{aligned} \tag{2.10}$$

□

Proposition 2.8. *The incomplete bivariate Lucas p -polynomials satisfy the following recurrence relation:*

$$L_{p,n}^{k+1}(x, y) = xL_{p,n-1}^{k+1}(x, y) + yL_{p,n-p-1}^k(x, y), \quad 0 \leq k \leq \frac{n-p-2}{p+1}. \tag{2.11}$$

Proof. We write by using (2.3) and (2.9)

$$\begin{aligned}
L_{p,n}^{k+1}(x, y) &= F_{p,n+1}^{k+1}(x, y) + pyF_{p,n-p}^k(x, y) \\
&= xF_{p,n}^{k+1}(x, y) + yF_{p,n-p}^k(x, y) + py \left[xF_{p,n-p-1}^k(x, y) + yF_{p,n-2p-1}^{k-1}(x, y) \right] \\
&= x \left[F_{p,n}^{k+1}(x, y) + pyF_{p,n-p-1}^k(x, y) \right] + y \left[F_{p,n-p}^k(x, y) + pyF_{p,n-2p-1}^{k-1}(x, y) \right] \\
&= xL_{p,n-1}^{k+1}(x, y) + yL_{p,n-p-1}^k(x, y).
\end{aligned} \tag{2.12}$$

□

Proposition 2.9. *The nonhomogeneous recurrence relation of incomplete bivariate Lucas p -polynomials is*

$$\begin{aligned}
L_{p,n}^k(x, y) &= xL_{p,n-1}^k(x, y) + yL_{p,n-p-1}^k(x, y) \\
&\quad - \frac{n-p-1}{n-p(k+1)-1} \binom{n-p(k+1)-1}{k} x^{n-(p+1)(k+1)} y^{k+1}.
\end{aligned} \tag{2.13}$$

Proof. The proof can be done by using (2.2) and (2.11). □

Proposition 2.10. For $0 \leq k \leq (n - p - h) / (p + 1)$, one has

$$\sum_{j=0}^h \binom{h}{j} x^j y^{h-j} L_{p,n+p(j-1)}^{k+j}(x, y) = L_{p,n+(p+1)h-p}^{k+h}(x, y). \quad (2.14)$$

Proof. Proof is similar to the proof of Proposition 2.5. $\square$

Proposition 2.11. For $n \geq (k + 1)(p + 1)$, one has

$$\sum_{j=0}^{h-1} \frac{y}{x^j} L_{p,n-p+j}^k(x, y) = \frac{1}{x^{h-1}} L_{p,n+h}^{k+1}(x, y) - x L_{p,n}^{k+1}(x, y). \quad (2.15)$$

Proof. Proof is obtained immediately by using (2.11) and induction h . $\square$

Proposition 2.12. One has

$$\sum_{k=0}^{\lfloor n/(p+1) \rfloor} L_{p,n}^k(x, y) = \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) L_{p,n}(x, y) + \frac{n}{p+1} [x F_{p,n}(x, y) - L_{p,n}(x, y)]. \quad (2.16)$$

Proof. We can write from (2.2)

$$\begin{aligned} \sum_{k=0}^{\lfloor n/(p+1) \rfloor} L_{p,n}^k(x, y) &= L_{p,n}^0(x, y) + L_{p,n}^1(x, y) + L_{p,n}^2(x, y) + \cdots + L_{p,n}^{\lfloor n/(p+1) \rfloor}(x, y) \\ &= \frac{n}{n} \binom{n}{0} x^n + \left[\frac{n}{n} \binom{n}{0} x^n + \frac{n}{n-p} \binom{n-p}{1} x^{n-(p+1)} y \right] \\ &\quad + \left[\frac{n}{n} \binom{n}{0} x^n + \frac{n}{n-p} \binom{n-p}{1} x^{n-(p+1)} y + \frac{n}{n-2p} \binom{n-2p}{2} x^{n-2(p+1)} y^2 \right] + \cdots \\ &\quad + \left[\frac{n}{n} \binom{n}{0} x^n + \cdots + \frac{n}{n - \lfloor n/(p+1) \rfloor p} \right. \\ &\quad \left. \times \binom{n - \lfloor \frac{n}{p+1} \rfloor p}{\lfloor \frac{n}{p+1} \rfloor} x^{n-(p+1)\lfloor n/(p+1) \rfloor} y^{\lfloor n/(p+1) \rfloor} \right] \end{aligned}$$

$$\begin{aligned}
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) \frac{n}{n} \binom{n}{0} x^n + \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - 1 \right) \frac{n}{n-p} \binom{n-p}{1} x^{n-(p+1)} y \\
&\quad + \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - 2 \right) \frac{n}{n-2p} \binom{n-2p}{1} x^{n-2(p+1)} y^2 + \dots \\
&\quad + \frac{n}{n - \lfloor n/(p+1) \rfloor p} \binom{n - \lfloor \frac{n}{p+1} \rfloor p}{\lfloor \frac{n}{p+1} \rfloor} x^{n-(p+1)\lfloor n/(p+1) \rfloor} y^{\lfloor n/(p+1) \rfloor} \\
&= \sum_{j=0}^{\lfloor n/(p+1) \rfloor} \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - j \right) \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j \\
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) \sum_{j=0}^{\lfloor n/(p+1) \rfloor} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j \\
&\quad - \sum_{j=0}^{\lfloor n/(p+1) \rfloor} j \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j.
\end{aligned} \tag{2.17}$$

Equation (2.17) is calculated using the formula $L_{p,n}(x, y)$ and $\partial L_{p,n}(x, y)/\partial x = nF_{p,n}(x, y)$ [5]

$$\begin{aligned}
\sum_{k=0}^{\lfloor n/(p+1) \rfloor} L_{p,n}^k(x, y) &= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) \sum_{j=0}^{\lfloor n/(p+1) \rfloor} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^j \\
&\quad + \frac{nF_{p,n}(x, y) - nx^{-1}L_{p,n}(x, y)}{(p+1)x^{-1}} \\
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) L_{p,n}(x, y) + \frac{n}{p+1} [xF_{p,n}(x, y) - L_{p,n}(x, y)].
\end{aligned} \tag{2.18}$$

□

Then we have the following conclusion.

Conclusion 1. When $x = y = p = 1$ in (2.16), we obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} L_n(k) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) L_n + \frac{n}{2} (F_n - L_n) \tag{2.19}$$

which is Proposition 11 in [2].

3. Generating Functions of the Incomplete Bivariate Fibonacci and Lucas p -Polynomials

Lemma 3.1 (see [3]). Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:

$$s_n = xs_{n-1} + ys_{n-p-1} + r_n, \quad n > p, \quad (3.1)$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S_p^k(x, y; t)$ of the sequence $\{s_n\}$ is

$$S_p^k(x, y; t) = \left[s_0 - r_0 + \sum_{i=1}^p (s_i - xs_{i-1} - r_i)t^i + G(t) \right] [1 - xt - yt^{p+1}]^{-1}, \quad (3.2)$$

where $G(t)$ denotes the generating function of $\{r_n\}$.

Theorem 3.2. The generating function of the incomplete bivariate Fibonacci p -polynomials is

$$\begin{aligned} R_p^k(x, y; t) = t^{k(p+1)+1} & \left[F_{p, k(p+1)+1}(x, y) + \sum_{i=1}^p t^i (F_{p, k(p+1)+1+i}(x, y) - xF_{p, k(p+1)+i}(x, y)) \right. \\ & \left. + \frac{y^{k+1}t^{p+1}}{(1-xt)^{k+1}} \right] [1 - xt - yt^{p+1}]^{-1}. \end{aligned} \quad (3.3)$$

Proof. From (2.1) and (2.5), $F_{p,n}^k(x, y) = 0$ for $0 \leq n < k(p+1) + 1$,

$$\begin{aligned} F_{p, k(p+1)+1}^k(x, y) &= F_{p, k(p+1)+1}(x, y), \\ F_{p, k(p+1)+2}^k(x, y) &= F_{p, k(p+1)+2}(x, y), \\ &\vdots \\ F_{p, k(p+1)+p+1}^k(x, y) &= F_{p, k(p+1)+p+1}(x, y), \end{aligned} \quad (3.4)$$

and for $n \geq k(p+1) + p + 2$

$$F_{p,n}^k(x, y) = xF_{p, n-1}^k(x, y) + yF_{p, n-p-1}^k(x, y) - \binom{n - p(k+1) - 2}{n - k(p+1) - p - 2} x^{n-p(k+1)-k-2} y^{k+1}. \quad (3.5)$$

Now let

$$s_0 = F_{p, k(p+1)+1}^k(x, y), \quad s_1 = F_{p, k(p+1)+2}^k(x, y), \dots, \quad s_p = F_{p, k(p+1)+p+1}^k(x, y), \quad (3.6)$$

and

$$s_n = F_{p,n+k(p+1)+1}^k(x, y). \quad (3.7)$$

Also

$$r_0 = r_1 = \cdots = r_p = 0, \quad r_n = \binom{n+k-p-1}{n-p-1} x^{n-p-1} y^{k+1}. \quad (3.8)$$

We obtained that $G(t) = y^{k+1}t^{p+1}/(1-xt)^{k+1}$ is the generating function of the sequence $\{r_n\}$. From Lemma 3.1, we get that the generating function $S_p^k(x, y; t)$ of sequence $\{s_n\}$ is

$$\begin{aligned} S_p^k(x, y; t) = & \left[F_{p,k(p+1)+1}^k(x, y) + \sum_{i=1}^p t^i \left(F_{p,k(p+1)+1+i}^k(x, y) - x F_{p,k(p+1)+i}^k(x, y) \right) \right. \\ & \left. + \frac{y^{k+1}t^{p+1}}{(1-xt)^{k+1}} \right] [1-xt-yt^{p+1}]^{-1}. \end{aligned} \quad (3.9)$$

Therefore,

$$R_p^k(x, y; t) = t^{k(p+1)+1} S_p^k(x, y; t). \quad (3.10)$$

□

Theorem 3.3. *The generating function of the incomplete bivariate Lucas p -polynomials is*

$$\begin{aligned} W_p^k(x, y; t) = & t^{k(p+1)} \left[L_{p,k(p+1)}(x, y) + \sum_{i=1}^p t^i (L_{p,k(p+1)+i}(x, y) - x L_{p,k(p+1)+i-1}(x, y)) \right. \\ & \left. + \frac{t^{p+1}y^{k+1}[p(1-xt)+1]}{(1-xt)^{k+1}} \right] [1-xt-yt^{p+1}]^{-1}. \end{aligned} \quad (3.11)$$

Proof. From (2.9) and (3.3),

$$\begin{aligned} W_p^k(x, y; t) &= \sum_{n=0}^{\infty} L_{p,n}^k(x, y) t^n \\ &= \sum_{n=0}^{\infty} \left[F_{p,n+1}^k(x, y) + p y F_{p,n-p}^{k-1}(x, y) \right] t^n \\ &= \sum_{n=0}^{\infty} F_{p,n+1}^k(x, y) t^n + p y \sum_{n=0}^{\infty} F_{p,n-p}^{k-1}(x, y) t^n \\ &= t^{-1} R_p^k(x, y; t) + p y t^p R_p^{k-1}(x, y; t). \end{aligned} \quad (3.12)$$

For the general case in Theorems 3.2 and 3.3, we find the generating functions of some special numbers by the special cases x , y , p . For example, $x = y = 1$ in (3.3) we obtain the generating function of incomplete Fibonacci p -numbers. $\square$

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