Research Article

Nearly Derivations on Banach Algebras

M. Eshaghi Gordji,¹ H. Khodaei,¹ and G. H. Kim²

¹ Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

² Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com
and G. H. Kim, ghkim@kangnam.ac.kr

Received 11 December 2011; Accepted 23 January 2012

Academic Editor: John Rassias

Copyright © 2012 M. Eshaghi Gordji et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let n be a fixed integer greater than 3 and let λ be a real number with $\lambda \neq (n^2 - n + 4)/2$. We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality $\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq l, j \leq n}} f((x_i + x_j)/2 + \sum_{l=1}^{n-2} x_{k_l}) + f(\sum_{i=2}^n x_i) + f(x_1) \| \leq \| \lambda f(\sum_{i=1}^n x_i) \|$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$ -bimodule. Then X^* , the dual space of X , is also a Banach $\mathcal{A}$ -bimodule with module multiplications defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle, \quad (a \in \mathcal{A}, x \in X, x^* \in X^*). \quad (1.1)$$

A bounded linear operator $D : \mathcal{A} \rightarrow X$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}). \quad (1.2)$$

Let $x \in X$. We define $\delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. δ_x is a derivation from $\mathcal{A}$ into X , which is called *inner derivation*. A Banach algebra $\mathcal{A}$ is *amenable* if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$ -bimodule X^* is inner. This definition was introduced by Johnson in [1]. A Banach algebra $\mathcal{A}$ is *weakly amenable* if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner. Bade et al. [2] have introduced the concept of weak amenability for commutative Banach algebras.

The stability problem of functional equations originated from a question of Ulam [3, 4] concerning the stability of group homomorphisms.

A famous talk presented by Ulam in 1940 triggered the study of stability problems for various functional equations.

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \rightarrow G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \rightarrow G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In the following year, Hyers was able to give a partial solution to Ulam's question that was the first significant breakthrough and step toward more solutions in this area (see [5]). Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers' theorem.

Let n be a fixed integer greater than 3 and let λ be a real number with $|\lambda| \neq (n^2 - n + 4)/2$. We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality:

$$\left\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq i, j \leq n}} f\left(\frac{x_i + x_j}{2} + \sum_{l=1}^{n-2} x_{k_l}\right) + f\left(\sum_{i=2}^n x_i\right) + f(x_1) \right\| \leq \left\| \lambda f\left(\sum_{i=1}^n x_i\right) \right\|. \quad (1.3)$$

2. Main Results

Let A be a Banach algebra and let X be a Banach A -module. From now on, the sum of $f(x)$ and $f(-x)$ will be denoted by $\tilde{f}(x)$. Also, $f(ab) - f(a)b - af(b)$ will be denoted by $\Delta f(a, b)$. In the following, we will use the Pascal formula:

$$C(r, k) = C(r-1, k) + C(r-1, k-1) \quad (2.1)$$

here, $C(r, k)$ denotes $r!/k!(r-k)!$. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}_{1/n_0}^1 := \{e^{i\theta}; 0 \leq \theta \leq 2\pi/n_0\}$.

Lemma 2.1. *Let $f : A \rightarrow X$ be a mapping such that*

$$\left\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq i, j \leq n}} f\left(\frac{x_i + x_j}{2} + \sum_{l=1}^{n-2} x_{k_l}\right) + f\left(\sum_{i=2}^n x_i\right) + f(x_1) \right\| \leq \left\| \lambda f\left(\sum_{i=1}^n x_i\right) \right\| \quad (2.2)$$

for all $x_1, \dots, x_n \in A$. Then f is Cauchy additive.

Proof. Substituting $x_1, \dots, x_n = 0$ in the functional inequality (2.2), we get

$$\|(C(n, 2) + 2)f(0)\| \leq \|\lambda f(0)\|. \quad (2.3)$$

Since $n \geq 3$ and $|\lambda| \neq (n^2 - n + 4)/2$, $f(0) = 0$. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \dots = x_n = 0$ in (2.2) and using Pascal formula, we get

$$\left\| (n-2)\tilde{f}\left(\frac{x}{2}\right) + (C(n-2, 2) + 1)f(0) + \tilde{f}(x) \right\| \leq \|\lambda f(0)\|, \quad (2.4)$$

for all $x \in A$. Hence

$$(n-2)\tilde{f}\left(\frac{x}{2}\right) + \tilde{f}(x) = 0 \quad (2.5)$$

for all $x \in A$. Letting $x_1 = 2x$, $x_2 = -x$, $x_3 = -x$ and $x_4 = \dots = x_n = 0$ in (2.2), we get

$$\left\| 2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + C(n-3, 2)f(0) + \tilde{f}(2x) \right\| \leq \|\lambda f(0)\| \quad (2.6)$$

for all $x \in A$. Hence

$$\begin{aligned} 2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + \tilde{f}(2x) &= 0, \\ 2f\left(\frac{x}{2}\right) + (n-3)f(x) + f(-x) + 2(n-3)f\left(\frac{-x}{2}\right) + \tilde{f}(-2x) &= 0 \end{aligned} \quad (2.7)$$

for all $x \in A$. Since $\tilde{f}(-x) = \tilde{f}(x)$, we obtain from (2.7) and (2.4) that

$$2(n-2)\tilde{f}\left(\frac{x}{2}\right) + (n-2)\tilde{f}(x) + 2\tilde{f}(2x) = 0 \quad (2.8)$$

for all $x \in A$. It follows from (2.5) and (2.8) that

$$2\tilde{f}\left(\frac{x}{2}\right) - \tilde{f}(x) = 0 \quad (2.9)$$

for all $x \in A$. By using (2.5) and (2.9), we get $n\tilde{f}(x/2) = 0$ and so $f(-x) = -f(x)$ for all $x \in A$. Hence, we obtain from (2.7) that $f(x/2) = (1/2)f(x)$ for all $x \in A$. Letting $x_1 = x+y$, $x_2 = -x$, $x_3 = -y$ and $x_4 = \dots = x_n = 0$ in (2.2), we get

$$\begin{aligned} \left\| f\left(\frac{-y}{2}\right) + f\left(\frac{-x}{2}\right) + (n-3)f\left(\frac{-x-y}{2}\right) + f\left(\frac{x+y}{2}\right) + (n-3)f\left(\frac{x}{2}\right) + (n-3)f\left(\frac{y}{2}\right) \right. \\ \left. + C(n-3, 2)f(0) + \tilde{f}(x+y) \right\| \leq \|\lambda f(0)\| \end{aligned} \quad (2.10)$$

for all $x, y \in A$. Next, notice that, using oddness of f and $f(x/2) = (1/2)f(x)$, we have

$$f(x+y) = f(x) + f(y) \quad (2.11)$$

for all $x, y \in A$, as desired. $\square$

We can prove the following lemma by the same reasoning as in the proof of Theorem 2.2 of [6].

Lemma 2.2. *Let $f : A \rightarrow X$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in T_{1/n_0}^1$ and all $x \in A$. Then the mapping f is $\mathbb{C}$ -linear.*

Theorem 2.3. *Let $f : A \rightarrow X$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\left\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq i, j \leq n}} f\left(\frac{\mu x_i + \mu x_j}{2} + \sum_{l=1}^{n-2} \mu x_{k_l}\right) + f\left(\sum_{i=2}^n \mu x_i\right) + \mu f(x_1) + \Delta f(a, b) \right\| \leq \left\| \lambda f\left(\sum_{i=1}^n \mu x_i\right) \right\| + \delta \quad (2.12)$$

for some $\delta > 0$, for all $\mu \in T_{1/n_0}^1$ and all $a, b, x_1, \dots, x_n \in A$. Then there exists a unique derivation $\mathfrak{D} : A \rightarrow X$ such that

$$\|f(x) - \mathfrak{D}(x)\| \leq \frac{13n-24}{n(n-4)} \delta \quad (2.13)$$

for all $x \in A$.

Proof. Letting $a = b = 0$, $x_1 = 2x$, $x_2 = -2x$, $x_3 = \dots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\|(n-2)\tilde{f}(x) + \tilde{f}(2x)\| \leq \delta \quad (2.14)$$

for all $x \in X$. Letting $a = b = 0$, $x_1 = 2x$, $x_2 = -x$, $x_3 = -x$, $x_4 = \dots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\left\| 2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + \tilde{f}(2x) \right\| \leq \delta \quad (2.15)$$

for all $x \in X$. Letting $a = b = 0$, $x_1 = -2x$, $x_2 = x$, $x_3 = x$, $x_4 = \dots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\left\| 2f\left(\frac{x}{2}\right) + (n-3)f(x) + f(-x) + 2(n-3)f\left(\frac{-x}{2}\right) + \tilde{f}(-2x) \right\| \leq \delta \quad (2.16)$$

for all $x \in X$. It follows from (2.15) and (2.16) that

$$\left\| (n-2)\tilde{f}\left(\frac{x}{2}\right) + \frac{(n-2)}{2}\tilde{f}(x) + \tilde{f}(2x) \right\| \leq \delta \quad (2.17)$$

for all $x \in X$. It follows from (2.14) and (2.17) that

$$\|\tilde{f}(x)\| \leq \frac{6}{n} \delta \quad (2.18)$$

for all $x \in X$. It follows from (2.15) and (2.18) that

$$\left\| 2\tilde{f}\left(\frac{x}{2}\right) + \tilde{f}(x) + (n-4)f(-x) + 2(n-4)f\left(\frac{x}{2}\right) \right\| \leq \frac{n+6}{n}\delta \quad (2.19)$$

for all $x \in X$. From the last two inequalities, we have

$$\|f(2x) + 2f(-x)\| \leq \frac{n+24}{n(n-4)}\delta \quad (2.20)$$

for all $x \in X$. It follows from (2.18) and (2.20) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{13n-24}{2n(n-4)}\delta \quad (2.21)$$

for all $x \in X$. Hence

$$\left\| \frac{1}{2^r}f(2^r x) - \frac{1}{2^m}f(2^m x) \right\| \leq \frac{13n-24}{2n(n-4)} \sum_{k=r}^{m-1} \frac{\delta}{2^k} \quad (2.22)$$

for all $x \in X$ and integers $m > r \geq 0$. Thus it follows that a sequence $\{(1/2^m)f(2^m x)\}$ is Cauchy in Y and so it converges. Therefore we can define a mapping $\mathfrak{D} : X \rightarrow Y$ by $\mathfrak{D}(x) := \lim_{m \rightarrow \infty} (1/2^m)f(2^m x)$ for all $x \in X$. In addition it is clear from (2.12) that the following inequality:

$$\begin{aligned} & \left\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq i, j \leq n}} \mathfrak{D}\left(\frac{\mu x_i + \mu x_j}{2} + \sum_{l=1}^{n-2} \mu x_{k_l}\right) + \mathfrak{D}\left(\sum_{i=2}^n \mu x_i\right) + \mu \mathfrak{D}(x_1) \right\| \\ &= \lim_{m \rightarrow \infty} \frac{1}{2^m} \left\| \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k_l \neq i, j \leq n}} f\left(2^{m-1}\mu(x_i + x_j) + \sum_{l=1}^{n-2} 2^m \mu x_{k_l}\right) + f\left(\sum_{i=2}^n 2^m \mu x_i\right) + \mu f(2^m x_1) \right\| \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{2^m} \left\| \lambda f\left(\sum_{i=1}^n 2^m \mu x_i\right) \right\| + \lim_{m \rightarrow \infty} \frac{\delta}{2^m} \\ &= \left\| \lambda \mathfrak{D}\left(\sum_{i=1}^n \mu x_i\right) \right\| \end{aligned} \quad (2.23)$$

holds for all $\mu \in T_{1/n_0}^1$ and all $x_1, \dots, x_n \in X$. If we put $\mu = 1$ in the last inequality, then $\mathfrak{D}$ is additive by Lemma 2.1. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \dots = x_n = 0$ in last inequality and using Lemma 2.1, we get

$$(n-2)\tilde{\mathfrak{D}}\left(\frac{\mu x}{2}\right) + \mathfrak{D}(-\mu x) + \mu \mathfrak{D}(x) = \mu \mathfrak{D}(x) - \mathfrak{D}(\mu x). \quad (2.24)$$

So $\mathfrak{D}(\mu x) = \mu \mathfrak{D}(x)$ for all $x \in X$ and all $\mu \in T_{1/n_0}^1$. Now by using Lemmas 2.1 and 2.2, we infer that the mapping $\mathfrak{D} : X \rightarrow Y$ is $\mathbb{C}$ -linear. Taking the limit as $m \rightarrow \infty$ in (2.22) with $r = 0$, we get (2.13).

To prove the afore-mentioned uniqueness, we assume now that there is another $\mathbb{C}$ -linear mapping $\mathfrak{L} : A \rightarrow X$ which satisfies the inequality (2.13). Then we get

$$\left\| \frac{1}{2^m} f(2^m x) - \mathfrak{L}(x) \right\| = \frac{1}{2^m} \|f(2^m x) - \mathfrak{L}(2^m x)\| \leq \frac{13n - 24}{2^m n(n - 4)} \delta \quad (2.25)$$

for all $x \in A$ and integers $m \geq 1$. Thus from $m \rightarrow \infty$, one establishes

$$\mathfrak{D}(x) - \mathfrak{L}(x) = 0 \quad (2.26)$$

for all $x \in A$, completing the proof of uniqueness.

Now, we have to show that $\mathfrak{D}$ is a derivation. To this end, let $x_1 = x_2 = \dots = x_n = 0$ in (2.12), we get

$$\|f(ab) - f(a)b - af(b)\| \leq \delta \quad (2.27)$$

for all $a, b \in A$. It follows from linearity of $\mathfrak{D}$ and (2.27) that

$$\begin{aligned} \|\mathfrak{D}(ab) - \mathfrak{D}(a)b - a\mathfrak{D}(b)\| &= \left\| \frac{1}{2^m} \mathfrak{D}(2^m ab) - \mathfrak{D}(a) \frac{1}{2^m} (2^m b) - \frac{1}{2^m} (2^m a) \mathfrak{D}(b) \right\| \\ &= \lim_{m \rightarrow \infty} \left\| \frac{1}{4^m} f(4^m ab) - f(2^m a) \frac{1}{4^m} (2^m b) - \frac{1}{4^m} (2^m a) f(2^m b) \right\| \\ &= \lim_{m \rightarrow \infty} \frac{1}{4^m} \|f(2^m a 2^m b) - f(2^m a)(2^m b) - (2^m a)f(2^m b)\| \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{4^m} \delta \\ &= 0 \end{aligned} \quad (2.28)$$

for all $a, b \in A$. This means that $\mathfrak{D}$ is a derivation from A into X . Therefore the mapping $\mathfrak{D} : A \rightarrow X$ is a unique derivation satisfying (2.13), as desired. $\square$

Theorem 2.4. *Let A be an amenable Banach algebra and let $f : A \rightarrow X^*$ be a mapping such that $f(0) = 0$ and (2.12). If*

$$\sup \{ \|f(x)\| : \|x\| \leq 1 \} < \infty, \quad (2.29)$$

then there exists $x_0 \in X^$ such that*

$$\|f(a) - ax_0 - x_0 a\| \leq \frac{13n - 24}{n(n - 4)} \delta \quad (2.30)$$

for all $a \in A$.

Proof. Let $\sup\{\|f(x)\| : \|x\| \leq 1\} = M_f$. Then by (2.29), we have $M_f < \infty$. By Theorem 2.3, there exists a derivation $D : A \rightarrow X^*$ satisfying (2.13). Then we have

$$\sup\{\|D(x)\| : \|x\| \leq 1\} \leq M_f + \frac{13n-24}{n(n-4)}\delta. \quad (2.31)$$

This means that D is bounded, and hence D is continuous. On the other hand, A is amenable. Then every continuous derivation from A into X^* is an inner derivation. It follows that D is and an inner derivation. In the other words, there exists $x_0 \in X^*$ such that $D(a) = ax_0 - x_0a$ for all $a \in A$. This completes the proof. $\square$

We know that every nuclear C^* -algebra is amenable (see [7]). Then we have the following result.

Corollary 2.5. *Let A be a nuclear C^* -algebra and let $f : A \rightarrow X^*$ be a mapping such that $f(0) = 0$, and (2.12) and (2.29). Then there exists $x_0 \in X^*$ such that*

$$\|f(a) - ax_0 - x_0a\| \leq \frac{13n-24}{n(n-4)}\delta \quad (2.32)$$

for all $a \in A$.

Theorem 2.6. *Let A be a C^* -algebra and let $f : A \rightarrow A^*$ be a mapping such that $f(0) = 0$, and (2.12) and (2.29). Then there exists $a' \in A^*$ such that*

$$\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n-24}{n(n-4)}\delta\|b\| \quad (2.33)$$

for all $a, b \in A$.

Proof. We know that every C^* -algebra is weakly amenable (see, e.g., [7]). Then every continuous derivation from A into A^* is an inner derivation. By the same reasoning as in the proof of Theorem 2.4, there exists a $a' \in A^*$ such that $D(a) = aa' - a'a$ for all $a \in A$, and

$$\|f(a) - aa' - a'a\| \leq \frac{13n-24}{n(n-4)}\delta \quad (2.34)$$

for all $a \in A$. By definition of module actions of A on A^* , we have

$$\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n-24}{n(n-4)}\delta\|b\| \quad (2.35)$$

for all $a, b \in A$. $\square$

Corollary 2.7. *Let A be a commutative C^* -algebra and let $f : A \rightarrow A^*$ be a mapping such that $f(0) = 0$, and (2.12) and (2.29). Then*

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{2^m} f(2^m a) &= 0, \\ \|f(a)\| &\leq \frac{13n - 24}{n(n - 4)} \delta \end{aligned} \tag{2.36}$$

for all $a \in A$.

References

- [1] B. E. Johnson, "Cohomology in Banach algebras," *Memoirs of the American Mathematical Society*, vol. 127, 1972.
- [2] W. G. Bade, P. G. Curtis, and H. G. Dales, "Amenability and weak amenability for Beurling and Lipschitz algebra," *Proceedings London Mathematical Society*, vol. 55, no. 3, pp. 359–377, 1987.
- [3] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publisher, New York, NY, USA, 1960.
- [4] S. M. Ulam, *Problems in Modern Mathematics*, vol. 6, Wiley-Interscience, New York, 1964.
- [5] D.H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [6] M. E. Gordji, "Nearly involutions on Banach algebras; A fixed point approach," *Fixed Point Theory*. In press.
- [7] H. G. Dales, *Banach Algebras and Automatic Continuity*, vol. 24 of *London Mathematical Society Monographs, New Series*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, NY, USA, 2000.