Research Article

Positive Solutions for a Class of Third-Order Three-Point Boundary Value Problem

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We investigate the problem of existence of positive solutions for the nonlinear third-order three-point boundary value problem $u'''(t) + \lambda a(t)f(u(t)) = 0$, $0 < t < 1$, $u(0) = u'(0) = 0$, $u''(1) = \alpha u''(\eta)$, where λ is a positive parameter, $\alpha \in (0, 1)$, $\eta \in (0, 1)$, $f : (0, \infty) \rightarrow (0, \infty)$, $a : (0, 1) \rightarrow (0, \infty)$ are continuous. Using a specially constructed cone, the fixed point index theorems and Leray-Schauder degree, this work shows the existence and multiplicities of positive solutions for the nonlinear third-order boundary value problem. Some examples are given to demonstrate the main results.

1. Introduction

This paper deals with the following third-order nonlinear boundary value problem:

$$\begin{aligned} u'''(t) + \lambda a(t)f(u(t)) &= 0, & 0 < t < 1, \\ u(0) = u'(0) &= 0, & u''(1) = \alpha u''(\eta). \end{aligned} \quad (1.1)$$

Third-order boundary value problems arise in a variety of different areas of applied mathematics and physics. In the few years, there has been increasing interest in studying certain third-order boundary value problems for nonlinear differential equation and have received much attention. To identify a few, we refer the reader to [1–6].

Recently, El-Shahed [1] discussed the following third-order two-point boundary value problem:

$$\begin{aligned} u'''(t) + \lambda a(t)f(u(t)) &= 0, & 0 < t < 1, \\ u(0) = u'(0) &= 0, & \alpha u'(1) + \beta u''(1) = 0. \end{aligned} \quad (1.2)$$

The methods employed in [1] are Kransnoselskii's fixed-point theorem of cone.

In later work, by placing restrictions on the nonlinear term f , Sun [2] studied the following boundary value problems and obtained the three solution via leggett-williams fixed point theorem:

$$\begin{aligned} u'''(t) &= a(t)f(t, u(t), u'(t), u''(t)), & 0 < t < 1, \\ u(0) = \delta u(\eta) &= 0, & u'(\eta) = 0, & u''(1) = 0. \end{aligned} \quad (1.3)$$

The upper and lower solution is a powerful tool for proving existence for boundary value problems, Ma [7] studied the multiplicity of positive solutions of three-point boundary value problem of second-order ordinary differential equations. Du et al. [5] investigated a class of third-order nonlinear problem.

Motivated by the work of the above papers, the purpose of this article is to study the existence of solution for boundary value problem (1.1) using a new technique (different from the proof of [1, 2, 7]) and we get a new existence result. The tools are based on the fixed point index theorems and Leray-Schauder degree.

The paper is organized as follows: Section 2 states some definitions and some lemmas which are important to obtain our main result. Section 3 is devoted to the existence result of BVP (1.1). Section 4 gives some examples to illustrate our main results.

2. Preliminary

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a cone of E if it satisfies the following two conditions:

- (1) $x \in K, \lambda \geq 0$ implies $\lambda x \in K$;
- (2) $x \in K, -x \in K$ implies $x = 0$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Lemma 2.3. Let $y \in C[0, 1]$, then the following boundary value problem:

$$u'''(t) + y(t) = 0, \quad 0 < t < 1, \quad (2.1)$$

$$u(0) = u'(0) = 0, \quad u''(1) = \alpha u''(\eta), \quad (2.2)$$

has the unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds, \quad (2.3)$$

where

$$G(t, s) = \begin{cases} -\frac{1}{2}(t-s)^2 + \frac{t^2}{2}, & s \leq \eta, s \leq t, \\ \frac{t^2}{2}, & t \leq s \leq \eta, \\ -\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)}, & \eta \leq s \leq t, \\ \frac{t^2}{2(1-\alpha)}, & \eta \leq s, t \leq s. \end{cases} \quad (2.4)$$

Proof. From (2.1), we have

$$u(t) = -\frac{1}{2} \int_0^t (t-s)^2 y(s) ds + At^2 + Bt + C. \quad (2.5)$$

In particular,

$$\begin{aligned} u(t) &= -\frac{1}{2} \int_0^t (t-s)^2 y(s) ds + At^2 + Bt + C, \\ u'(t) &= -t \int_0^t y(s) ds + \int_0^t sy(s) ds + 2At + B, \\ u''(t) &= -\int_0^t y(s) ds + 2A. \end{aligned} \quad (2.6)$$

Combining this with boundary conditions (2.2), we conclude that

$$\begin{aligned} A &= \frac{\int_0^1 y(s) ds}{2(1-\alpha)} - \frac{\alpha \int_0^\eta y(s) ds}{2(1-\alpha)}, \\ B &= 0, \\ C &= 0. \end{aligned} \quad (2.7)$$

Therefore, BVP (2.1)-(2.2) has a unique solution:

$$\begin{aligned} u(t) &= -\frac{1}{2} \int_0^t (t-s)^2 y(s) ds - \frac{\alpha t^2 \int_0^\eta y(s) ds}{2(1-\alpha)} + \frac{t^2 \int_0^1 y(s) ds}{2(1-\alpha)} \\ &= \begin{cases} \int_0^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2} \right] y(s) ds + \int_t^\eta \frac{t^2}{2} y(s) ds + \int_\eta^1 \frac{t^2}{2(1-\alpha)} y(s) ds, & t \leq \eta, \\ \int_0^\eta \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2} \right] y(s) ds + \int_\eta^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)} \right] y(s) ds \\ \quad + \int_t^1 \frac{t^2}{2(1-\alpha)} y(s) ds, & t \geq \eta, \end{cases} \\ &= \int_0^1 G(t, s) y(s) ds. \end{aligned} \quad (2.8)$$

The proof is completed. $\square$

Lemma 2.4. For all $(t, s) \in [0, 1] \times [0, 1]$, one has $G(t, s) \geq 0$.

Lemma 2.5. for all $(t, s) \in [\tau, 1] \times [0, 1]$, one has

$$\gamma G(1, s) \leq G(t, s) \leq G(1, s), \quad (2.9)$$

where $\gamma = \alpha\tau^2/2$, and τ statisfies $\int_\tau^1 G(t, s) a(s) ds > 0$.

Proof. For $s \leq t$, $s \leq \eta$,

$$\begin{aligned} G(t, s) &= -\frac{1}{2}(t-s)^2 + \frac{t^2}{2} = \frac{s(2t-s)}{2} \leq G(1, s), \\ \frac{G(t, s)}{G(1, s)} &= \frac{2t-s}{2-s} = \frac{t+t-s}{2-s} \geq \frac{t}{2}. \end{aligned} \quad (2.10)$$

For $t \leq s \leq \eta$,

$$\begin{aligned} G(t, s) &= \frac{t^2}{2} \leq G(1, s), \\ \frac{G(t, s)}{G(1, s)} &= \frac{t^2/2}{1/2} = t^2. \end{aligned} \quad (2.11)$$

For $\eta \leq s \leq t$,

$$\begin{aligned} G(t, s) &= -\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)} = \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{2(1-\alpha)} \leq G(1, s), \\ \frac{G(t, s)}{G(1, s)} &= \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{\alpha + 2s(1-\alpha) + s^2(1-\alpha)} \geq \alpha t^2. \end{aligned} \quad (2.12)$$

For $\eta \leq s$, $t \leq s$,

$$\begin{aligned} G(t, s) &= \frac{t^2}{2(1-\alpha)} \leq G(1, s), \\ \frac{G(t, s)}{G(1, s)} &= t^2. \end{aligned} \quad (2.13)$$

Thus,

$$\frac{\alpha t^2}{2} G(1, s) \leq G(t, s) \leq G(1, s), \quad \text{for } (t, s) \in [0, 1] \times [0, 1]. \quad (2.14)$$

Therefore,

$$\gamma G(1, s) \leq G(t, s) \leq G(1, s), \quad \forall (t, s) \in [\tau, 1] \times [0, 1]. \quad (2.15)$$

The proof is completed. $\square$

Lemma 2.6. *If $y \in C[0, 1]$ and $y \geq 0$, then the unique solution $u(t)$ of the BVP (2.1)-(2.2) is non-negative and satisfies*

$$\min_{t \in [\tau, 1]} u(t) \geq \gamma \|u\|. \quad (2.16)$$

Proof. Let $y \in C^+[0, 1]$, it is obvious that it is nonnegative. For any $t \in [0, 1]$, by (2.3) and Lemma 2.5, it follows that

$$u(t) = \int_0^1 G(t, s)y(s)ds \leq \int_0^1 G(1, s)y(s)ds, \quad (2.17)$$

and thus,

$$\|u\| \leq \int_0^1 G(1, s)y(s)ds. \quad (2.18)$$

On the other hand, (2.3) and Lemma 2.5 imply, for any $t \in [\tau, 1]$,

$$u(t) = \int_0^1 G(t, s)y(s)ds \geq \gamma \int_0^1 G(1, s)y(s)ds. \quad (2.19)$$

Therefore,

$$\min_{t \in [\tau, 1]} u(t) \geq \gamma \|u\|. \quad (2.20)$$

This completes the proof. $\square$

Let $E = C[0, 1]$ with the usual normal $\|u\| = \max_{t \in [0, 1]} |u(t)|$.
Define the cone K by

$$K = \left\{ u \in C^+[0, 1] : \min_{t \in [\tau, 1]} u(t) \geq \gamma \|u\| \right\}. \quad (2.21)$$

Define an operator T by

$$Tu(t) = \lambda \int_0^1 G(t, s)a(s)f(u(s))ds. \quad (2.22)$$

By Lemma 2.3, BVP (1.1) has a positive solution $u = u(t)$ if and only if u is a fixed point of T .

Lemma 2.7. *Assume that $0 < \lambda < \infty$. Then, $T : K \rightarrow K$ is completely continuous.*

Proof. Firstly, it is easy to check that $T : K \rightarrow K$ is well defined. By Lemma 2.6, we know that $T(K) \subset K$.

Let Ω be any boundary subset of K , then there exists $r > 0$, $\|u\| \leq r$, for all $u \in \Omega$. Therefore, we have

$$|Tu| = \lambda \left| \int_0^1 G(t, s)a(s)f(u(s))ds \right| \leq \lambda \left| \int_0^1 G(1, s)a(s)f(u(s))ds \right|. \quad (2.23)$$

So $T\Omega$ is boundary. Moreover, for any $t_1, t_2 \in [0, 1]$, $|t_1 - t_2| \leq \delta$, $\delta > 0$, we have

$$|Tu(t_1) - Tu(t_2)| \leq \lambda \int_0^1 |G(t_1, s) - G(t_2, s)|a(s)f(u(s))ds. \quad (2.24)$$

By the continuity of f and a , we have $a(t)$ and $f(u(t))$ are boundary on $u \in \Omega, t \in [0, 1]$, which means that there exists a constant $M_a^f > 0$, depending only on Ω such that

$$|a(t)f(u(t))| < M_a^f, \quad (2.25)$$

and thus for any $\varepsilon > 0$,

$$\begin{aligned} |G(t_1, s) - G(t_2, s)| &\leq \frac{\varepsilon}{\lambda M_a^f}, \\ |Tu(t_1) - Tu(t_2)| &< \varepsilon. \end{aligned} \quad (2.26)$$

Therefore, we can get $T\Omega$ is equicontinuity. Thirdly, we prove that T is continuous. Let $u_n \rightarrow u$ as $n \rightarrow \infty, u_n \in K$. Then, the continuity of f , we can get

$$\begin{aligned} |Tu_n(t) - Tu(t)| &= \left| \lambda \int_0^1 G(t, s) a(s) f(u_n(s)) ds - \lambda \int_0^1 G(t, s) a(s) f(u(s)) ds \right| \\ &= \left| \lambda \int_0^1 G(t, s) a(s) (f(u_n(s)) - f(u(s))) ds \right| \\ &\leq \left| \lambda \int_0^1 G(1, s) a(s) (f(u_n(s)) - f(u(s))) ds \right| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (2.27)$$

Then, $Tu_n(t) \rightarrow Tu(t)$. Therefore, T is continuous. The operator T is completely continuous by an application of the Ascoli-Arzelà theorem. This completes the proof. $\square$

Lemma 2.8 (see [7, 8]). *Let E be a real Banach space and let K be a cone in E . For $r \geq 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume $T : \overline{K}_r \rightarrow K$ is a completely continuous operator such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*

(1) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0. \quad (2.28)$$

(2) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1. \quad (2.29)$$

3. Main Results

Theorem 3.1. *Assume that*

- (A1) λ is a positive parameter, $\eta \in (0, 1)$ and $\alpha \in (0, 1)$;
- (A2) $a : [0, 1] \rightarrow (0, \infty)$ is continuous;
- (A3) $f : [0, \infty) \rightarrow (0, \infty)$ is continuous;
- (A4) $f_\infty := \lim_{u \rightarrow \infty} (f(u)/u) = \infty$.

When λ is sufficiently small, (1.1) has at least one positive solution, whereas for λ is sufficiently large, (1.1) has no positive solution.

Proof. If $q > 0$, then

$$\beta(q) = \max_{u \in K, \|u\|=q} \left[\int_0^1 G(t,s)a(s)f(u(s))ds \right] > 0. \quad (3.1)$$

For any number $0 < r_1$, let $\delta_1 = r_1/\beta(r_1)$, and set

$$K_{r_1} = \{u \in K : \|u\| < r_1\}. \quad (3.2)$$

Then, for $\lambda \in (0, \delta_1)$ any $u \in \partial K_{r_1}$, we have

$$Tu(t) < \delta_1 \left[\int_0^1 G(t,s)f(u(s))ds \right] \leq \delta_1 \beta(r_1) = r_1. \quad (3.3)$$

Thus, Lemma 2.8 implies

$$i(T, K_{r_1}, K) = 1. \quad (3.4)$$

Since $f_\infty = \infty$, there is $M > 0$, such that $f(u) \geq \mu u$, for $u > M$, where μ is chosen so that

$$\lambda \mu \gamma \int_\tau^1 G(1,s)a(s)ds > 1. \quad (3.5)$$

Let $r_2 > M/\gamma$, and set

$$K_{r_2} = \{u \in K : \|u\| < r_2\}. \quad (3.6)$$

If $u \in \partial K_{r_2}$, then

$$\min_{t \in [\tau, 1]} u(t) \geq \gamma \|u\| \geq M. \quad (3.7)$$

Therefore,

$$\begin{aligned} Tu(1) &= \lambda \int_0^1 G(1,s)a(s)f(u(s))ds \\ &\geq \lambda \int_\tau^1 G(1,s)a(s)f(u(s))ds \\ &\geq \lambda \int_\tau^1 G(1,s)a(s)\mu u(s)ds \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \mu \int_{\tau}^1 G(1, s) a(s) ds \gamma \|u\| \\
&\geq \lambda \mu \gamma \int_{\tau}^1 G(1, s) a(s) ds \|u\| \\
&> \|u\|,
\end{aligned} \tag{3.8}$$

which implies that

$$\|Tu\| \geq \|u\|, \tag{3.9}$$

for $u \in \partial K_{r_2}$. An application of Lemma 2.8 again shows that

$$i(T, K_{r_2}, K) = 0. \tag{3.10}$$

Since we can adjust r_1, r_2 so that $r_1 < r_2$, it follows the additivity of the fixed-point index that

$$i(T, K_{r_2} \setminus \overline{K}_{r_1}, K) = -1. \tag{3.11}$$

Thus, T has a fixed point in $K_{r_2} \setminus \overline{K}_{r_1}$ which is the desired positive solution of (1.1).

We verify that BVP of (1.1) has no positive solution for λ large enough.

Otherwise, there exist $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$, with $\lim_{n \rightarrow \infty} \lambda_n = +\infty$, such that for any positive integer n , the BVP,

$$\begin{aligned}
u'''(t) + \lambda_n a(t) f(u(t)) &= 0, \quad 0 < t < 1, \\
u(0) = u'(0) &= 0, \quad u''(1) = \alpha u''(\eta),
\end{aligned} \tag{3.12}$$

has a positive solution $u_n(t)$. By (2.22), we have

$$u_n = \lambda_n \int_0^1 G(t, s) a(s) f(u_n(s)) ds \longrightarrow +\infty, \quad (n \longrightarrow \infty). \tag{3.13}$$

Thus,

$$u_n \longrightarrow \infty, \quad (n \longrightarrow \infty). \tag{3.14}$$

Since $f_{\infty} > 0$, for $c_0 > 0$, there exists $r_3 > 0$, such that $f(u)/u > c_0$, for $u \in [r_3, \infty)$, which implies that

$$f(u) > c_0 u, \quad \text{for } u \in [r_3, \infty). \tag{3.15}$$

Let n be large enough that $\|u_n\| \geq r_3$, then

$$\begin{aligned}
 \|u_n\| &\geq u_n(1) \\
 &= \lambda_n \int_0^1 G(1, s) a(s) f(u_n(s)) ds \\
 &\geq \lambda_n \gamma \int_0^1 G(1, s) a(s) ds c_0 \|u_n\| \\
 &> \|u_n\|.
 \end{aligned} \tag{3.16}$$

Choose n so that $c_0 \lambda_n \gamma \int_0^1 G(1, s) a(s) ds > 1$ which is a contradiction. The proof is completed. $\square$

Theorem 3.2. Assume that

- (B1) λ is a positive parameter; $\eta \in (0, 1)$ and $\alpha \in (0, 1)$;
- (B2) $a : [0, 1] \rightarrow (0, \infty)$ is continuous and there exists $m > 0$ such that $a(t) \geq m$;
- (B3) $f : [0, \infty) \rightarrow (0, \infty)$ is continuous;
- (B4) $f_\infty = \lim_{u \rightarrow \infty} (f(u)/u) = 0$, $f_0 = \lim_{u \rightarrow 0} (f(u)/u) = 0$;
- (B5) there exists $\sigma > 0$, for $u \geq \sigma$, such that $f(u) \geq \beta$, where $\beta > 0$, then there exists $\delta_2 > 0$, such that, for $\lambda > \delta_2$, BVP (1.1) has at least two positive solutions u_λ^1, u_λ^2 and $\max u_\lambda^1 > \sigma$.

Proof. Let $\delta_2 = (M\gamma m\beta)^{-1}\sigma$, then for $\lambda > \delta_2$, Lemma 2.7 implies that $T : K \rightarrow K$ is completely continuous. Considering (B4), there exists $0 < r < \sigma$ such that $f(u) \leq u/2\Lambda\lambda$, for $0 \leq u \leq r$, where $\Lambda = \int_0^1 G(1, s) a(s) ds$.

So, for $u \in \partial K_r$, we have from (2.4)

$$\begin{aligned}
 (Tu)(t) &= \lambda \left[\int_0^1 G(t, s) a(s) f(s) ds \right] \\
 &\leq \lambda \int_0^1 G(1, s) a(s) f(u(s)) ds \\
 &\leq \lambda \left[\int_0^1 G(1, s) a(s) ds \right] \frac{\|u\|}{2\Lambda\lambda} \\
 &= \frac{\|u\|}{2} < \|u\| = r.
 \end{aligned} \tag{3.17}$$

Consequently, for $u \in \partial K_r$, we have $\|Tu\| < \|u\|$, by Lemma 2.8,

$$i(T, K_r, K) = 1. \tag{3.18}$$

Now considering (B4), there exists $h > 0$, for $u > h$, such that $f(u) \leq u/2\Lambda\lambda$. Letting $\rho = \max_{0 \leq u \leq h} f(u)$, then

$$0 \leq f(u) \leq \frac{u}{2\Lambda\lambda} + \rho. \tag{3.19}$$

Choose

$$R > \max\{r, 2\Lambda\rho\lambda\}. \quad (3.20)$$

So for $u \in \partial K_R$, from (3.18) and (3.19), we have

$$\begin{aligned} (Tu)(t) &= \lambda \left[\int_0^1 G(t, s) a(s) f(u) ds \right] \\ &\leq \lambda \left[\int_0^1 G(1, s) a(s) f(u) ds \right] \\ &\leq \lambda \left[\int_0^1 G(1, s) a(s) ds \right] \left(\frac{1}{2\Lambda\lambda} \|u\| + \rho \right) \\ &< \frac{\|u\|}{2} + \frac{R}{2} = \|u\|, \end{aligned} \quad (3.21)$$

That is, by Lemma 2.8,

$$i(T, K_R, K) = 1. \quad (3.22)$$

On the other hand, for $u \in \overline{K}_\sigma^R = \{u \in K : \|u\| \leq R, \min_{t \in J_\theta} u(t) \geq \sigma, \theta \in (0, 1/2), J_\theta = [\theta, 1 - \theta]\}$, (2.3) and (2.4) yield that

$$\|Tu\| \leq \lambda \left[\int_0^1 G(t, s) a(s) ds \right] \left(\frac{1}{2\Lambda\lambda} \|u\| + \rho \right) < R. \quad (3.23)$$

Furthermore, for $u \in \overline{K}_\sigma^R$, from (2.3) and (2.4), we obtain

$$\begin{aligned} \min_{t \in J_\theta} (Tu)(t) &= \min_{t \in J_\theta} \lambda \left[\int_0^1 G(1, s) a(s) f(u(s)) ds \right] \\ &\geq \min_{t \in J_\theta} \lambda \int_\theta^{1-\theta} G(t, s) a(s) f(u(s)) ds \\ &\geq \lambda \gamma \int_\theta^{1-\theta} G(1, s) a(s) f(u(s)) ds \\ &\geq \lambda M \gamma m \beta > \delta_2 M \gamma m \beta = \sigma, \end{aligned} \quad (3.24)$$

where $M = \int_\theta^{1-\theta} G(1, s) ds$. Let $u_0 \equiv (\sigma + R)/2$ and $H(t, u) = (1 - t)Tu + tu_0$, then $H : [0, 1] \times \overline{K}_\sigma^R \rightarrow K$ is continuous, and from the analysis above, we obtain for $(t, u) \in [0, 1] \times \overline{K}_\sigma^R$:

$$H(t, u) \in K_\sigma^R. \quad (3.25)$$

Therefore, for $u \in \partial K_\sigma^R$, we have $H(t, u) \neq u$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i(T, K_\sigma^R, K) = i(u_0, K_\sigma^R, K) = 1. \quad (3.26)$$

Consequently, by the solution property of the fixed point index, T has a fixed point u_λ^1 and $u_\lambda^1 \in K_\sigma^R$. By Lemma 2.4, it follows that u_λ^1 is a solution to BVP (1.1), and

$$\max_{t \in [0,1]} u_\lambda^1 \geq \min_{t \in J_\theta} u_\lambda^1 > \gamma. \quad (3.27)$$

On the other hand, from (3.18) and (3.19) together with the additivity of the fixed point index, we get

$$i(T, K_R \setminus (\overline{K_r} \cup \overline{K_\sigma^R})) = i(T, K_R, K) - i(T, K_\sigma^R, K) - i(T, K_r, K) = 1 - 1 - 1 = -1. \quad (3.28)$$

Hence, by the solution property of the fixed point index, T has a fixed point u_λ^2 and $u_\lambda^2 \in K_R \setminus (\overline{K_r} \cup \overline{K_\sigma^R})$. By Lemma 2.3, it follows that u_λ^2 is also a solution to BVP (1.1), and $u_\lambda^1 \neq u_\lambda^2$. The proof is completed. $\square$

4. Examples

Example 4.1. We consider the following third-order boundary value problems:

$$\begin{aligned} u'''(t) + \lambda(2t+1)e^u &= 0, \\ u(0) = u'(0) &= 0, \quad u''(1) = \frac{3}{4}u''\left(\frac{1}{4}\right), \end{aligned} \quad (4.1)$$

here $\eta = 1/4$, $\alpha = 3/4$, $f(u(t)) = e^u$, $a(t) = 2t+1$, $f_\infty = \lim_{u \rightarrow \infty} (f(u)/u) = \infty$, f is continuous, $a(t)$ is continuous. By direct calculations, we obtain that $\lambda < r_1(1-\alpha)$, for $r_1 > 0$. Therefore, by Theorem 3.1, there exists at least one solution $u(t)$ for BVP (4.1), whereas for λ large enough, (4.1) has no solution.

Example 4.2. Consider the following third-order ordinary differential equation:

$$\begin{aligned} u''' + \lambda(2t+1)f(u(t)) &= 0, \\ u(0) = u'(0) &= 0, \quad u''(1) = \frac{1}{4}u''\left(\frac{1}{2}\right), \end{aligned} \quad (4.2)$$

where

$$f(u(t)) = \begin{cases} u^2 e^{-u}, & \text{if } u \leq a, \\ a^{3/2} \sqrt{u} e^{-a}, & \text{if } u > a, \end{cases} \quad (4.3)$$

f is continuous, $a(t)$ is continuous. Here, $m = 1$, $\alpha = 1/4$, $\beta = a^2 e^{-a}$, $\sigma = a$, $a > 0$. Choose $\delta_2 = 6a/(2\theta^3 - 3\theta^2 + 3\theta - 1)$, $\theta \in (0, 1/2)$, $\tau \in (0, 1)$, when $\lambda > \delta_2$, by Theorem 3.2, there exist at least two solutions $u_\lambda^1(t), u_\lambda^2(t)$ for BVP (4.1).

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