Research Article

Hermite Polynomials and their Applications Associated with Bernoulli and Euler Numbers

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We derive some interesting identities and arithmetic properties of Bernoulli and Euler polynomials from the orthogonality of Hermite polynomials. Let $P_n = \{ p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n \}$ be the $(n + 1)$-dimensional vector space over $\mathbb{Q}$. Then we show that $\{ H_0(x), H_1(x), \ldots, H_n(x) \}$ is a good basis for the space $P_n$ for our purpose of arithmetical and combinatorial applications.

1. Introduction

As is well known, the Euler polynomials, $E_n(x)$, are defined by the generating function as follows:

$$
\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}
$$

(1.1)

(see [1–8]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ is called the $n$th Euler number. From (1.1) and definition of Euler numbers, we note that

$$
E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l
$$

(1.2)

with the usual convention about replacing $E^n$ by $E_n$. 


The Bernoulli numbers are defined as

$$B_0 = 1, \quad (B + 1)^n - B_n = \delta_{1,n}$$ (1.3)

(see [9–14]), where $\delta_{k,n}$ is a Kronecker symbol.

As is well known, Bernoulli polynomials are also defined by

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l$$ (1.4)

with the usual convention about replacing $B^n$ by $B_n$ (see [1, 15–18]).

The Hermite polynomials are defined by the generating function as follows:

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$ (1.5)

(see [5, 19]), with the usual convention about replacing $H^n(x)$ by $H_n(x)$.

From (1.5), we can derive the following identities:

$$H_n(x) = \left. \left( \frac{d}{dt} \right)^n e^{2xt-t^2} \right|_{t=0} = \left. e^{x^2} \left( \frac{d}{dt} \right)^n e^{-(x-t)^2} \right|_{t=0}$$

$$= (-1)^n e^{x^2} \left( \frac{d}{dx} \right)^n e^{-(x-t)^2} \right|_{t=0} = (-1)^n e^{x^2} \left( \frac{d^n}{dx^n} e^{-x^2} \right).$$ (1.6)

Let us consider two operators as follows:

$$f \mapsto O_1 f = -\left( e^{x^2} \frac{d}{dx} e^{-x^2} \right) f = 2xf - \frac{df}{dx},$$

$$f \mapsto O_2 f = \left( e^{x^2/2} \left( x - \frac{d}{dx} \right) e^{-x^2/2} \right) f = 2xf - \frac{df}{dx}.\quad (1.7)$$

By (1.7), we get $O_1 = O_2$. In particular, if we take $f = 1$, then we have

$$-e^{x^2} \left( \frac{d}{dx} e^{-x^2} \right) = e^{x^2/2} \left( x - \frac{d}{dx} \right) e^{-x^2/2}.\quad (1.8)$$

We note that

$$(-1)^n e^{x^2} \left( \frac{d^n}{dx^n} e^{-x^2} \right) = \left( -e^{x^2} \frac{d}{dx} e^{-x^2} \right)^n.$$ (1.9)
From (1.8), we note that
\[ (-1)^n e^{x^2} \left( \frac{d^n e^{-x^2}}{dx^n} \right) = \left( -e^{x^2} \frac{d}{dx} e^{-x^2} \right)^n = \left( e^{x^2/2} \left( x - \frac{d}{dx} \right) e^{-x^2/2} \right)^n \]
\[ = e^{x^2/2} \left( x - \frac{d}{dx} \right)^n e^{-x^2/2}. \]

Thus, by (1.10), we get
\[ H_n(x) = e^{x^2/2} \left( x - \frac{d}{dx} \right)^n e^{-x^2/2} \]
(1.11)

(see [5, 19–23]). In the special case, \( x = 0 \), \( H_n(0) = H_n \) are called the Hermite numbers.

From (1.5), we can derive the following identities:
\[ H_n(x) = (H + 2x)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}2^l x^l \]
(1.12)

(cf. [5, 19]), with the usual convention about replacing \( H^n \) by \( H_n \). It is easy to show that
\[ \sum_{n=0}^{\infty} H_n \frac{t^n}{n!} e^{-2t} = \sum_{l=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}. \]
(1.13)

By comparing coefficients on the both sides of (1.13), we get
\[ H_{2n} = (-1)^n 2n(2n - 1) \cdots (n + 1) = \frac{(-1)^n (2n)!}{n!}, \quad H_{2n-1} = 0, \]
(1.14)

where \( n \in \mathbb{N} \). From (1.12), we have
\[ \frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \quad (n \in \mathbb{N}). \]
(1.15)

Let \( \mathbf{P}_n = \{ p \in \mathbb{Q}[x] \mid \deg p(x) \leq n \} \) be the \((n + 1)\)-dimensional vector space over \( \mathbb{Q} \). Probably, \( \{1, x, x^2, \ldots, x^n\} \) is the most natural basis for this space. But \( \{ H_0(x), H_1(x), H_2(x), \ldots, H_n(x) \} \) is also a good basis for the space \( \mathbf{P}_n \), for our purpose of arithmetical and combinatorial applications.

For \( p(x) \in \mathbf{P}_n \),
\[ p(x) = \sum_{k=0}^{n} C_k H_k(x), \]
(1.16)

for some uniquely determined \( b_k \in \mathbb{Q} \).

The purpose of this paper is to develop methods for computing \( C_k \) from the information of \( p(x) \). By using these methods, we define some interesting identities.
2. Properties of Hermite Polynomials

From (1.5) and (1.13), we note that

\[
1 = \left( \sum_{m=0}^{\infty} \frac{H_m t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \frac{t^{2l}}{l!} \right)
\]

\[
= \left( \sum_{m=0}^{\infty} \frac{H_2m}{(2m)!} \right) \left( \sum_{l=0}^{\infty} \frac{(2l)(2l-1) \cdots (l+1) t^{2l}}{(2l)!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(2l)(2l-1) \cdots (l+1)}{(2l)!(2n-2l)!} H_{2n-2l}(2n)! \right) \frac{t^{2n}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{l!}{l!} \left( \frac{2n}{2l} \right) H_{2n-2l} \right) \frac{t^{2n}}{(2n)!}.
\]

Thus, by (2.1), we obtain the following recurrence formula.

**Proposition 2.1.** For \( n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), one has

\[
\sum_{l=0}^{n} \frac{l!}{l!} \left( \frac{2n}{2l} \right) H_{2n-2l} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}.
\]

By, (1.5), we get

\[
\sum_{n=0}^{\infty} H_n(-x) \frac{t^n}{n!} = e^{2t(-x)-t^2} = e^{2x(-t)-(t)^2} = \sum_{n=0}^{\infty} H_n(x)(-1)^n \frac{t^n}{n!}.
\]

From (2.3), we can derive the following reflection symmetric identity of \( H_n(x) \):

\[
H_n(-x) = (-1)^n H_n(x).
\]

By (1.5), we easily see that

\[
\frac{\partial}{\partial t} \left( e^{2xt-t^2} \right) = (2x - 2t)e^{2xt-t^2}.
\]
Thus, by (1.5) and (2.5), we get

\[
\frac{\partial}{\partial t} \left( \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \right) = (2x - 2t) \left( \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \right).
\tag{2.6}
\]

LHS of (2.5) = \( \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!} \),
\tag{2.7}

RHS of (2.5) = \( \sum_{n=0}^{\infty} \left( 2xH_n(x) \frac{t^n}{n!} \right) - \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} \)
\[
= \sum_{n=0}^{\infty} \left( 2xH_n(x) \frac{t^n}{n!} \right) - \sum_{n=1}^{\infty} 2H_{n-1}(x) \frac{t^n}{(n-1)!}
\]
\[
= \sum_{n=0}^{\infty} (2xH_n(x)) \frac{t^n}{n!} - \sum_{n=1}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}.
\tag{2.8}
\]

Thus, by (2.6) and (2.7), we get

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad (n \in \mathbb{N}). \tag{2.9} \]

From (1.15) and (2.9), we note that

\[ H_{n+1}(x) - 2xH_n(x) + H'_n(x) = 0. \tag{2.10} \]

Differentiating on both sides, we have

\[ 2(n + 1)H_n(x) - 2H_n(x) - 2xH'_n(x) + H''_n(x) = 0. \tag{2.11} \]

Thus, we have

\[ H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0. \tag{2.12} \]

From (2.12), we note that \( H_n(x) \) is a solution of the following second-order linear differential equation:

\[ u'' - 2xu' + 2nu = 0. \tag{2.13} \]

From (1.5), we note that

\[
\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx - t^2} = \left( \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} t^k \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!} \right) \frac{t^n}{n!}.
\tag{2.14} \]
Thus, by (2.14), we get

\[
H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}
\]

\[
= \begin{cases} 
\sum_{l=0}^{n/2} (-1)^{n/2-l} n! 2^{2l} \frac{(n/2-l)!}{(2l)!} x^{2l}, & \text{if } n \equiv 0 \pmod{2}, \\
\sum_{l=0}^{(n-1)/2} (-1)^{(n-1)/2-l} n! 2^{2l+1} \frac{((n-1)/2-l)!}{(2l+1)!} x^{2l+1}, & \text{if } n \equiv 1 \pmod{2}.
\end{cases}
\]

(2.15)

### 3. Main Results

By (1.6), we easily get

\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = (-1)^n \int_{-\infty}^{\infty} \left( \frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx.
\]

(3.1)

From (3.1), we note that

\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{m,n}.
\]

(3.2)

It is easy to show that

\[
\int_{-\infty}^{\infty} e^{-x^2} x^l dx = \begin{cases} 
0 & \text{if } l \equiv 1 \pmod{2}, \\
\frac{l! \sqrt{\pi}}{2^{l/2} (l/2)!} & \text{if } l \equiv 0 \pmod{2},
\end{cases}
\]

(3.3)

where \( l \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). By (3.3), we get

\[
\int_{-\infty}^{\infty} \left( \frac{d^n}{dx^n} e^{-x^2} \right) x^m dx = \begin{cases} 
0 & \text{if } n > m \text{ or } n < m \text{ with } n - m \equiv 1 \pmod{2}, \\
\frac{m!(-1)^n \sqrt{\pi}}{2^{m-n} ((m-n)/2)!} & \text{if } n \leq m \text{ with } n - m \equiv 0 \pmod{2}.
\end{cases}
\]

(3.4)

From (3.2), we note that \( H_0(x), H_1(x), \ldots, H_n(x) \) are orthogonal basis for the space \( \mathbb{P}_n = \{ p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n \} \) with respect to the inner product

\[
\langle p(x), q(x) \rangle = \int_{-\infty}^{\infty} e^{-x^2} p(x) q(x) dx.
\]

(3.5)
For \( p(x) \in \mathbb{P}_n \), the polynomial \( p(x) \) is given by

\[
p(x) = \sum_{k=0}^{\infty} C_k H_k(x),
\]

where

\[
C_k = \frac{1}{2^k k! \sqrt{\pi}} \langle p(x), H_k(x) \rangle
= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) p(x) dx.
\]

Let us take \( p(x) = x^n \in \mathbb{P}_n \). For \( n \equiv 0 \pmod{2} \), we compute \( C_k \) in (3.6) as follows

\[
C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) x^n dx
= \begin{cases} 
(-1)^k \frac{n! \sqrt{\pi}}{2^n k! ((n - k)/2)!} & \text{if } k \equiv 0 \pmod{2}, \\
0 & \text{if } k \equiv 1 \pmod{2}.
\end{cases}
\]

Let \( n \equiv 1 \pmod{2} \). Then we have

\[
C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) x^n dx
= \begin{cases} 
\frac{n!}{2^n k! ((n - k)/2)!} & \text{if } k \equiv 1 \pmod{2}, \\
0 & \text{if } k \equiv 0 \pmod{2}.
\end{cases}
\]

Therefore, by (3.6), (3.8), and (3.9), we obtain the following proposition.

**Proposition 3.1.** One has

\[
\begin{align*}
\chi^{2n} &= \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{1}{(2k)!(n-k)!} H_{2k}(x), \\
\chi^{2n+1} &= \frac{(2n+1)!}{2^{2n+1}} \sum_{k=0}^{n} \frac{1}{(2k+1)!(n-k)!} H_{2k+1}(x).
\end{align*}
\]

Let us take \( p(x) = B_n(x) \). From (3.4), \( P(x) \) can be rewritten by

\[
B_n(x) = \sum_{k=0}^{n} C_k H_k(x),
\]

\[
\chi^{2n+1} = \sum_{k=0}^{n} C_k H_k(x).
\]
where

\[ C_k = \frac{(-1)^k}{2^{k+1} \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_n(x) dx. \]  

(3.12)

By integrating by parts, we get

\[ \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_n(x) dx = (-n)(-n-1) \cdots (-n-k+1) \int_{-\infty}^{\infty} e^{-x^2} B_{n-k}(x) dx \]

\[ = (-1)^k \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \left( \begin{array}{c} n-k \\ l \end{array} \right) B_{n-k-l} \int_{-\infty}^{\infty} e^{-x^2} x^l dx \]

\[ = (-1)^k \frac{n!}{(n-k)!} \sum_{\substack{l \in \mathbb{Z} \in n-k \mod 2 \l l \in 0 \mod 2}} \frac{(n-k)! B_{n-k-l}}{l!(n-k-l)!} \times \frac{l! \sqrt{\pi}}{2^{l/2} l!} \]

\[ = (-1)^k n! \sqrt{\pi} \sum_{\substack{l \in \mathbb{Z} \in n-k \mod 2 \l l \in 0 \mod 2}} \frac{B_{n-k-l}}{(n-k-l)! 2^{l/2} l!}. \]  

(3.13)

Thus, from (3.11) and (3.13), we have

\[ C_k = \frac{n!}{2^{k+1} k!} \sum_{\substack{l \in \mathbb{Z} \in n-k \mod 2 \l l \in 0 \mod 2}} \frac{B_{n-k-l}}{(n-k-l)! 2^{l/2} l!}. \]  

(3.14)

Therefore, by (3.11) and (3.14), we obtain the following theorem.

**Theorem 3.2.** For \( n \in \mathbb{Z}_+ \), one has

\[ B_n(x) = n! \sum_{k=0}^{n} \sum_{\substack{l \in \mathbb{Z} \in n-k \mod 2 \l l \in 0 \mod 2}} \frac{B_{n-k-l}}{2^{k+l} k!(n-k-l)! (l/2)!} H_k(x). \]

(3.15)

**Remark 3.3.** Let us take \( p(x) = E_n(x) \). Then, by the same method, we obtain the following identity:

\[ E_n(x) = n! \sum_{k=0}^{n} \sum_{\substack{l \in \mathbb{Z} \in n-k \mod 2 \l l \in 0 \mod 2}} \frac{E_{n-k-l}}{2^{k+l} k!(n-k-l)! (l/2)!} H_k(x). \]

(3.16)

Now, we consider \( p(x) = H_n(x) \). From (3.6), we note that \( p(x) \) can be rewritten as

\[ H_n(x) = \sum_{k=0}^{n} C_k H_k(x), \]

(3.17)
By integrating by parts, we get

\[
\int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) H_n(x) dx = (-2n) \cdots (-2(n-k+1)) \int_{-\infty}^{\infty} e^{-x^2} H_{n-k}(x) dx
\]

\[
= (-1)^k 2^k n! \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l H_{n-k-l} \int_{-\infty}^{\infty} e^{-x^2} x^l dx
\]

\[
= (-1)^k 2^k n! \sum_{l=0}^{n-k} \frac{2^l (n-k)!}{l! (n-k-l)!(l/2)!} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!}
\]  

From (3.17) and (3.19), we note that

\[
C_k = \left( \frac{(-1)^k}{2k! \sqrt{\pi}} \right) \times \left( (-1)^k 2^k n! \sqrt{\pi} \sum_{l=0}^{n-k} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!} \right)
\]

\[
= \frac{n!}{k!} \sum_{l=0}^{n-k} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!}
\]

Therefore, by (3.17) and (3.20), we obtain the following theorem.

**Theorem 3.4.** For \( n \in \mathbb{Z}_+ \), one has

\[
H_n(x) = n! \sum_{k=0}^{n} \sum_{0 \leq s \leq n-k} \frac{H_{n-k-l}}{k!(n-k-l)!(l/2)!} H_k(x).
\]  

From Theorem 3.4, we note that

\[
H_n(x) = n! \sum_{k=0}^{n-1} \sum_{0 \leq s \leq n-k} \frac{H_{n-k-l}}{k!(n-k-l)!(l/2)!} H_k(x) + \frac{n! H_n(x)}{n!}.
\]
Thus, we have, for $0 \leq k \leq n - k$,

$$\sum_{0 \leq j \leq n-k \atop j \equiv l \mod 2} \frac{H_{n-k-j}}{(n-k-j)!(l/2)!} = 0. \quad (3.23)$$

Let $l, k \in \mathbb{Z}_+$ with $k \leq l$. Then we easily see that

$$\int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_l(x) dx = (-1)^k! \sqrt{\pi} \sum_{0 \leq j \leq l-k \atop j \equiv l \mod 2} \frac{B_{l-k-j}}{(l-k-j)!(j/2)!}. \quad (3.24)$$

$$\int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) E_l(x) dx = (-1)^k! \sqrt{\pi} \sum_{0 \leq j \leq l-k \atop j \equiv l \mod 2} \frac{E_{l-k-j}}{(l-k-j)!(j/2)!}. \quad (3.25)$$

Let us consider the following polynomial of degree $n$ in $\mathbb{P}_n$:

$$p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x). \quad (3.26)$$

From (3.6), we note that $p(x)$ can be rewritten as

$$p(x) = \sum_{k=0}^{n} C_k H_k(x), \quad (3.27)$$

where

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) p(x) dx. \quad (3.28)$$

In [15], it is known that

$$p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x)$$

$$= \frac{2}{n+2} \sum_{l=0}^{n-2} \binom{n+2}{l} B_{n-l} B_l(x) + (n+1) B_n(x). \quad (3.29)$$

From (3.23) and (3.29), we have the following:

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \left\{ \frac{2}{n+2} \sum_{l=0}^{n-2} \binom{n+2}{l} \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_l(x) dx + (n+1) \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_n(x) dx \right\}, \quad (3.30)$$
By (3.24) and (3.30), we get

\[ C_n = \left( \frac{(-1)^n}{2^n n! \sqrt{\pi}} \right) \times (n + 1) \int_{-\infty}^{\infty} \left( \frac{d^n e^{-x^2}}{dx^n} \right) B_n(x) dx \]

\[ = \left( \frac{(-1)^n}{2^n n! \sqrt{\pi}} \right) \times \left( (n + 1) \frac{(-1)^n n! \sqrt{\pi} B_0}{0!2^n} \right) = \frac{n + 1}{2^n}, \]

\[ C_{n-1} = \left( \frac{(-1)^{n-1}}{2^{n-1} (n-1)! \sqrt{\pi}} \right) \times \left( (n + 1) \frac{(-1)^{n-1} (n-1)! \sqrt{\pi} B_1}{1!2^n} \right) \]

\[ = \left( \frac{(-1)^{n-1}}{2^{n-1} (n-1)! \sqrt{\pi}} \right) \times \left( (n + 1) \frac{(-1)^{n-1} n! \sqrt{\pi} B_1}{(1-j)2! (j/2)!} \right) \]

For \( 0 \leq k \leq n - 2 \), we have

\[ C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \left\{ \frac{2}{n + 2} \sum_{l=1}^{n-2} \binom{n + 2}{l} B_{n-l} \int_{-\infty}^{\infty} \left( \frac{d^l e^{-x^2}}{dx^l} \right) B_l(x) dx + (n + 1) \int_{-\infty}^{\infty} \left( \frac{d^k e^{-x^2}}{dx^k} \right) B_k(x) dx \right\} \]

\[ = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \left\{ \frac{2}{n + 2} \sum_{l=1}^{n-2} \binom{n + 2}{l} B_{n-l} (-1)^{l-k} \frac{2!}{l!} \sum_{0 \leq j \leq l-k} \frac{B_{l-k-j}}{(l-k-j)2! (j/2)!} \right\} \]

\[ + (n + 1) (-1)^k n! \sqrt{\pi} \sum_{0 \leq j \leq n-k} \frac{B_{n-k-j}}{(n-k-j)2! (j/2)!} \]

\[ = \frac{2}{n + 2} \sum_{l=1}^{n-2} \sum_{0 \leq j \leq l-k \atop j=0 \text{ (mod 2)}} \binom{n + 2}{l} \frac{B_{n-l} B_{l-k-j}!}{2^k l! (l-k-j)! (j/2)!} \]

\[ + (n + 1)! \sum_{0 \leq j \leq n-k \atop j=0 \text{ (mod 2)}} \frac{B_{n-k-j}}{k! (n-k-j)! (j/2)! 2^k j!}. \]

(3.32)

Therefore, by (3.27) and (3.32), we obtain the following theorem.
Theorem 3.5. For $n \in \mathbb{Z}_+$, one has

$$
\sum_{k=0}^{n} B_k(x)B_{n-k}(x) = \sum_{k=0}^{n-2} \left\{ \frac{2}{n+2} \sum_{0 \leq j \leq n-k \atop j=0 \ (\text{mod} \ 2)} \sum_{l=k-j}^{n-2} \binom{n+2}{l} \frac{l!B_{n-l}B_{l-k-j}}{2^{k+j}k!(l-k-j)!(j/2)!} \right\} + (n+1)! \sum_{0 \leq j \leq n-k \atop j=0 \ (\text{mod} \ 2)} \frac{B_{n-k-j}}{2^{k+j}k!(n-k-j)!(j/2)!} \right\} H_k(x) 
- \frac{n(n+1)}{2^n} H_{n-1}(x) + \frac{n+1}{2^n} H_n(x).
$$

(3.33)

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References


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