Research Article

A Fixed Point Theorem for Contraction Type Maps in Partially Ordered Metric Spaces and Application to Ordinary Differential Equations

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We present a fixed point theorem for generalized contraction in partially ordered complete metric spaces. As an application, we give an existence and uniqueness for the solution of a periodic boundary value problem.

1. Introduction

The contraction mapping theorem and the abstract monotone iterative technique are well known and are applicable to a variety of situations. Recently, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order (see [1–7]). It is of interest to determine if it is still possible to establish the existence of a unique fixed point assuming that the operator considered is monotone in such a setting. Such a fixed point theorem is useful, for example, in establishing the existence of a unique solution to periodic boundary value problems, besides many others.

That approach was initiated by Ran and Reurings in [8], where some applications to matrix equations were studied. This fixed point theorem was refined and extended in [7, 9] and applied to the periodic boundary value problem in the monotone case. In this paper, we consider a special case of the following periodic boundary value problem

$$\begin{aligned}u'(t) &= f(t, u(t)) \quad \text{if } t \in I = [0, T], \\u(0) &= u(T),\end{aligned}\tag{1.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function.

Definition 1.1. A lower solution for (1.1) is a function $u \in C^1(I, \mathbb{R})$ such that

$$\begin{aligned} u'(t) &\leq f(t, u(t)) \quad \text{for } t \in I = [0, T], \\ u(0) &\leq u(T). \end{aligned} \tag{1.2}$$

Very recently, Harjani and Sadarangani [4] proved the following existence theorem.

Theorem 1.2. Consider problem (1.1) with $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \geq x$

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \tag{1.3}$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ can be written by $\phi(x) = x - \psi(x)$ with $\psi : [0, \infty) \rightarrow [0, \infty)$ continuous increasing positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Then the existence of a lower solution of (1.1) provides the existence of a unique solution of (1.1).

In Section 2, we prove a new fixed point theorem in partially ordered complete metric spaces. In Section 3, existence of a unique solution for problem (1.1) is obtained under suitable conditions.

2. Fixed Point Theorem

Let S denote the class of those functions $\alpha : [0, \infty) \rightarrow [0, 1)$ which satisfies the condition

$$\limsup_{s \rightarrow t^+} \alpha(s) < 1, \quad \forall t \in [0, \infty). \tag{2.1}$$

We prove the main theorem of the paper as follows.

Theorem 2.1. Let $(X, \preccurlyeq)$ be a partially order metric space that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \rightarrow X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$. Suppose that there exists $\alpha \in S$ such that

$$d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y), \tag{2.2}$$

for all comparable $x, y \in X$. If

$$f \text{ is continuous} \tag{2.3}$$

or

$$\text{if an increasing sequence } \{x_n\} \rightarrow x \text{ in } X, \text{ then } x_n \preccurlyeq x, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Besides, if

$$\text{for each } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \quad (2.5)$$

then f have a unique fixed point.

Proof. We first show that f has a fixed point. Since $x_0 \preceq f(x_0)$ and f is increasing function, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq f^3(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \cdots. \quad (2.6)$$

Put $x_n = f^n(x_0)$, $n = 1, 2, \dots$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$ then by (2.2)

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(f^{n+1}(x_0), f^{n+2}(x_0)) \\ &\leq \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \\ &\leq d(x_n, x_{n+1}). \end{aligned} \quad (2.7)$$

And so the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\tau \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \tau$. Since $\limsup_{s \rightarrow \tau^+} \alpha(s) < 1$ and $\alpha(\tau) < 1$ then there exist $r \in [0, 1)$ and $\epsilon > 0$ such that $\alpha(s) < r$ for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that $\tau \leq d(x_{n+1}, x_n) \leq \tau + \epsilon$ for all $n \in \mathbb{N}$ with $n \geq \nu$. Then since

$$d(x_{n+1}, x_{n+2}) \leq \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq r d(x_n, x_{n+1}), \quad (2.8)$$

for all $n \in \mathbb{N}$ with $n \geq \nu$ we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\nu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_{\nu}, x_{\nu+1}) < \infty, \quad (2.9)$$

and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$. To prove that z is a fixed point of f , if f is a continuous, then

$$z = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f^n(x_0) = \lim_{n \rightarrow \infty} f^{n+1}(x_0) = f\left(\lim_{n \rightarrow \infty} f^n(x_0)\right) = f(z); \quad (2.10)$$

hence $z = f(z)$. If case (2.4) holds then

$$\begin{aligned} d(f(z), z) &\leq d(f(z), f(x_n)) + d(f(x_n), z) \\ &\leq \alpha(d(x_n, z))d(x_n, z) + d(x_{n+1}, z) \\ &\leq d(x_n, z) + d(x_{n+1}, z). \end{aligned} \quad (2.11)$$

Since $d(x_n, z) \rightarrow 0$ then we get $f(z) = z$. To prove the uniqueness of the fixed point, let y be another fixed point of f . From (2.5) there exists $x \in X$ which is comparable to y and z . Monotonicity implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n = 0, 1, \dots$. Moreover,

$$\begin{aligned} d(z, f^n(x)) &= d(f^n(z), f^n(x)) \\ &\leq \alpha\left(d\left(f^{n-1}(z), f^{n-1}(x)\right)\right)d\left(f^{n-1}(z), f^{n-1}(x)\right) \\ &\leq d\left(f^{n-1}(z), f^{n-1}(x)\right) \\ &= d\left(z, f^{n-1}(x)\right). \end{aligned} \tag{2.12}$$

Consequently, the sequence $\zeta_n^z := d(z, f^n(x))$ is nonnegative and decreasing and so $\lim_{n \rightarrow \infty} d(z, f^n(x)) = \zeta_z \in \mathbb{R}$. Similarly we can show that the sequence $\zeta_n^y := d(y, f^n(x))$ is nonnegative and decreasing and so $\lim_{n \rightarrow \infty} d(y, f^n(x)) = \zeta_y \in \mathbb{R}$. Now similarly the above method we can choose r_1, r_2 in $[0, 1)$ and $\tau_1 \in \mathbb{N}$ such that

$$\begin{aligned} d(z, f^n(x)) &\leq \alpha\left(d\left(z, f^{n-1}(x)\right)\right)d\left(z, f^{n-1}(x)\right) \leq r_1 d\left(z, f^{n-1}(x)\right), \\ d(y, f^n(x)) &\leq \alpha\left(d\left(y, f^{n-1}(x)\right)\right)d\left(y, f^{n-1}(x)\right) \leq r_2 d\left(y, f^{n-1}(x)\right), \end{aligned} \tag{2.13}$$

for all $n \in \mathbb{N}$ with $n > \tau_1$. Finally

$$d(z, y) \leq d(z, f^n(x)) + d(f^n(x), y) \leq r_1^{n-\tau_1} d(z, f^{\tau_1} x_0) + r_2^{n-\tau_1} d(y, f^{\tau_1} x_0), \tag{2.14}$$

for all $n \in \mathbb{N}$ with $n > \tau_1$. Therefor if in (2.14) taking $n \rightarrow \infty$ yields $d(z, y) = 0$. $\square$

3. Application to Ordinary Differential Equation

In this section we present an example where Theorem 2.1 can be applied. This example is inspired in [2, 4, 7].

Definition 3.1. Let $\mathfrak{B}$ denote the class of those functions $\phi : [0, \infty) \rightarrow [0, \infty)$ which satisfies the following condition:

- (i) ϕ is increasing,
- (ii) for each $x > 0$, $\phi(x) < x$,
- (iii) $\beta(x) = \phi(x)/x \in S$.

For example, $\phi(x) = ax$, where $0 \leq a < 1$, $\phi(x) = x/(x+1)$, and $\phi(x) = \ln(1+x)$ are in $\mathfrak{B}$.

Theorem 3.2. Consider problem (1.1) with $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \geq x$

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \tag{3.1}$$

where $\phi \in \mathfrak{B}$. Then the existence of a lower solution of (1.1) provides the existence of a unique solution of (1.1).

Proof. Problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s)] ds, \quad (3.2)$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s < t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t < s \leq T. \end{cases} \quad (3.3)$$

Define $F : C(I, \mathbb{R}) \rightarrow C(I, \mathbb{R})$ by

$$(Fu)(t) = \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s)] ds. \quad (3.4)$$

Note that if $u \in C(I, \mathbb{R})$ is a fixed point of F then $u \in C^1(I, \mathbb{R})$ is a solution of (1.1). Now, we check that hypotheses in Theorem 2.1 are satisfied. Indeed, $X = C(I, \mathbb{R})$ is a partially ordered set if we define the following order relation in X :

$$x, y \in C(I, \mathbb{R}), \quad x \leq y \quad \text{iff} \quad x(t) \leq y(t), \quad \forall t \in I. \quad (3.5)$$

The mapping F is increasing since, by hypotheses, for $u \geq v$

$$f(t, u) + \lambda u \geq f(t, v) + \lambda v \quad (3.6)$$

which implies for $t \in I$, using that $G(t, s) > 0$ for $(t, s) \in I \times I$, that

$$\begin{aligned} (Fu)(t) &= \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s)] ds \\ &\geq \int_0^T G(t, s) [f(s, v(s)) + \lambda v(s)] ds = (Fv)(t). \end{aligned} \quad (3.7)$$

Beside, for $u \geq v$

$$\begin{aligned} d(Fu, Fv) &= \sup_{t \in I} |(Fu)(t) - (Fv)(t)| \\ &\leq \sup_{t \in I} \int_0^T G(t, s) |f(s, u(s)) + \lambda u(s) - (f(s, v(s)) + \lambda v(s))| ds \\ &\leq \sup_{t \in I} \int_0^T G(t, s) \cdot \lambda \phi(u(s) - v(s)) ds. \end{aligned} \quad (3.8)$$

As the function $\phi(x)$ is increasing and $u \geq v$ then $\phi(u(s) - v(s)) \leq \phi(d(u, v))$ we obtain

$$\begin{aligned}
 d(Fu, Fv) &\leq \sup_{t \in I} \int_0^T G(t, s) \cdot \lambda \phi(u(s) - v(s)) ds \\
 &\leq \lambda \phi(d(u, v)) \sup_{t \in I} \int_0^T G(t, s) ds \\
 &= \lambda \phi(d(u, v)) \sup_{t \in I} \frac{1}{e^{\lambda T} - 1} \left(\frac{1}{\lambda} e^{\lambda(T+s-t)} \Big|_0^t + \frac{1}{\lambda} e^{\lambda(s-t)} \Big|_t^T \right) \\
 &= \lambda \phi(d(u, v)) \cdot \frac{1}{\lambda(e^{\lambda T} - 1)} (e^{\lambda T} - 1) = \phi(d(u, v)).
 \end{aligned} \tag{3.9}$$

Then for $u \geq v$

$$d(Fu, Fv) \leq \alpha(d(u, v))d(u, v). \tag{3.10}$$

Finally, let $\beta(t)$ be a lower solution of (1.1), and we will show that $\beta \leq F(\beta)$.
Indeed,

$$\beta'(t) + \lambda \beta(t) \leq f(t, \beta(t)) + \lambda \beta(t), \quad \text{for } t \in I. \tag{3.11}$$

Multiplying by $e^{\lambda t}$ we get

$$(\beta(t)e^{\lambda t})' \leq [f(t, \beta(t)) + \lambda \beta(t)]e^{\lambda t}, \quad \text{for } t \in I, \tag{3.12}$$

and this gives us

$$\beta(t)e^{\lambda t} \leq \beta(0) + \int_0^t [f(s, \beta(s)) + \lambda \beta(s)]e^{\lambda s} ds, \quad \text{for } t \in I \tag{3.13}$$

which implies that

$$\beta(0)e^{\lambda T} \leq \beta(T)e^{\lambda T} \leq \beta(0) + \int_0^T [f(s, \beta(s)) + \lambda \beta(s)]e^{\lambda s} ds, \tag{3.14}$$

and so

$$\beta(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \beta(s)) + \lambda \beta(s)] ds. \tag{3.15}$$

From this equality and (3.13) we obtain

$$\beta(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1} [f(s, \beta(s)) + \lambda \beta(s)] ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \beta(s)) + \lambda \beta(s)] ds, \tag{3.16}$$

and, consequently,

$$\beta(t) \leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} [f(s, \beta(s)) + \lambda\beta(s)] ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} [f(s, \beta(s)) + \lambda\beta(s)] ds. \quad (3.17)$$

Hence

$$\beta(t) \leq \int_0^T G(t, s) [f(s, \beta(s)) + \lambda\beta(s)] ds = (F(\beta))(t), \quad \text{for } t \in I. \quad (3.18)$$

Finally, Theorem 2.1 gives that F has a unique fixed point. $\square$

Example 3.3. Let $\phi_0 : [0, \infty) \rightarrow [0, \infty)$ be a defined

$$\phi_0(t) = \begin{cases} 0, & 0 \leq t \leq 3, \\ 3t - 9, & 3 < t \leq 4, \\ \frac{3}{4}t, & 4 < t. \end{cases} \quad (3.19)$$

Let $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \geq x$

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi_0(y - x). \quad (3.20)$$

Then by Theorem 2.1, the existence of a lower solution for (1.1) provides the existence of a unique solution of (1.1).

The example discussed above cannot be the result of Harjani and Sadarangani noted Theorem 1.2, because $\psi(x) = x - \phi_0(x)$ is not increasing.

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References

- [1] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.
- [2] A. Amini-Harandi and H. Emami, "A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations," *Nonlinear Analysis*, vol. 72, no. 5, pp. 2238–2242, 2010.
- [3] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis*, vol. 65, no. 7, pp. 1379–1393, 2006.
- [4] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

- [5] V. Lakshmikantham and L. B. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis*, vol. 70, no. 12, pp. 4341–4349, 2009.
- [6] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, "Fixed point theorems in ordered abstract spaces," *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.
- [7] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [8] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.
- [9] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.