Research Article

H_∞ Filtering of Time-Varying Nonlinear Systems with Randomly Occurring Output Degradation

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This paper deals with the H_∞ filtering problem for nonlinear systems with randomly occurring output degradation phenomenon. Such a phenomenon is described by a stochastic variable which obeys the Bernoulli distribution with probability known priorly. A sufficient condition is derived for the nonlinear system to reach the required H_∞ performance. An iterative algorithm is then proposed to obtain the filter parameters recursively by solving the corresponding linear matrix inequality. A numerical example is presented to show the effectiveness of the proposed method.

1. Introduction

Many practical engineering systems, like the radar systems which are used for tracking the hostile weapon systems, are always encountering failures. For example, the radar systems will fail from time to time due to the electromagnetic interference from the enemy. Moreover, other reasons that lead the sensors to failures mainly include the external disturbance and changes of working conditions, to name just a few; see [1–3], for example. It is worth pointing out that, in the systems mentioned above, the sensor failure is not persistent all the time but is intermittent stochastically. In other words, the sensor failure occurs at random time points in a probabilistic way. Such phenomena are called the randomly occurring phenomena which would drastically degrade the system performance. Therefore, in recent years, the randomly occurring phenomena have stirred quite a lot of research interests due to its clear engineering insights and many results have been reported in the literature; see [4–11] for some latest publications. However, in spite of its clear physical insight and importance in engineering application, the filtering problem for nonlinear time-varying systems under the circumstance of randomly occurring output degradation has not yet been studied sufficiently.

On another research frontier, it is well known that the nonlinearities are inevitable in practical engineering systems, and the analysis and synthesis of nonlinear systems have been

attracting more and more research attention, among which the sector bound nonlinearity which could cover several class of well-studied nonlinearities has drawn particular research focus since many sensor failures like missing measurements, signal saturations can be easily converted into the nonlinearity belonging to a known sector; see [11–15] and the references therein. On the other hand, in recent years, time-varying systems have started to receive attention due to the fact that there are virtually no strictly time-invariant systems since the working circumstances, operating points, or equipment deterioration levels are inherently time-varying in nature; see [7, 8, 16, 17] for some latest results. For time-varying Markov jumping system, in [7], the fault detection problem has been solved in the presence of sensor saturation and randomly occurring nonlinearities. The distributed H_∞ estimation algorithm has been proposed in [8] for a class of nonlinear time-varying stochastic systems. A distributed filter has been designed by [16] for a type of time-varying systems while quantization errors and packet dropouts are happening. Recently, in [10, 17], mixed H_2/H_∞ controllers have been designed for a special type of nonlinear stochastic systems over a finite horizon. Unfortunately, despite the importance of the time-varying nature in system modeling, the H_∞ filtering problem for nonlinear time-varying systems with randomly occurring output degradation has not yet received enough research.

Motivated by the above discussion and in order to meet the ever-increasing practical engineering requirements toward the system reliability, in this paper, we aim to design an H_∞ filter for nonlinear time-varying stochastic systems in the presence of randomly occurring output degradation. The main contributions of this paper lie in the following two aspects: (1) for the first time, the phenomena of randomly occurring output degradation are considered to better illustrate the system's complex working environment and its impact on the performance. Moreover, such a model can stand for a wide range of output degradations, including the signal quantization phenomenon and saturation as special cases; (2) by virtue of the iterative linear matrix inequality method, the filtering problem is solved for a class of nonlinear time-varying systems which are much more general than those investigated in the existing literature. The rest of the paper is arranged as follows. Section 2 formulates the H_∞ filtering problem for the nonlinear time-varying systems with randomly occurring output degradation. In Section 3, the H_∞ performance is analyzed in terms of certain matrix inequality. Then, it gives the methodology to solve the addressed filtering problem and outlines the computational algorithm to recursively obtain the required filter parameters. A numerical example is presented in Section 4 to show the effectiveness and applicability of the proposed algorithm. Section 5 draws the conclusion.

2. Problem Formulation

Let us consider the discrete-time nonlinear stochastic system and measurement equations defined on $k \in [0, N]$:

$$\begin{aligned} x_{k+1} &= A_k x_k + f(x_k) + g(x_k) + D_k \omega_k, \\ y_k &= \beta(C_k x_k) + E_k \omega_k, \\ z_k &= M_k x_k, \end{aligned} \quad (1)$$

where x_k , y_k , z_k , and ω_k represent the state, measured output, estimated output, and disturbance belonging to l_2 , respectively. A_k , C_k , D_k , E_k , and M_k are known real time-varying matrices with appropriate dimensions. $f(x_k)$ and $g(x_k)$ are nonlinear functions.

It is assumed that the nonlinear function $g(x_k)$ has the following constraint:

$$g^T(x_k) S_k^{-1} g(x_k) \leq 1, \quad (2)$$

where S_k is known positive-definite matrix sequence with appropriate dimensions describing the shape of the ellipsoids with 0 being the center of the ellipsoids.

Remark 1. The nonlinear function $g(x_k)$ stands for the nonlinearity that is unknown, bounded, and deterministic but reside within an ellipsoidal set. Such a type of nonlinearity is usually occurring in practical engineering practice and always being a main origin for the degradation of the system performance.

Definition 2. A nonlinear function $\psi(\cdot)$ is said to satisfy the sector-bounded condition if

$$(\psi(v) - V_1 v)^T (\psi(v) - V_2 v) \leq 0 \quad (3)$$

for some real matrices V_1 and V_2 , where $V = V_2 - V_1$ is a symmetric positive-definite matrix. In this case, we say that $\psi(\cdot)$ belongs to $[V_1, V_2]$.

The nonlinear function $f(x_k)$ in (1) belongs to sector $[R_1, R_2]$ where R_1 and R_2 are real matrices of appropriate dimensions; that is

$$(f(x_k) - R_1 x_k)^T (f(x_k) - R_2 x_k) \leq 0. \quad (4)$$

Remark 3. The nonlinear function $f(x_k)$ is used to describe the nonlinearity which belongs to a specified sector. Such a kind of nonlinearity is frequently seen in both theoretical research and engineering practice and could cover quite a lot of nonlinearities as special cases, for example, the Lipschitz type nonlinearity.

The nonlinear function $\beta(\cdot)$ has the following form:

$$\beta(C_k x_k) = \alpha_k \sigma(C_k x_k) + (1 - \alpha_k) C_k x_k, \quad (5)$$

where α_k is a Bernoulli distributed random sequence taking values between 0 and 1 with the following probability:

$$\text{Prob}\{\alpha_k = 1\} = \mathbb{E}\{\alpha_k\} = \bar{\alpha}. \quad (6)$$

Remark 4. Equation (5) is introduced to illustrate the constraints on the output sampling when the output signal is confronted with degradation. The nonlinear function $\sigma(\cdot)$ which will be interpreted in detail later stands for the sector bound type output degradation phenomenon.

Remark 5. The Bernoulli distributed random variable α_k is employed to describe the phenomenon of the randomly occurring output degradations. To be specific, when $\alpha_k = 0$, the output signal will not be confronted with degradation; otherwise there might be possible signal missing during the output sampling. Compared to the traditional models where the output signals are assumed to be always available, such a model proposed in this paper could obviously better reflect the real-world engineering practice especially when the working condition is changing.

As discussed in [11], there exist matrices K_1 and K_2 such that $0 \leq K_1 < I \leq K_2$; the sensor fault function $\sigma(C_k x_k)$ can be written as

$$\sigma(C_k x_k) = K_1 C_k x_k + \varphi_y(C_k x_k), \quad (7)$$

where $\varphi_y(C_k x_k)$ is a nonlinear vector-valued function satisfying the sector-bounded condition with $V_1 = 0$ and $V_2 = K$. In this case, $\varphi_y(C_k x_k)$ can be represented as follows:

$$\varphi_y^T(C_k x_k) (\varphi_y(C_k x_k) - K C_k x_k) \leq 0, \quad (8)$$

where $K = K_2 - K_1$.

The objective of this paper is to design a filter with the following form for system (1):

$$\begin{aligned}\hat{x}_{k+1} &= G_k \hat{x}_k + N_k y_k, \\ \hat{z}_k &= M_k \hat{x}_k,\end{aligned}\quad (9)$$

where $\hat{x}_k$ is the state estimate, $\hat{z}_k$ is the output to be estimated, and G_k and N_k are the filter parameters to be determined.

Let us define the estimation error by $\tilde{z}_k = z_k - \hat{z}_k$ and then obtain the following filtering error system:

$$\begin{aligned}\xi_{k+1} &= (\bar{A}_k + \hat{A}_k) \xi_k + F_k f(x_k) + F_k g(x_k) \\ &\quad + (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) + \bar{D}_k \omega_k, \\ \tilde{z}_k &= \bar{M}_k \xi_k,\end{aligned}\quad (10)$$

where

$$\begin{aligned}\xi_k &= \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} A_k & 0 \\ N_k C_k + \bar{\alpha} N_k (K_1 - I) C_k & G_k \end{bmatrix}, \\ \hat{A}_k &= \begin{bmatrix} 0 & 0 \\ \bar{\alpha}_k N_k (K_1 - I) C_k & 0 \end{bmatrix}, \quad F_k = \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ \bar{D}_k &= \begin{bmatrix} D_k \\ N_k E_k \end{bmatrix}, \quad \bar{\Phi}_k = \begin{bmatrix} 0 \\ \bar{\alpha} N_k \end{bmatrix}, \\ \hat{\Phi}_k &= \begin{bmatrix} 0 \\ \bar{\alpha}_k N_k \end{bmatrix}, \quad \check{A}_k = \begin{bmatrix} 0 & 0 \\ \sigma_\alpha N_k (K_1 - I) C_k & 0 \end{bmatrix}, \\ \check{\Phi}_k &= \begin{bmatrix} 0 \\ \sigma_\alpha N_k \end{bmatrix}, \quad \bar{M}_k = \begin{bmatrix} M_k & -M_k \end{bmatrix}.\end{aligned}\quad (11)$$

Given a H_∞ performance parameter $\gamma > 0$, the objective of this paper is to obtain G_k and N_k in (9) such that

$$\mathbb{E} \left\{ \sum_{k=0}^N \tilde{z}_k^T \tilde{z}_k \right\} \leq \gamma^2 \sum_{k=0}^N \omega_k^T \omega_k + \gamma^2 x_0^T R x_0 \quad (12)$$

are satisfied when the sensor faults are randomly happening.

3. Main Results

Before giving the main results, some useful lemmas are firstly introduced.

Lemma 6. Let $W_0(\cdot)$, $W_1(\cdot)$, $\dots$, $W_p(\cdot)$ be quadratic functions of the variable η : $W_i(\eta) \triangleq \eta^T T_i \eta$ with $T_i^T = T_i$ ($i = 0, 1, \dots, p$). If there exist scalars $\tau_1 \geq 0$, $\tau_2 \geq 0, \dots, \tau_p \geq 0$ such that

$$T_0 - \sum_{i=1}^p \tau_i T_i \leq 0, \quad (13)$$

then the following is true:

$$W_1(\eta) \leq 0, \quad W_2(\eta) \leq 0, \dots, W_p(\eta) \leq 0 \implies W_0(\eta) \leq 0. \quad (14)$$

Lemma 7 (Schur complement equivalence). Given constant matrices Q_1 , Q_2 , and Q_3 where $Q_1 = Q_1^T$ and $Q_2 = Q_2^T > 0$, then

$$Q_1 + Q_3^T Q_2^{-1} Q_3 < 0 \quad (15)$$

if and only if

$$\begin{bmatrix} Q_1 & Q_3^T \\ Q_3 & -Q_2 \end{bmatrix} < 0 \quad (16)$$

or equivalently

$$\begin{bmatrix} -Q_2 & Q_3 \\ Q_3^T & Q_1 \end{bmatrix} < 0. \quad (17)$$

3.1. H_∞ Performance Analysis. The following theorem gives a sufficient condition under which the required H_∞ performance can be satisfied.

Theorem 8. Given a H_∞ performance $\gamma > 0$ and the initial matrix $R > 0$, let the filter parameters G_k and N_k be given. If there exist three sequences of positive scalars τ_{1k} , τ_{2k} , and τ_{3k} ($0 \leq k \leq N$) and a sequence of positive definite matrices P_k ($0 \leq k \leq N+1$) satisfying $\xi_0^T P_0 \xi_0 \leq \gamma^2 \xi_0^T \bar{R} \xi_0$ such that the following matrix inequality

$$\begin{bmatrix} \Xi_{11} & \tau_{1k} \frac{\bar{R}_1^T + \bar{R}_2^T}{2} & 0 & \tau_{3k} \frac{F_k C_k^T K^T}{2} & 0 & \bar{A}_k^T & \check{A}_k^T \\ * & -\tau_{1k} I & 0 & 0 & 0 & F_k^T & 0 \\ * & * & -\tau_{2k} S_k^{-1} & 0 & 0 & F_k^T & 0 \\ * & * & * & -\tau_{3k} I & 0 & \bar{\Phi}_k^T & \check{\Phi}_k^T \\ * & * & * & * & -\gamma^2 I & \bar{D}_k^T & 0 \\ * & * & * & * & * & -P_{k+1}^{-1} & 0 \\ * & * & * & * & * & * & -P_{k+1}^{-1} \end{bmatrix} \leq 0,$$

$$\Xi_{11} = -P_k + \bar{M}_k^T \bar{M}_k - \tau_{1k} \frac{\bar{R}_1^T \bar{R}_2 + \bar{R}_2^T \bar{R}_1}{2} + \tau_{2k} I,$$

$$\bar{R}_1 = \begin{bmatrix} R_1 & 0 \end{bmatrix}, \quad \bar{R}_2 = \begin{bmatrix} R_2 & 0 \end{bmatrix}, \quad \hat{R} = \text{diag} \{R, 0\} \quad (18)$$

is feasible, then H_∞ performance can be satisfied.

Proof. Defining

$$J_k = \xi_{k+1}^T P_{k+1} \xi_{k+1} - \xi_k^T P_k \xi_k, \quad (19)$$

one can get from (10) that

$$\begin{aligned}
J_k &= \xi_{k+1}^T P_{k+1} \xi_{k+1} - \xi_k^T P_k \xi_k \\
&= ((\bar{A}_k + \hat{A}_k) \xi_k + F_k f(x_k) + F_k g(x_k) \\
&\quad + (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) + \bar{D}_k \omega_k)^T P_{k+1} \\
&\quad \times ((\bar{A}_k + \hat{A}_k) \xi_k + F_k f(x_k) + F_k g(x_k) \\
&\quad + (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) + \bar{D}_k \omega_k) - \xi_k^T P_k \xi_k \\
&= \xi_k^T (\bar{A}_k + \hat{A}_k)^T P_{k+1} ((\bar{A}_k + \hat{A}_k) \xi_k) \\
&\quad + f^T(x_k) F_k^T P_{k+1} F_k f(x_k) + g^T(x_k) F_k^T P_{k+1} F_k g(x_k) \\
&\quad + ((\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k))^T \\
&\quad \times P_{k+1} ((\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k)) \\
&\quad + \omega_k^T \bar{D}_k^T P_{k+1} \bar{D}_k \omega_k \\
&\quad + 2\xi_k^T (\bar{A}_k + \hat{A}_k)^T P_{k+1} F_k f(x_k) \\
&\quad + 2\xi_k^T (\bar{A}_k + \hat{A}_k)^T P_{k+1} F_k g(x_k) \\
&\quad + 2\xi_k^T (\bar{A}_k + \hat{A}_k)^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) \\
&\quad + 2\xi_k^T (\bar{A}_k + \hat{A}_k)^T P_{k+1} \bar{D}_k \omega_k \\
&\quad + 2f^T(x_k) F_k^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) \\
&\quad + 2f^T(x_k) F_k^T P_{k+1} F_k g(x_k) \\
&\quad + 2f^T(x_k) F_k^T P_{k+1} \bar{D}_k \omega_k \\
&\quad + 2g^T(x_k) F_k^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k) \\
&\quad + 2g^T(x_k) F_k^T P_{k+1} \bar{D}_k \omega_k \\
&\quad + 2((\bar{\Phi}_k + \hat{\Phi}_k) \varphi_y(C_k x_k))^T P_{k+1} \bar{D}_k \omega_k - \xi_k^T P_k \xi_k \\
&= \eta_k^T \Theta_k \eta_k,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\eta_k &= [\xi_k^T \ f^T(x_k) \ g^T(x_k) \ \varphi_y^T(C_k x_k) \ \omega_k^T]^T, \\
\Theta_k &= \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} \\ * & F_k^T P_{k+1} F_k & F_k^T P_{k+1} F_k & \Theta_{24} & F_k^T P_{k+1} \bar{D}_k \\ * & * & F_k^T P_{k+1} F_k & \Theta_{34} & F_k^T P_{k+1} \bar{D}_k \\ * & * & * & \Theta_{44} & \Theta_{45} \\ * & * & * & * & \bar{D}_k^T P_{k+1} \bar{D}_k \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Theta_{11} &= (\bar{A}_k + \hat{A}_k)^T P_{k+1} (\bar{A}_k + \hat{A}_k) - P_k, \\
\Theta_{12} &= (\bar{A}_k + \hat{A}_k)^T P_{k+1} F_k, \\
\Theta_{13} &= (\bar{A}_k + \hat{A}_k)^T P_{k+1} F_k, \\
\Theta_{14} &= (\bar{A}_k + \hat{A}_k)^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k), \\
\Theta_{15} &= (\bar{A}_k + \hat{A}_k)^T P_{k+1} \bar{D}_k, \\
\Theta_{24} &= F_k^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k), \\
\Theta_{34} &= F_k^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k), \\
\Theta_{44} &= (\bar{\Phi}_k + \hat{\Phi}_k)^T P_{k+1} (\bar{\Phi}_k + \hat{\Phi}_k), \\
\Theta_{45} &= (\bar{\Phi}_k + \hat{\Phi}_k)^T P_{k+1} \bar{D}_k.
\end{aligned} \tag{21}$$

Adding the zero term

$$\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^T \omega_k - (\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^T \omega_k) \tag{22}$$

to both sides of (20), one can get

$$J_k = \eta_k^T \Lambda_k \eta_k - (\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^T \omega_k), \tag{23}$$

where

$$\Lambda_k = \Theta_k + \text{diag} \{ \bar{M}_k^T \bar{M}_k, 0, 0, 0, -\gamma^2 I \}. \tag{24}$$

Next, by summing up (23) on both sides from 0 to N with respect to k , one can have

$$\sum_{k=0}^N J_k = \xi_{N+1}^T P_{N+1} \xi_{N+1} - \xi_0^T P_0 \xi_0 \tag{25}$$

$$= \sum_{k=0}^N \eta_k^T \Lambda_k \eta_k - \sum_{k=0}^N (\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^T \omega_k)$$

which is equivalent to

$$\begin{aligned}
\sum_{k=0}^N \tilde{z}_k^T \tilde{z}_k &= -\gamma^2 \omega_k^T \omega_k + \sum_{k=0}^N \eta_k^T \Lambda_k \eta_k \\
&\quad - \xi_{N+1}^T P_{N+1} \xi_{N+1} + \xi_0^T P_0 \xi_0.
\end{aligned} \tag{26}$$

Taking the property of α_k which is contained in random matrices $\bar{A}_k$ and $\bar{\Phi}_k$ in Λ_k into consideration, by taking the mathematical expectation of both sides of (26), one can have

$$\begin{aligned}
\mathbb{E} \left\{ \sum_{k=0}^N \tilde{z}_k^T \tilde{z}_k \right\} &= -\gamma^2 \omega_k^T \omega_k + \sum_{k=0}^N \eta_k^T \bar{\Lambda}_k \eta_k \\
&\quad - \xi_{N+1}^T P_{N+1} \xi_{N+1} + \xi_0^T P_0 \xi_0,
\end{aligned} \tag{27}$$

where

$$\bar{\Lambda}_k = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{A}_k^T P_{k+1} F_k & \bar{A}_k^T P_{k+1} F_k & \bar{A}_k^T P_{k+1} \bar{D}_k \\ * & F_k^T P_{k+1} F_k & F_k^T P_{k+1} F_k & F_k^T P_{k+1} \bar{\Phi}_k & F_k^T P_{k+1} \bar{D}_k \\ * & * & F_k^T P_{k+1} F_k & F_k^T P_{k+1} \bar{\Phi}_k & F_k^T P_{k+1} \bar{D}_k \\ * & * & * & \bar{\Phi}_k^T P_{k+1} \bar{D}_k \\ * & * & * & * & \bar{D}_k^T P_{k+1} \bar{D}_k - \gamma^2 I \end{bmatrix},$$

$$\bar{\Lambda}_{11} = \bar{A}_k^T P_{k+1} \bar{A}_k + \check{A}_k^T P_{k+1} \check{A}_k - P_k + \bar{M}_k^T \bar{M}_k,$$

$$\bar{\Lambda}_{14} = \bar{A}_k^T P_{k+1} \bar{\Phi}_k + \check{A}_k^T P_{k+1} \check{\Phi}_k,$$

$$\bar{\Lambda}_{44} = \bar{\Phi}_k^T P_{k+1} \bar{\Phi}_k + \check{\Phi}_k^T P_{k+1} \check{\Phi}_k. \quad (28)$$

From (27), since $P_{N+1} > 0$ and the initial condition $\xi_0^T P_0 \xi_0 \leq \gamma^2 \xi_0^T \bar{R} \xi_0 = \gamma^2 x_0^T R x_0$, one can see that the H_∞ performance can be satisfied if $\eta_k^T \bar{\Lambda}_k \eta_k \leq 0$ is satisfied.

Next, one can see from the sector-bounded condition of nonlinear function $f(x_k)$ (4) that

$$\eta_k^T Y_k \eta_k \leq 0, \quad (29)$$

where

$$Y_k = \frac{1}{2} \begin{bmatrix} \bar{R}_1^T \bar{R}_2^T + \bar{R}_2^T \bar{R}_1 & -\bar{R}_1^T + \bar{R}_2^T & 0 & 0 & 0 \\ * & 2I & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}. \quad (30)$$

Thirdly, one can get from the ellipsoidal nonlinear function $g(x_k)$ (2) that

$$\eta_k^T \Omega_k \eta_k \leq 0, \quad (31)$$

where

$$\Omega_k = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & S_k^{-1} & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}. \quad (32)$$

Moreover, it also can be obtained from the sensor fault constraint (8) that

$$\eta_k^T \Pi_k \eta_k \leq 0, \quad (33)$$

where

$$\Pi_k = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -F_k C_k^T K^T & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 2I & 0 \\ * & * & * & * & 0 \end{bmatrix}. \quad (34)$$

By Schur complement equivalence lemma, the matrix inequality (18) can be rewritten as follows:

$$\bar{\Lambda}_k - \tau_{1k} Y_k - \tau_{2k} \Omega_k - \tau_{3k} \Pi_k \leq 0. \quad (35)$$

By Lemma 6, $\eta_k^T \bar{\Lambda}_k \eta_k \leq 0$ can be satisfied, which means that the required H_∞ performance can be satisfied. The proof is complete. $\square$

3.2. Filter Design

Theorem 9. Let the H_∞ performance $\gamma > 0$ and the initial condition matrix $R > 0$ be given. If there exist a sequence of matrices G_k ($0 \leq k \leq N$), a sequence of matrices N_k ($0 \leq k \leq N$), three sequences of positive scalars τ_{1k} , τ_{2k} , and τ_{3k} ($0 \leq k \leq N$), a sequence of positive-definite matrices X_k ($0 \leq k \leq N+1$), and a sequence of positive-definite matrices Y_k ($0 \leq k \leq N+1$) satisfying $\xi_0^T \text{diag}\{X_0^{-1}, Y_0^{-1}\} \xi_0 \leq \gamma^2 \xi_0^T \bar{R} \xi_0$ such that the following matrix inequality

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ * & \Gamma_3 \end{bmatrix} \leq 0, \quad (36)$$

where

$$\Gamma_1 = \begin{bmatrix} \Gamma_{11} & -M_k^T M_k & \Gamma_{13} & 0 & \frac{1}{2} \tau_{3k} C_k^T K^T & 0 \\ * & \Gamma_{22} & 0 & 0 & 0 & 0 \\ * & * & -\tau_{1k} I & 0 & 0 & 0 \\ * & * & * & -\tau_{2k} S_k^{-1} & 0 & 0 \\ * & * & * & * & -\tau_{3k} I & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} A_k^T & \Gamma_{18} & 0 & \Gamma_{110} \\ G_k^T & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & \bar{\alpha} N_k^T & 0 & \sigma_\alpha N_k^T \\ D_k^T & E_k^T N_k^T & 0 & 0 \end{bmatrix},$$

$$\Gamma_3 = \text{diag}\{-X_{k+1}, -Y_{k+1}, -X_{k+1}, -Y_{k+1}\},$$

$$\Gamma_{11} = -H_k - \frac{1}{2} \tau_{1k} (R_1^T R_2 + R_2^T R_1) + \tau_{2k} I + M_k^T M_k,$$

$$\Gamma_{13} = \frac{1}{2} \tau_{1k} (R_1^T + R_2^T),$$

$$\Gamma_{22} = -L_k + \tau_{2k} I + M_k^T M_k,$$

$$\Gamma_{18} = C_k^T ((1 - \bar{\alpha}) I + \bar{\alpha} K_1) N_k^T,$$

$$\Gamma_{110} = \sigma_\alpha C_k^T (K_1 - I) N_k^T \quad (37)$$

is satisfied, then the required H_∞ filter parameters at each time k can be obtained by solving the corresponding matrix inequality.

Proof. The proof is quite straightforward. First, define $P_k^{-1} = \text{diag}\{H_k^{-1}, L_k^{-1}\} = \text{diag}\{X_k, Y_k\}$. From Theorem 8, (36) can be obtained by simply expanding the matrix inequality (18). $\square$

3.3. Computational Algorithm

Algorithm 10 (H_∞ filter design algorithm).

Step 1. Set the required γ , the initial condition R , the initial $\xi(0)$, and N . Select the initial values for H_0 and L_0 satisfying the initial condition constraint.

Step 2. Set $k = 0$. Solve (36) for G_0 , N_0 , X_1 , and Y_1 . Then H_1 and L_1 can also be obtained.

Step 3. Set $k = k + 1$. Using the obtained H_k and L_k , solve (36) for G_k , N_k , X_{k+1} , and Y_{k+1} .

Step 4. If $k = N + 1$, then stop. Else go to Step 3.

Remark 11. The H_∞ filter design algorithm gives a recursive way to obtain the numerical values of the desired filter parameters at each time point k . It should be pointed out that the existence of the filter is expressed as the feasibility of certain linear matrix inequalities that can be solved forward in time. The possible research topic in the future is to consider more performance indices and give the filtering schemes that are able to satisfy multiple requirements simultaneously.

4. An Illustrative Example

In this section, an illustrative example is presented to show the effectiveness of the proposed H_∞ filtering design algorithm.

Let us consider the nonlinear system (1) with the parameters given below:

$$\begin{aligned} A_k &= \begin{bmatrix} 0.5 + 0.1 \sin(k) & -1 \\ 0.2 & -0.5 \end{bmatrix}, & D_k &= \begin{bmatrix} 0.2 \\ -0.4 \end{bmatrix}, \\ C_k &= [1 - \cos(5k) \quad 0.7], & E_k &= 0.3, \\ M_k &= [1.5 \quad 0.75]. \end{aligned} \quad (38)$$

Let $g(x_k) = [\sin(10x_k) \quad 0.5 \cos(10x_k)]^T$ and set $S_k = 2$. Let $\omega_k = \sin(5k)/3(k+1)$.

Assume that the nonlinear function $f(x_k)$ which is chosen as

$$f(x_k) = \begin{bmatrix} -0.1x_{1k} + 0.25x_{2k} + \frac{0.1x_{2k} \sin(x_{1k})}{\sqrt{x_{1k}^2 + x_{2k}^2 + 10}} \\ -0.05x_{1k} + 0.05x_{2k} \end{bmatrix} \quad (39)$$

belongs to sector $[R_1, R_2]$ with

$$R_1 = \begin{bmatrix} -0.4 & 0 \\ -0.2 & -0.3 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}. \quad (40)$$

TABLE 1: The filter parameters.

	G_k	N_k
$k = 0$	$\begin{bmatrix} 0.25 & -0.1 \\ 0.23 & 0.7 \end{bmatrix}$	$\begin{bmatrix} 0.54 \\ 1.25 \end{bmatrix}$
$k = 1$	$\begin{bmatrix} -0.65 & -0.03 \\ 0.38 & 1.12 \end{bmatrix}$	$\begin{bmatrix} 1.40 \\ -0.25 \end{bmatrix}$
$k = 2$	$\begin{bmatrix} -0.16 & 0.45 \\ -0.28 & 0.07 \end{bmatrix}$	$\begin{bmatrix} 1.25 \\ 0.17 \end{bmatrix}$
$k = 3$	$\begin{bmatrix} 1.65 & -0.39 \\ -0.87 & 0.05 \end{bmatrix}$	$\begin{bmatrix} 1.90 \\ 0.15 \end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$

FIGURE 1: The state x_k^1 and its estimate.

Set $\gamma = 0.9$, $R = I$, $\xi_0 = [0.5 \quad 0.5 \quad 0 \quad 0]^T$, $N = 4$, and $\bar{\alpha} = 0.8$. Set $H_0 = 0.8I$ and $L_0 = I$; hence the initial condition can be satisfied. The required filter parameters can be obtained by Algorithm 10, and some of the results are listed in Table 1.

Set the initial value of the system state and its estimate by $x_0 = [1 \quad 0]^T$ and $\hat{x}_0 = [0 \quad 0]^T$. We can see the effectiveness and applicability of the proposed filtering algorithm from the simulation results, Figures 1 and 2.

5. Conclusion

The problem of H_∞ filtering for nonlinear stochastic systems when the measurement output is confronted with possible randomly occurring degradation is considered. Such a phenomenon is described by a Bernoulli sequence with known probability. A sufficient condition is proposed under which the required H_∞ performance can be satisfied even if the randomly occurring output degradation happens. A recursive

FIGURE 2: The state x_k^2 and its estimate.

algorithm is given in order to obtain the required filter parameters iteratively. Finally, an illustrative example is presented to show the effectiveness of the proposed method.

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