Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 360182, 6 pages
http://dx.doi.org/10.1155/2013/360182

Research Article

Periodic Solutions of a Nonautonomous Plant-Hare Model with Impulses

Haihui Wu[1] and Yan Zhou[2]

[1] *Sunshine College, Fuzhou University, Fuzhou, Fujian 350015, China*
[2] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350015, China*

Correspondence should be addressed to Yan Zhou; zhouyanlijunhao@163.com

Received 25 July 2013; Revised 14 September 2013; Accepted 15 September 2013

Academic Editor: Thabet Abdeljawad

A plant-hare model subjected by the effect of impulses is studied in this paper. Sufficient conditions are obtained for the existence of at least one positive periodic solution.

1. Introduction

Classical predator-prey model has been well studied (e.g., see [1–8] and the references cited therein). To explore the impact of plant toxicity on the dynamics of plant-hare interactions, Gao and Xia [9] consider a nonautonomous plant-herbivore dynamical system with a toxin-determined functional response:

$$\begin{aligned}
\dot{N}(t) &= r(t)N(t)\left[1-\frac{N(t)}{K}\right]-C(N(t))P(t),\\
\dot{P}(t) &= B(t)C(N(t))P(t)-d(t)P(t),\\
C(N(t)) &= f(N(t))\left[1-\frac{f(N(t))}{4G}\right],\\
f(N(t)) &= \frac{e\delta N(t)}{1+he\delta N(t)},
\end{aligned} \tag{1}$$

where $N(t)$ denotes the density of plant at time t, $P(t)$ denotes the herbivore biomass at time t, $r(t)$ is the plant intrinsic growth rate at time t, $d(t)$ is the per capita rate of herbivore death unrelated to plant toxicity at time t, $B(t)$ is the conversion rate at time t, e is the encounter rate per unit plant, δ is the fraction of food items encountered that the herbivore ingests, K is the carrying capacity of plant, G measures the toxicity level, and h is the time for handing one unit of plant. To explore the impact of environmental factors (e.g., seasonal effects of weather, food supplies, mating habits, harvesting, etc.), the assumption of periodicity of parameters is more realistic and important. To this reason, they assumed that $r(t)$, $d(t)$, and $B(t)$ are continuously positive periodic functions with period ω and e, δ, K, G, h are five positive real constants.

However, birth of many species is an annual birth pulse, for having more accurate description of the system, we need to consider using the impulsive differential equations. To see how impulses affect the differential equations, for examples, one can refer to [10–17]. Motivated by the above-mentioned works, in this paper, we consider the above system with impulses:

$$\begin{aligned}
\dot{N}(t) &= N(t)\left[r(t)\left(1-\frac{N(t)}{K}\right)\right.\\
&\qquad\left.-\frac{4Ge\delta P(t)+(4Gh-1)e^2\delta^2N(t)P(t)}{4G(1+he\delta N(t))^2}\right],\\
\dot{P}(t)&\\
&= P(t)\left[\left(4Ge\delta B(t)N(t)+(4Gh-1)e^2\delta^2\right.\right.\\
&\qquad\left.\left.\times B(t)N^2(t)\right)\left(4G(1+he\delta N(t))^2\right)^{-1}-d(t)\right],\\
&\qquad t\neq t_k;\\
\Delta N(t_k) &= N(t_k^+)-N(t_k^-)=c_{1k}N(t_k),\\
\Delta P(t_k) &= P(t_k^+)-P(t_k^-)=c_{2k}P(t_k),\quad t=t_k,\ k=1,2,\ldots,
\end{aligned} \tag{2}$$

where the assumptions on r, d, B, e, δ, K, G, and h are the same as before, $c_{jk} \in (-1, \infty)$ $(j = 1, 2,\ k \in \mathbb{N} = 1, 2, \ldots)$, $\{t_k\} \in \mathbb{N}$ is a strictly increasing sequence with $t_1 > 0$, and $\lim_{k\to\infty} t_k = \infty$. We further assume that there exists a $q \in \mathbb{N}$ such that $c_{j(k+q)} = c_{jk}$ $(j = 1, 2)$ and $t_{k+q} = t_k + \omega$ for $k \in \mathbb{N}$.

Without loss of generality, we will assume $t_k \neq 0$ for $k = 1, 2, \ldots$, and $[0, \omega] \cap \{t_k\} = \{t_1, t_2, \ldots, t_m\}$; hence $q = m$.

2. Preliminaries

In this section, we cite some definitions and lemmas.

Let PC_ω denote the space of ω-periodic functions $\psi : \mathbb{R} \to \mathbb{R}$ which are continuous for $t \neq t_k$, are continuous from the left for $t \in \mathbb{R}$, and have possible discontinuities of the first kind at points $t = t_k$; that is, the limit from the right of t_k exists but may be different from the value at t_k. We also denote $PC^1_\omega = \{\psi \in PC_\omega : \dot{\psi} \in PC_\omega\}$.

For the convenience, we list the following definitions and lemmas.

Definition 1 (see [10]). The set $\mathbb{F} \subset PC_\omega$ is said to be quasi-equicontinuous in $[0, \omega]$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathbb{F}$; $k \in \mathbb{Z}$; $\tau_1, \tau_2 \in (t_{k-1}, t_k) \cap [0, \omega]$ and $|\tau_1 - \tau_2| < \delta$, then

$$\left|x(\tau_1) - x(\tau_2)\right| < \epsilon. \tag{3}$$

Lemma 2 (see [10]). *The set $\mathbb{F} \subset PC_\omega$ is relatively compact if and only if*

(1) *$\mathbb{F}$ is bounded, that is, $\|x\| \leq M$, for each $x \in \mathbb{F}$, and some $M > 0$;*

(2) *$\mathbb{F}$ is quasi-equicontinuous in $[0, \omega]$.*

Lemma 3 (see [11]). *Assume that $\psi \in PC^1_\omega$, then the following inequality holds:*

$$\begin{aligned} &\sup_{s\in[0,\omega]} \psi(s) - \inf_{s\in[0,\omega]} \psi(s) \\ &\quad \leq \frac{1}{2}\left[\int_0^\omega \left|\dot{\psi}(s)\right| ds + \sum_{k=1}^m \left|\Delta\psi(t_k)\right|\right]. \end{aligned} \tag{4}$$

Before starting the main result, for the sake of convenience, one denotes

$$\begin{aligned} \overline{f} &= \frac{1}{\omega}\int_0^\omega f(t)\,dt \quad f \in PC_\omega, \\ c_j &= \sum_{k=1}^m \ln\left(1 + c_{jk}\right), \quad j = 1, 2, \\ C_1 &= \sum_{k=1}^m \left|\ln\left(1 + c_{1k}\right)\right| + c_1, \\ C_2 &= \sum_{k=1}^m \left|\ln\left(1 + c_{2k}\right)\right| - c_2. \end{aligned} \tag{5}$$

3. Existence of Positive Periodic Solutions

In order to obtain the existence of positive periodic solutions of (2), for convenience, we will summarize in the following a few concepts and results from [18] that will be basic for this section.

Let X, Y be normed vector spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that $L \mid \operatorname{dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, then the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 4 (see [18]). *Let $\Omega \subset X$ be an open and bounded set. Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*

(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

If f is a continuous ω-periodic function, then we set

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,dt. \tag{6}$$

The following assumptions are valid throughout this paper:

$(\overline{A}_1)$ $(4G\overline{d}h^2/(4Gh - 1)) < \overline{B} < 2h(\overline{d} - (c_2/\omega))$,

$(\overline{A}_2)$ $K(1 + (c_1/\overline{r}\omega)) - e^{H_1} > 0$, $H_1 = \max\{|\ln K + 2\overline{r}\omega|, |\ln(2(\overline{d} - (c_2/\omega) - (\overline{B}/2h))/e\delta\overline{B}) - 2\overline{r}\omega|\}$.

For convenience, we introduce two numbers as follows:

$$\begin{aligned} u_+ &= \frac{\left(4Ge\delta\overline{B} - 8Ghe\delta\left(\overline{d} - (c_2/\omega)\right)\right) + \sqrt{\Delta}}{2\left[4G\overline{d}h^2e^2\delta^2 - (4Gh - 1)e^2\delta^2\overline{B}\right]}, \\ u_- &= \frac{\left(4Ge\delta\overline{B} - 8Ghe\delta\left(\overline{d} - (c_2/\omega)\right)\right) - \sqrt{\Delta}}{2\left[4G\overline{d}h^2e^2\delta^2 - (4Gh - 1)e^2\delta^2\overline{B}\right]}, \end{aligned} \tag{7}$$

where $\Delta = (4Ge\delta\overline{B} - 8Ghe\delta(\overline{d} - (c_2/\omega)))^2 - 16G\overline{d}[4G\overline{d}h^2e^2\delta^2 - (4Gh - 1)e^2\delta^2\overline{B}]$.

Theorem 5. *In addition to $(\overline{A}_1)$, $(\overline{A}_2)$, suppose that*

$(\overline{A}_3)$ $\dfrac{1}{2h} < G < \dfrac{1}{h}$.

Then system (2) has at least one positive ω-periodic solution.

Remark 6. If the impulsive operators disappear, then $c_1 = c_2 = 0$. Then Theorem 5 reduces to the main results in Gao and Xia [9]. This implies that our result generalizes the previous one. It shows that the impulses do affect the system indeed.

Proof. Making the change of variables

$$N(t) = \exp(u_1(t)), \qquad P(t) = \exp(u_2(t)). \tag{8}$$

Then, system (2) can be rewritten as

$$\begin{aligned}
&\dot{u}_1(t) \\
&= r(t) - \frac{r(t)}{K}\exp(u_1(t)) \\
&\quad - \frac{4Ge\delta\exp(u_2(t)) + (4Gh-1)e^2\delta^2\exp(u_1(t)+u_2(t))}{4G(1+he\delta\exp(u_1(t)))^2} \\
&\equiv f_1(t), \\
&\dot{u}_2(t) = -d(t) + \big(4Ge\delta B(t)\exp(u_1(t)) + (4Gh-1) \\
&\qquad \times e^2\delta^2 B(t)\exp(2u_1(t))\big) \\
&\qquad \times \left(4G(1+he\delta\exp(u_1(t)))^2\right)^{-1} \\
&\equiv f_2(t), \quad t \neq t_k, \\
&\Delta u_1(t_k) = \ln(1+c_{1k}), \\
&\Delta u_2(t_k) = \ln(1+c_{2k}), \quad t = t_k.
\end{aligned} \tag{9}$$

Take

$$X = \left\{x = (u_1,u_2)^T : u_j \in PC_\omega, j = 1,2, x(t+\omega) = x(t)\right\}, \qquad Y = X \times \mathbb{R}^{2q} \tag{10}$$

and define

$$\begin{aligned}
\|x\|_0 &= \sum_{j=1}^{2}\sup_{t\in[0,\omega]}\left|u_j(t)\right|, \quad x = (u_1,u_2) \in X, \\
\|y\|_1 &= \|x\|_0 + \sum_{j=1}^{q}\left\|\xi_j\right\|, \quad y = \left[x,\xi_1,\ldots,\xi_q\right] \in Y.
\end{aligned} \tag{11}$$

Both $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ are Banach spaces.

Define

$$\begin{aligned}
&\operatorname{Dom} L = \{x \in X : \dot{x} \in X\}, \quad L : \operatorname{Dom} L \longrightarrow Y, \\
&L\left((u_1,u_2)^T\right) = \left(\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\end{pmatrix}, \begin{pmatrix}\Delta u_1(t_k)\\ \Delta u_2(t_k)\end{pmatrix}_{k=1}^{q}\right);
\end{aligned} \tag{12}$$

$N : X \to Y$,

$$N\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \left(\begin{pmatrix}f_1(t)\\ f_2(t)\end{pmatrix}, \begin{pmatrix}\ln(1+c_{1k})\\ \ln(1+c_{2k})\end{pmatrix}_{k=1}^{q}\right); \tag{13}$$

$P : X \to X$, $P((u_1,u_2)^T) = (\overline{u_1},\overline{u_2})^T$; $Q : Y \to Y$,

$$\begin{aligned}
&Q\left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \left\{\begin{pmatrix}m_k\\ n_k\end{pmatrix}\right\}_{k=1}^{q}\right) \\
&\quad = \left(\begin{pmatrix}\overline{u_1} + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q} m_k \\ \overline{u_2} + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q} n_k\end{pmatrix}, \left\{\begin{pmatrix}0\\ 0\end{pmatrix}\right\}_{k=1}^{q}\right).
\end{aligned} \tag{14}$$

It is not difficult to show that $\operatorname{Ker} L = \{x \mid x \in X, (u_1,u_2)^T \in X : (u_1(t),u_2(t))^T = (c_1,c_2)^T \in \mathbb{R}^2, t \in \mathbb{R}\}$.

$$\operatorname{Im} L = \left\{\left[\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \left\{\begin{pmatrix}m_k\\ n_k\end{pmatrix}\right\}_{k=1}^{q}\right] \in Y : \begin{array}{c}\omega\overline{u_1} + \displaystyle\sum_{k=1}^{q} m_k = 0 \\ \omega\overline{u_2} + \displaystyle\sum_{k=1}^{q} n_k = 0\end{array}\right\}. \tag{15}$$

Since $\operatorname{Im} L$ is closed in Y, P and Q are continuous projectors such that

$$\begin{aligned}
&\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q), \\
&\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L = 2.
\end{aligned} \tag{16}$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists, which is given by

$$\begin{aligned}
&K_p\left[\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \left\{\begin{pmatrix}m_k\\ n_k\end{pmatrix}\right\}_{k=1}^{q}\right] \\
&\quad = \begin{pmatrix}\displaystyle\int_0^t u_1(s)\,ds + \sum_{0<t_k<t} m_k - \frac{1}{\omega}\sum_{k=1}^{q} m_k - \overline{\int_0^t u_1(s)\,ds} \\ \displaystyle\int_0^t u_2(s)\,ds + \sum_{0<t_k<t} n_k - \frac{1}{\omega}\sum_{k=1}^{q} n_k - \overline{\int_0^t u_2(s)\,ds}\end{pmatrix}.
\end{aligned} \tag{17}$$

Then $QN : X \to Y$ and $K_p(I-Q)N : X \to X$ are defined by

$$QN\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \left(\begin{pmatrix}\overline{f_1} + \dfrac{c_1}{\omega}\\ \overline{f_2} + \dfrac{c_2}{\omega}\end{pmatrix}, \left\{\begin{pmatrix}0\\ 0\end{pmatrix}\right\}_{k=1}^{q}\right),$$

$$K_p(I-Q)N\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}\int_0^t f_1(s)\,ds + \sum_{0<t_k<t}\ln(1+c_{1k})\\ \int_0^t f_2(s)\,ds + \sum_{0<t_k<t}\ln(1+c_{2k})\end{pmatrix} + \left(\frac{1}{2}-\frac{t}{\omega}\right)\begin{pmatrix}\int_0^\omega f_1(t)\,dt + c_1\\ \int_0^\omega f_2(t)\,dt + c_2\end{pmatrix} - \frac{1}{\omega}\begin{pmatrix}\int_0^\omega\int_0^t f_1(s)\,ds\,dt + c_1\\ \int_0^\omega\int_0^t f_2(s)\,ds\,dt + c_2\end{pmatrix}. \tag{18}$$

Clearly, QN and $K_p(I-Q)N$ are continuous. By using the Arzela-Ascoli theorem (see [10]), it is not difficult to prove that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned}\dot{u}_1(t) &= \lambda\Big[r(t) - \frac{r(t)}{K}\exp(u_1(t)) - \Big(\big(4Ge\delta\exp(u_2(t)) + (4Gh-1)\times e^2\delta^2\exp(u_1(t)+u_2(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\Big],\\ \dot{u}_2(t) &= \lambda\Big[-d(t) + \Big(\big(4Ge\delta B(t)\exp(u_1(t)) + (4Gh-1)\times e^2\delta^2 B(t)\exp(2u_1(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\Big],\\ &\qquad t\neq t_k,\\ \Delta u_1(t_k) &= \lambda\ln(1+c_{1k}),\\ \Delta u_2(t_k) &= \lambda\ln(1+c_{2k}),\quad t=t_k.\end{aligned} \tag{19}$$

Suppose $x = (u_1(t), u_2(t))^T \in X$ is a solution of (19) for a certain $\lambda \in (0,1)$. Integrating the first equation of (19) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^\omega \frac{r(t)}{K}\exp(u_1(t))\,dt + \int_0^\omega\Big(\big(4Ge\delta\exp(u_2(t)) + (4Gh-1)\times e^2\delta^2\exp(u_1(t)+u_2(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\,dt\\ &= \overline{r}\omega + c_1.\end{aligned} \tag{20}$$

Similarly, integrating the second equation of (19) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^\omega\Big(\big(4Ge\delta B(t)\exp(u_1(t)) + (4Gh-1)\times e^2\delta^2 B(t)\exp(2u_1(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\,dt\\ &= \overline{d}\omega - c_2.\end{aligned} \tag{21}$$

It follows from the first equation of (19) and (20) and $(\overline{A}_2)$ that

$$\begin{aligned}&\int_0^\omega |\dot{u}_1(t)|\,dt\\ &= \lambda\int_0^\omega\Big|\Big[r(t) - \frac{r(t)}{K}\exp(u_1(t)) - \Big(\big(4Ge\delta\exp(u_2(t)) + (4Gh-1)\times e^2\delta^2\exp(u_1(t)+u_2(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\Big]\Big|\,dt\\ &< \int_0^\omega r(t)\,dt + \int_0^\omega\frac{r(t)}{K}\exp(u_1(t))\,dt + \int_0^\omega\Big(\big(4Ge\delta\exp(u_2(t)) + (4Gh-1)\times e^2\delta^2\exp(u_1(t)+u_2(t))\big)\times\big(4G(1+he\delta\exp(u_1(t)))^2\big)^{-1}\Big)\,dt\\ &= \int_0^\omega r(t)\,dt + \overline{r}\omega + c_1\\ &= 2\overline{r}\omega + c_1.\end{aligned} \tag{22}$$

That is,

$$\int_0^\omega |\dot{u}_1(t)|\,dt < 2\overline{r}\omega + c_1. \tag{23}$$

Similarly, it follows from the second equation of (19) and (21) and $(\overline{A}_2)$ that

$$\int_0^\omega \left|\dot{u}_2(t)\right| dt < 2\overline{d}\omega - c_2. \tag{24}$$

Since $(u_1(t), u_2(t))^T \in X$, there exists $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \inf_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \sup_{t\in[0,\omega]} u_i(t), \quad i = 1, 2. \tag{25}$$

From (20), we see that

$$\overline{r}\omega + c_1 \geq \int_0^\omega \frac{r(t)}{K} \exp(u_1(t))\, dt, \tag{26}$$

which implies

$$\overline{r} + \frac{c_1}{\omega} \geq \frac{\overline{r}}{K} \exp(u_1(\xi_1)). \tag{27}$$

So

$$u_1(\xi_1) \leq \ln K\left(1 + \frac{c_1}{\overline{r}\omega}\right) := \Delta_{11}. \tag{28}$$

This, combined with (23), gives

$$u_1(t) \leq u_1(\xi_1) + \int_0^\omega \left|\dot{u}_1(t)\right| dt < \Delta_{11} + 2\overline{r}\omega := H_{11}. \tag{29}$$

Similarly, it follows from (21) that

$$\begin{aligned}\overline{d}\omega - c_2 \leq &\int_0^\omega \frac{4Ge\delta B(t)\exp(u_1(t))}{8Ghe\delta\exp(u_1(t))} dt \\ &+ \int_0^\omega \frac{4Ghe^2\delta^2 B(t)\exp(2u_1(t))}{8Ghe\delta\exp(u_1(t))} dt,\end{aligned} \tag{30}$$

which implies

$$\overline{d} - \frac{c_2}{\omega} \leq \frac{\overline{B}}{2h} + \frac{e\delta\overline{B}}{2}\exp(u_1(\eta_1)). \tag{31}$$

It follows from $(\overline{A}_1)$ that

$$u_1(\eta_1) \geq \ln \frac{22\left(\overline{d} - \left(\overline{B}/2h\right) - (c_2/\omega)\right)}{e\delta\overline{B}} := \Delta_{12}. \tag{32}$$

This, combined with (23), gives

$$u_1(t) \geq u_1(\eta_1) - \int_0^\omega \left|\dot{u}_1(t)\right| dt > \Delta_{12} - 2\overline{r}\omega := H_{12}. \tag{33}$$

It follows from (29) and (33) that

$$\max_{t\in[0,\omega]} u_1(t) < \max\{|H_{11}|, |H_{12}|\} := H_1. \tag{34}$$

On the other hand, it follows from (21) and (34) that

$$\begin{aligned}\overline{r}\omega + c_1 \leq &\frac{\overline{r}\omega}{K}\exp(H_1) + \int_0^\omega \frac{4Ge\delta\exp(u_2(t))}{4G} dt \\ &+ \int_0^\omega \frac{4Ghe^2\delta^2\exp(u_1(t) + u_2(t))}{8Ghe\delta\exp(u_1(t))} dt,\end{aligned} \tag{35}$$

which implies

$$\overline{r} + \frac{c_1}{\omega} \leq \frac{\overline{r}}{K}\exp(H_1) + \frac{3}{2}e\delta\exp(u_2(\eta_2)). \tag{36}$$

It follows from $(\overline{A}_2)$ that

$$u_2(\eta_2) \geq \ln \frac{2\overline{r}\left(1 + (c_1/\overline{r}\omega) - \left(e^{H_1}/K\right)\right)}{3e\delta} := \Delta_{22}. \tag{37}$$

This, combined with (25), gives

$$u_2(t) \geq u_2(\eta_2) - \int_0^\omega \left|\dot{u}_2(t)\right| dt > \Delta_{22} - 2\overline{d}\omega := H_{22}. \tag{38}$$

Similarly, it follows from (21) and (34) that

$$\overline{r}\omega + c_1 \geq \int_0^\omega \frac{4Ge\delta\exp(u_2(\xi_2))}{4G\left(1 + he\delta\exp(H_1)\right)^2} dt, \tag{39}$$

which implies

$$\overline{r} + \frac{c_1}{\omega} \geq \frac{e\delta\exp(u_2(\xi_2))}{\left(1 + he\delta\exp(H_1)\right)^2}. \tag{40}$$

So

$$u_2(\xi_2) \leq \ln \frac{\overline{r}\left(1 + (c_1/\overline{r}\omega)\right)\left(1 + he\delta\exp(H_1)\right)^2}{e\delta} := \Delta_{21}. \tag{41}$$

This, combined with (25), gives

$$u_2(t) \leq u_2(\xi_2) + \int_0^\omega \left|\dot{u}_2(t)\right| dt < \Delta_{21} + 2\overline{d}\omega := H_{21}. \tag{42}$$

It follows from (38) and (42) that

$$\max_{t\in[0,\omega]} u_2(t) < \max\{|H_{21}|, |H_{22}|\} := H_2. \tag{43}$$

Now, let us consider QNx with $x = (u_1, u_2)^T \in \mathbb{R}^2$. Note that

$$\begin{aligned}&QN(u_1, u_2) \\ &= \Bigg[\overline{r} - \frac{\overline{r}}{K}\exp(u_1) \\ &\quad - \frac{4Ge\delta\exp(u_2) + (4Gh - 1)e^2\delta^2\exp(u_1 + u_2)}{4G\left(1 + he\delta\exp(u_1)\right)^2} + \frac{c_1}{\omega}, \\ &\quad - \overline{d} + \frac{4Ge\delta\overline{B}\exp(u_1) + (4Gh - 1)e^2\delta^2\overline{B}\exp(2u_1)}{4G\left(1 + he\delta\exp(u_1)\right)^2} + \frac{c_2}{\omega}\Bigg].\end{aligned} \tag{44}$$

It follows from $(\overline{A}_1)$, $(\overline{A}_2)$, and $(\overline{A}_3)$ that $u_+ < 0$, which implies that the equation $QN(u_1, u_2) = 0$ has only one solution

$$\widetilde{u} = \left(\ln u_-, \ln \frac{4G\left(\overline{r} - (\overline{r}/K)u_-\right)\left(1 + he\delta u_-\right)^2}{4Ge\delta + (4Gh - 1)e^2\delta^2 u_-}\right). \tag{45}$$

Choose $C > 0$ such that

$$C > \left| \ln \frac{4G\left(\bar{r} - (\bar{r}/K)\, u_-\right)\left(1 + he\delta u_-\right)^2}{4Ge\delta + (4Gh - 1)\, e^2\delta^2 u_-} \right|. \quad (46)$$

Set $H = \|(H_1, H_2)^T\| + C$; then $\|x\| < H$. Let

$$\Omega = \left\{ x(t) = \left(u_1(t), u_2(t)\right)^T \in X : \|x(t)\| < H \right\}. \quad (47)$$

It is clear that Ω verifies the requirement (a) in Lemma 4. When $x \in \operatorname{Ker} L \cap \partial\Omega$, x is a constant with $\|x\| = H$. Then $QNx \neq 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega$. Simple computation shows that $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$. Here, J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. □

By now, we have proved that Ω verifies all the requirements in Lemma 4. Hence, (2) has at least one ω-periodic solution in $\overline{\Omega}$.

Acknowledgment

This work is supported by NNSFC and the Natural Science Foundation of Fujiang Province (2013J01010).

References

[1] L. Chen, *Mathematical Models and Methods in Ecology*, Science Press, Beijing, China, 1998, (in Chinese).

[2] Y. Li, "Periodic solutions of a periodic delay predator-prey system," *Proceedings of the American Mathematical Society*, vol. 127, no. 5, pp. 1331–1335, 1999.

[3] Y. Song, Y. Peng, and J. Wei, "Bifurcations for a predator-prey system with two delays," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 466–479, 2008.

[4] Y.-H. Xia, "Periodic solution of certain nonlinear differential equations: via topological degree theory and matrix spectral theory," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 22, no. 8, Article ID 940287, 2012.

[5] T. Zhang, J. Liu, and Z. Teng, "Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 293–306, 2010.

[6] Z. Zhang and J. Luo, "Multiple periodic solutions of a delayed predator-prey system with stage structure for the predator," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 4109–4120, 2010.

[7] M. Fan, Q. Wang, and X. Zou, "Dynamics of a non-autonomous ratio-dependent predator-prey system," *Proceedings of the Royal Society of Edinburgh A*, vol. 133, no. 1, pp. 97–118, 2003.

[8] H.-F. Huo, "Periodic solutions for a semi-ratio-dependent predator-prey system with functional responses," *Applied Mathematics Letters*, vol. 18, no. 3, pp. 313–320, 2005.

[9] Y.-F. Gao and Y.-H. Xia, "Periodic solutions of a nonautonomous plant-hare model," *Journal of Zhejiang University*, vol. 39, no. 5, pp. 507–511, 2012.

[10] D. Baĭnov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, vol. 66 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1993.

[11] Q. Wang, B. Dai, and Y. Chen, "Multiple periodic solutions of an impulsive predator-prey model with Holling-type IV functional response," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1829–1836, 2009.

[12] L. Mahto, S. Abbas, and A. Favini, "Analysis of Caputo impulsive fractional order differential equations with applications," *International Journal of Differential Equations*, vol. 2013, Article ID 704547, 11 pages, 2013.

[13] A. M. A. El-Sayed, E. Ahmed, and H. A. A. El-Saka, "Dynamic properties of the fractional-order logistic equation of complex variables," *Abstract and Applied Analysis*, vol. 2012, Article ID 251715, 12 pages, 2012.

[14] J. Hui and L. S. Chen, "Existence of positive periodic solution of periodic time-dependent predator-prey system with impulsive effects," *Acta Mathematica Sinica (English Series)*, vol. 20, no. 3, pp. 423–432, 2004.

[15] Y. Shao, P. Li, and G. Tang, "Dynamic analysis of an impulsive predator-prey model with disease in prey and Ivlev-type functional response," *Abstract and Applied Analysis*, vol. 2012, Article ID 750530, 20 pages, 2012.

[16] J. O. Alzabut and T. Abdeljawad, "Exponential boundedness for solutions of linear impulsive differential equations with distributed delay," *International Journal of Pure and Applied Mathematics*, vol. 34, no. 2, pp. 203–217, 2007.

[17] J. O. Alzabut and T. Abdeljawad, "On existence of a globally attractive periodic solution of impulsive delay logarithmic population model," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 463–469, 2008.

[18] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, vol. 568 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.