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Research Article

Uniqueness Theorems of Difference Operator on Entire Functions

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We investigate the uniqueness questions of the difference operator on entire functions and obtain three uniqueness theorems using the idea of weight sharing.

1. Introduction

A function $f(z)$ is called meromorphic, if it is analytic in the complex plane except at poles. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory such as the characteristic function $T(r, f)$, and proximity function $m(r, f)$, counting function $N(r, f)$ (see [1, 2]). In addition we use $S(r, f)$ denotes any quantity that satisfies the condition: $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside an exceptional set of finite logarithmic measure.

Let f and g be two nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a IM (ignoring multiplicities) if $f-a$ and $g-a$ have the same zeros, they share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities. When $a = \infty$ the zeros of $f - a$ mean the poles of f (see [2]).

Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}(r, 1/(f - a))$ to denote the counting function of the zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not bigger than p, $N_{(p+1}(r, 1/(f - a))$ to denote the counting function of the zeros of $f - a$ whose multiplicities are not less than $p + 1$. $\overline{N}_{p)}(r, 1/(f - a))$ and $\overline{N}_{(p+1}(r, 1/(f - a))$ denote their corresponding reduced counting functions (ignoring multiplicities), respectively. We denote by $E(a, f)$ the set of zeros of $f - a$ with multiplicity, $E_{p)}(a, f)$ the set of zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not greater than p.

In 1997, Yang and Hua (see [3]) studied the uniqueness of the differential monomials and obtained the following result.

Theorem A. *Let f and g be nonconstant entire functions, and let $n \geq 3$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 \exp^{cz}$, $g(z) = c_2 \exp^{-cz}$, where c_1, c_2, and c are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant such that $t^{n+1} = 1$.*

Recently, a number of papers (including [1, 3–17]) have focused on complex difference equations and differences analogues of Nevanlinna theory.

In particular, Qi et al. (see [16]) proved Theorem B, which can be considered as a difference counterpart of Theorem A.

Theorem B. *Let f and g be transcendental entire functions with finite order, c be a nonzero complex constant and $n \geq 6$ be an integer. If $f^n f(z + c)$ and $g^n g(z + c)$ share 1 CM, then $fg = t_1$ or $f = t_2 g$ for some constant t_1 and t_2 which satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2011, Zhang et al. (see [17]) investigated the distribution of zeros and shared values of the difference operator on meromorphic functions and uniqueness of difference polynomials with the same 1 points or fixed points. They obtained the following results.

Theorem C. *Let f and g be nonconstant entire functions of finite order, and let $n \geq 5$ be an integer. Suppose that c is a*

nonzero complex constant such that $f(z+c)-f(z)\not\equiv 0$ and $g(z+c)-g(z)\not\equiv 0$. If $f^n[f(z+c)-f(z)]$ and $g^n[g(z+c)-g(z)]$ share 1 CM, and $g(z+c)$ and $g(z)$ share 0 CM, then $f(z)=c_1\exp^{az}$ and $g(z)=c_2\exp^{-az}$, where c_1, c_2, and a are constants satisfying $(c_1c_2)^{n+1}(\exp^{az}+\exp^{-az}-2)=-1$ or $f=tg$ for a constant such that $t^{n+1}=1$.

Theorem D. *Let f and g be nonconstant entire functions of finite order, and let $n\geq 5$ be an integer. Suppose that c is a nonzero complex constant such that $f(z+c)-f(z)\not\equiv 0$ and $g(z+c)-g(z)\not\equiv 0$. If $f^n[f(z+c)-f(z)]$ and $g^n[g(z+c)-g(z)]$ share z CM, and $g(z+c)$ and $g(z)$ share 0 CM, then $f=tg$, where t is a constant satisfying $t^{n+1}=1$.*

We investigate the uniqueness theorem of another differences polynomial and prove Theorem 1.

Theorem 1. *Let f and g be nonconstant transcendental entire functions of finite order, and let $n\geq 9$ be an integer. Suppose that c is a nonzero real constant such that $f(z+2c)+f(z+c)+f(z)\neq g(z+2c)+g(z+c)+g(z)$ and $f(z+2c)+f(z+c)+f(z)\not\equiv 0$, $g(z+2c)+g(z+c)+g(z)\not\equiv 0$. If $E_{3)}(z,f^n[f(z+2c)+f(z+c)+f(z)])=E_{3)}(z,g^n[g(z+2c)+g(z+c)+g(z)])$, then $f(z)=tg(z)$, where t is a constant satisfying $t^{n+1}=1$ except that $t=1$.*

In paper [15], Wang et al. improved the Theorem B and proved the following result.

Theorem E. *Let f and g be transcendental entire functions with finite order, c be a nonzero complex constant and $n\geq 6$ be an integer. If $E_{3)}(1,f^nf(z+c))=E_{3)}(1,g^ng(z+c))$, then $fg=t_1$ or $f=t_2g$ for some constant t_1 and t_2 which satisfy $t_1^{n+1}=1$ and $t_2^{n+1}=1$.*

The purpose of this paper is to induce the idea of weight sharing to Theorems C and D, the results as follow.

Theorem 2. *Let f and g be nonconstant entire functions of finite order, and let $n\geq 6$ be an integer. Suppose that c is a nonzero complex constant such that $f(z+c)-f(z)\not\equiv 0$ and $g(z+c)-g(z)\not\equiv 0$. If $E_{3)}(1,f^n[f(z+c)-f(z)])=E_{3)}(1,g^n[g(z+c)-g(z)])$ share 1 CM, $g(z+c)$ and $g(z)$ share 0 CM, then $f(z)=c_1\exp^{az}$ and $g(z)=c_2\exp^{-az}$, where c_1, c_2, and a are constants satisfying $(c_1c_2)^{n+1}(\exp^{az}+\exp^{-az}-2)=-1$ or $f=tg$ for a constant such that $t^{n+1}=1$.*

Theorem 3. *Let f and g be nonconstant entire functions of finite order, and let $n\geq 6$ be an integer. Suppose that c is a non-zero complex constant such that $f(z+c)-f(z)\not\equiv 0$ and $g(z+c)-g(z)\not\equiv 0$. If $E_{3)}(1,f^n[f(z+c)-f(z)])=E_{3)}(1,g^n[g(z+c)-g(z)])$ share z CM, $g(z+c)$ and $g(z)$ share 0 CM, then $f=tg$, where t is a constant satisfying $t^{n+1}=1$.*

Remark 4. Some ideas of this paper are based on [15, 17].

2. Some Lemmas

In order to prove our theorems, we need the following Lemmas.

Lemma 5 is a difference analogue of the logarithmic derivative lemma, given by Halburd and Korhonen [9] and Chiang and Feng [7] independently.

Lemma 5 (see [9]). *Let $f(z)$ be a meromorphic function of finite order, and let $c\in\mathbb{C}$ and $\delta\in(0,1)$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O\left(\frac{T(r,f)}{r^{\delta}}\right)=S(r,f), \tag{1}$$

for all r outside of a possibly exceptional set with finite logarithmic measure.

Lemma 6 (see [2]). *Let $f(z)$ be a nonconstant meromorphic function, and let $P(f)=a_0f^n+a_1f^{n-1}+\cdots a_n$, where $a_0(\neq 0)$, $a_1,\ldots,a_n$ are small functions of f. Then*

$$T(r,P(f))=nT(r,f)+S(r,f). \tag{2}$$

Lemma 7 (see [18]). *Let f and g be two nonconstant meromorphic function satisfying $E_{k)}(1,f)=E_{k)}(1,g)$ for some positive integer $k\in\mathbb{N}$. Define H as follow*

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right). \tag{3}$$

If $H\not\equiv 0$, then

$$\begin{aligned}N(r,H)&\leq \overline{N}_{(2}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\&\quad+\overline{N}_{(2}(r,g)+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+\overline{N}_0\left(r,\frac{1}{f'}\right)\\&\quad+\overline{N}_0\left(r,\frac{1}{g'}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right)\\&\quad+\overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g),\end{aligned} \tag{4}$$

where $\overline{N}_0(r,1/f')$ denotes the counting function of zeros of f' but not zeros of $f(f-1)$, and $\overline{N}_0(r,1/g')$ is similarly defined.

Lemma 8 (see [19]). *Under the conditions of Lemma 7, we have*

$$\begin{aligned}N_{1)}\left(r,\frac{1}{f-1}\right)&=N_{1)}\left(r,\frac{1}{g-1}\right)\\&\leq N(r,H)+S(r,f)+S(r,g).\end{aligned} \tag{5}$$

Lemma 9 (see [19]). *If* $H = (f''/f' - 2f'/(f-1)) - (g''/g' - 2g'/(g-1)) \equiv 0$, *then either* $f \equiv g$ *or* $fg \equiv 1$ *provided that*

$$\limsup_{r\to\infty, r\in I} \frac{\overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,1/f) + \overline{N}(r,1/g)}{T(r)} \le 1, \tag{6}$$

where $T(r) := \max\{T(r,f), T(r,g)\}$ *and* I *is a set with infinite linear measure.*

Lemma 10. *Let* $f(z)$ *be a meromorphic function of finite order,* $c \in \mathbb{C}$. *Then*

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)). \tag{7}$$

Proof. Using Lemma 5 and the formula (12) in [12]

$$N(r, f(z+c)) \le N(r, f(z)) + S(r, f(z)). \tag{8}$$

Replacing $f(z)$ with $f(z-c)$, we have

$$\begin{aligned} N(r, f(z)) &\le N(r, f(z-c)) + S(r, f(z-c)) \\ &= N(r, f(z-c)) + S(r, f(z)), \end{aligned} \tag{9}$$

for every $c \in \mathbb{C}$, so we deduce that

$$N(r, f(z)) \le N(r, f(z+c)) + S(r, f(z)). \tag{10}$$

From (8) and (10), we obtain that

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)). \tag{11}$$

Thus we completed the proof. □

Lemma 11 (see [9]). *Let* $T : (0,+\infty) \to (0,+\infty)$ *be a nondecreasing continuous function,* $s > 0$, $0 < \alpha < 1$, *and let* $F \subset R^+$ *be the set of all* r *satisfy*

$$T(r) \le \alpha T(r+s). \tag{12}$$

If the logarithmic measure of F *is infinite, then*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \infty. \tag{13}$$

3. Proof of Theorems

Proof of Theorem 1. We define

$$\begin{aligned} F(z) &= \frac{f^n(z)\left[f(z+2c) + f(z+c) + f(z)\right]}{z}, \\ G(z) &= \frac{g^n(z)\left[g(z+2c) + g(z+c) + g(z)\right]}{z}. \end{aligned} \tag{14}$$

In Lemma 7, we replace f and g, by F and G respectively, we claim that $H \equiv 0$. If it is not true, then $H \not\equiv 0$. From Lemma 8 we have that

$$\begin{aligned} &rlN_{1)}\left(r, \frac{1}{F-1}\right) \\ &\quad = N_{1)}\left(r, \frac{1}{G-1}\right) \le N(r,H) \\ &\qquad + S(r,f) + S(r,g) \\ &\quad \le \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) \\ &\qquad + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) \\ &\qquad + \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{G-1}\right) \\ &\qquad + S(r,f) + S(r,g). \end{aligned} \tag{15}$$

From the Nevanlinna second foundational theorem, we can get that

$$\begin{aligned} &T(r,F) + T(r,G) \\ &\quad \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\qquad + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) \\ &\qquad + \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ &\qquad + S(r,f) + S(r,g). \end{aligned} \tag{16}$$

From the definitions of $N_{k)}$ and $N_{(k}$, the following inequalities are obvious:

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F-1}\right) - \frac{1}{2}N_{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) \\ &\quad \le \frac{1}{2}N\left(r, \frac{1}{F-1}\right), \\ &\overline{N}\left(r, \frac{1}{G-1}\right) - \frac{1}{2}N_{1)}\left(r, \frac{1}{G-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{G-1}\right) \\ &\quad \le \frac{1}{2}N\left(r, \frac{1}{G-1}\right). \end{aligned} \tag{17}$$

Combining (15), (16), and (17), we deduce that

$$\begin{aligned} &T(r,F) + T(r,G) \\ &\quad \le 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{F}\right) \end{aligned}$$

$$
\begin{aligned}
&+ 2\overline{N}\left(r, \frac{1}{G}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g) \\
&\leq 4\overline{N}\left(r, \frac{1}{f}\right) + 4\overline{N}\left(r, \frac{1}{g}\right) + 2N\left(r, \frac{1}{f(z+c)}\right) \\
&+ 2N\left(r, \frac{1}{g(z+c)}\right) \\
&+ 2N\left(r, \frac{1}{f(z+2c)}\right) + 2N\left(r, \frac{1}{g(z+2c)}\right) \\
&+ S(r, f) + S(r, g) \\
&\leq 8N\left(r, \frac{1}{f}\right) + 8N\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g) \\
&\leq 8T\left(r, \frac{1}{f}\right) + 8T\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g).
\end{aligned} \tag{18}
$$

We can apply Lemma 5, Lemma 6, and Lemma 10 to show that

$$
\begin{aligned}
&(n+1)T(r, f) \\
&= T\left(r, f^{n+1}\right) \leq T(r, F(z)) \\
&\quad + T\left(r, \frac{f(z+2) + f(z+1) + f(z)}{f(z)}\right) + S(r, f(z)) \\
&= T(r, F(z)) + N\left(r, \frac{f(z+2) + f(z+1) + f(z)}{f(z)}\right) \\
&\quad + S(r, f(z)) \\
&\leq T(r, F(z)) + T(r, f(z)) + S(r, f(z)),
\end{aligned} \tag{19}
$$

which implies

$$T(r, F) \geq nT(r, f) + S(r, f). \tag{20}$$

The same augment as above, we have that

$$T(r, G) \geq nT(r, g) + S(r, g). \tag{21}$$

From (18), (20), and (21), we can deduce that

$$(n-8)\left[T(r, f) + T(r, g)\right] \geq S(r, f) + S(r, g). \tag{22}$$

which is a contraction. Therefore, $H \equiv 0$.

Noting that

$$
\begin{aligned}
&\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\
&\quad \leq 4\left[T(r, f) + T(r, g)\right] \\
&\quad\quad + S(r, f) + S(r, g) \leq T(r),
\end{aligned} \tag{23}
$$

where $T(r) = \max T(r, F), T(r, G)$.

Because of Lemma 9, we have that $F \equiv G$ or $FG \equiv 1$. We will consider the following two cases.

Case 1. Suppose that $F(z) = G(z)$. Then

$$
\begin{aligned}
&f^n(z)\left[f(z+2c) + f(z+c) + f(z)\right] \\
&\quad = g^n(z)\left[g(z+2c) + g(z+c) + g(z)\right].
\end{aligned} \tag{24}
$$

Let $h(z) = f(z)/g(z)$, we deduce that

$$
\begin{aligned}
&h^n(z)\left[h(z+2c)\, g(z+2c)\right. \\
&\quad \left. + h(z+c)\, g(z+c) + h(z)\, g(z)\right] \\
&\quad = g(z+2c) + g(z+c) + g(z).
\end{aligned} \tag{25}
$$

If $h(z+c) \not\equiv h(z)$, by the hypothesis $f(z+2c) + f(z+c) + f(z) \neq g(z+2c) + g(z+c) + g(z)$, we get that $h(z) \neq 1$. So

$$
\begin{aligned}
&T(r, h^n) \leq \overline{N}\left(r, \frac{1}{h^n}\right) + \overline{N}(r, h^n) \\
&\quad + \overline{N}\left(r, \frac{1}{h^n - 1}\right) \leq 2T(r, h) + S(r, h),
\end{aligned} \tag{26}
$$

which means h is a constant, because of $n \geq 10$.

Then $h(z) = t$ and t is a constant satisfying $t^{n+1} = 1$ except that $t = 1$.

Case 2. Suppose that $F(z) \cdot G(z) \equiv 1$. Then

$$
\begin{aligned}
&f^n(z)\left[f(z+2c) + f(z+c) + f(z)\right] \\
&\quad \times g^n(z)\left[g(z+2c) + g(z+c) + g(z)\right] = z^2.
\end{aligned} \tag{27}
$$

Note that zero is a Picard exceptional value of f and g, then $f(z) = e^{P(z)}$ and $g(z) = e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. In (27), we let $z = 0$, then

$$e^{nP(0)}\left[e^{P(2c)} + e^{P(c)} + e^{P(0)}\right] e^{nQ(0)}\left[e^{Q(2c)} + e^{Q(c)} + e^{Q(0)}\right] = 0. \tag{28}$$

It is impossible, because of c is a real number. □

Proof of Theorem 2. Denoting

$$
\begin{aligned}
F(z) &= f^n(z)\left[f(z+c) - f(z)\right], \\
G(z) &= g^n(z)\left[g(z+c) - g(z)\right].
\end{aligned} \tag{29}
$$

In Lemma 7, we replace f and g, by F and G respectively. If $H \not\equiv 0$, by Lemma 8 we deduce that

$$
\begin{aligned}
&N_{1)}\left(r, \frac{1}{F-1}\right) \\
&\quad = N_{1)}\left(r, \frac{1}{G-1}\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq N(r,H)+S(r,f)+S(r,g)\\
&\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&\quad+\overline{N}_{0}\left(r,\frac{1}{F'}\right)+\overline{N}_{0}\left(r,\frac{1}{G'}\right)\\
&\quad+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)\\
&\quad+S(r,f)+S(r,g).
\end{aligned}\tag{30}$$

The same reasons as in the proof of Theorem 1, we have that

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\
&\qquad+\overline{N_0}\left(r,\frac{1}{F'}\right)+\overline{N_0}\left(r,\frac{1}{G'}\right)\\
&\qquad+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\
&\qquad+S(r,f)+S(r,g),\\
&\overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)\\
&\qquad+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right),\\
&\overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)\\
&\qquad+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{G-1}\right).
\end{aligned}\tag{31}$$

Combining (30) and (31), we deduce that

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\leq 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}_{(2}\left(r,\frac{1}{F}\right)\\
&\qquad+2\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}_{(2}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g)\\
&\quad\leq 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{f(z+c)-f(z)}\right)\\
&\qquad+2N\left(r,\frac{1}{g(z+c)-g(z)}\right)+S(r,f)+S(r,g)\\
&\quad\leq 4\overline{N}\left(r,\frac{1}{f}\right)+2T(r,f(z+c)-f(z))\\
&\qquad+6N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
&\quad=4\overline{N}\left(r,\frac{1}{f}\right)+2m(r,f(z+c)-f(z))\\
&\qquad+6N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
&\quad\leq 6T\left(r,\frac{1}{f}\right)+6T\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{32}$$

We can apply Lemmas 5, 6, and 10 to show that

$$\begin{aligned}
&(n+1)T(r,f)\\
&\quad=T\left(r,f^{n+1}\right)\leq T(r,F(z))\\
&\qquad+T\left(r,\frac{f(z+c)-f(z)}{f(z)}\right)+S(r,f(z))\\
&\quad=T(r,F(z))+N\left(r,\frac{f(z+c)-f(z)}{f(z)}\right)+S(r,f(z))\\
&\quad\leq T(r,F(z))+T(r,f(z))+S(r,f(z)),
\end{aligned}\tag{33}$$

which implies

$$T(r,F)\geq nT(r,f)+S(r,f).\tag{34}$$

We have that $N(r,(g(z+c)-g(z))/g(z))$, Since $g(z+c)$ and $g(z)$ share 0 CM, then

$$T(r,G)\geq(n+1)T(r,g)+S(r,g).\tag{35}$$

From (32), (34), and (35), we can deduce that

$$\begin{aligned}
&(n-6)T(r,f)+(n-5)T(r,g)\\
&\quad\geq S(r,f)+S(r,g).
\end{aligned}\tag{36}$$

which is a contraction. Therefore, $H\equiv 0$.

Noting that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\\
&\quad\leq 3\left[T(r,f)+T(r,g)\right]+S(r,f)\\
&\qquad+S(r,g)\leq T(r),
\end{aligned}\tag{37}$$

where $T(r)=\max T(r,F),T(r,G)$.

Because of Lemma 9, we have that $F\equiv G$ or $FG\equiv 1$.

By using the same methods as in the proof of Theorem 1.10 in [17], we can complete the proof of Theorem 2. □

Proof of Theorem 3. The proof is almost literally the same as the proof of Theorem 2, with the methods as in the proof of Theorem 1.9 in [17] replacing the methods as in the proof of Theorem 1.10 in [17]. □

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