Research Article

Periodic Solutions for Gause-Type Ratio-Dependent Predator-Prey Systems with Delays on Time Scales

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This paper is devoted to periodic Gause-type ratio-dependent predator-prey systems with monotonic or nonmonotonic numerical responses on time scales. By using a continuation theorem based on coincidence degree theory, we establish easily verifiable criteria for the existence of periodic solutions. In particular, our results improve and generalize some known ones.

1. Introduction

In this paper, we consider the following periodic Gause-type ratio-dependent predator-prey system with time delays on a time scale $\mathbb{T}$:

$$\begin{aligned} u_1^\Delta(t) &= f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)}), \\ u_2^\Delta(t) &= -d(t) + h\left(t, e^{u_1(t-\tau(t))-u_2(t-\tau(t))}\right). \end{aligned} \quad (1)$$

Here $\mathbb{T}$ is a periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. The symbol Δ stands for the delta-derivative which gives the ordinary derivative if $\mathbb{T} = \mathbb{R}$, and the forward difference operator if $\mathbb{T} = \mathbb{Z}$.

The theory of dynamic equations on time scales, which was introduced in order to unify differential and difference equations, has recently received a lot of attention, and many results on the issue have been documented in monographs [1, 2] and papers [3–12].

In system (1), set $x(t) = \exp[u_1(t)]$, $y(t) = \exp[u_2(t)]$. If $\mathbb{T} = \mathbb{R}$, then system (1) reduces to the standard Gause-type ratio-dependent predator-prey system governed by the ordinary differential equations

$$\begin{aligned} x'(t) &= x(t) f(t, x(t-\sigma(t))) - y(t) g\left(t, \frac{x(t)}{y(t)}\right), \\ y'(t) &= y(t) \left[-d(t) + h\left(t, \frac{x(t-\tau(t))}{y(t-\tau(t))}\right)\right], \end{aligned} \quad (2)$$

where $x(t)$ and $y(t)$ stand for the population of the prey and the predator, respectively. The function $f(t, v)$ is the growth rate of the prey in the absence of the predator. The function $d(t)$ is the death rate of the predator. The function $g(t, v) =: vp(t, v)$, called functional response of predator to prey, describes the change in the rate of exploitation of prey by a predator as a result of a change in the prey density. The function $h(t, v)$, called numerical response of predator to prey, describes the change in reproduction rate with changing prey density. If $\mathbb{T} = \mathbb{Z}$, then system (1) is reformulated as

$$\begin{aligned} x(k+1) &= x(k) \exp\left[f(k, x(k-\sigma(k))) - p\left(k, \frac{x(k)}{y(k)}\right)\right], \\ y(k+1) &= y(k) \exp\left[-d(k) + h\left(k, \frac{x(k-\tau(k))}{y(k-\tau(k))}\right)\right], \end{aligned} \quad (3)$$

which is the discrete time ratio-dependent predator-prey system and is a discrete analogue of (2).

We note that Ding [13], Ding and Jiang [14], Fan and Li [15], Fan et al. [16], Fan and Wang [17], Fan et al. [18], Hsu et al. [19], Kuang [20], Wang and Li [21], and Xia and Han [22] considered some special cases of system (2). Fan and Wang [23], and Xia et al. [24] discussed some special cases of system (3). Bohner et al. [4], and Shao [10] investigated some special cases of system (1). So far as we know, there is no published paper concerned system (1).

The main purpose of this paper is, by using the coincidence degree theory developed by Gaines and Mawhin [25],

to derive sufficient conditions for the existence of periodic solutions of system (1). Furthermore, we will see that our results for the above system can be easily extended to the one with distributed or state-dependent delays. Our results improve or generalize some theorems in [10, 13–18, 21, 23].

2. Preliminaries

In this section, we briefly give some elements of the time scale calculus, recall the continuation theorem from coincidence degree theory, and state an auxiliary result that will be used in this paper.

First, let us present some foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, we refer the reader to [1, 2].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Thus, the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, and the natural numbers $\mathbb{N}$ are examples of time scales, while the rational numbers $\mathbb{Q}$ and the open interval $(1, 2)$ are no time scales.

Let $\omega > 0$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be ω -periodic; that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. In particular, the time scale $\mathbb{T}$ under consideration is unbounded above and below.

For $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{R}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad (4)$$

respectively.

If $\sigma(t) = t$, t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{\text{rd}}(\mathbb{T})$.

For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$ we define $f^\Delta(t)$, the delta-derivative of f at t , to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) in $\mathbb{T}$ such that

$$\begin{aligned} & \left| [f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s] \right| \\ & \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \end{aligned} \quad (5)$$

f is said to be delta-differentiable if its delta-derivative exists for all $t \in \mathbb{T}$. The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by $C_{\text{rd}}^1(\mathbb{T})$.

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}$. Then, we define the delta integral by

$$\int_a^b f(t) \Delta t = F(b) - F(a), \quad \forall a, b \in \mathbb{T}. \quad (6)$$

If f is continuous, then f is rd-continuous, and if f is delta-differentiable at t , then f is continuous at t . Moreover, every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative.

Next, let us recall the continuation theorem in coincidence degree theory. To do so, we need to introduce the following notation.

Let X, Y be real Banach spaces, $L : \text{Dom } L \subset X \rightarrow Y$ be a linear mapping, and $N : X \rightarrow Y$ be a continuous mapping.

The mapping L is said to be a Fredholm mapping of index zero, if $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$ and $\text{Im } L$ is closed in Y .

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$, such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$. It follows that the restriction L_P of L to $\text{Dom } L \cap \text{Ker } P : (I - P)X \rightarrow \text{Im } L$ is invertible. Denote the inverse of L_P by K_P .

The mapping N is said to be L -compact on $\overline{\Omega}$, if Ω is an open bounded subset of X , $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact.

Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Here we state the Gaines-Mawhin theorem, which is a main tool in the proof of our main result.

Lemma 1 (Continuation Theorem [25, page 40]). *Let $\Omega \subset X$ be an open bounded set, L a Fredholm mapping of index zero, and N L -compact on $\overline{\Omega}$. Assume*

- (a) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom } L$, $Lx \neq \lambda Nx$;
- (b) for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$;
- (c) $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom } L$.

For convenience and simplicity in the following discussion, we always use the following notation:

$$\begin{aligned} \kappa &= \min\{[0, +\infty) \cap \mathbb{T}\}, & I_\omega &= [\kappa, \kappa + \omega] \cap \mathbb{T}, \\ \hat{a} &= \frac{1}{\omega} \int_\kappa^{\kappa+\omega} a(t) \Delta t, & \hat{A} &= \frac{1}{\omega} \int_\kappa^{\kappa+\omega} |a(t)| \Delta t, \\ \tilde{b}(x) &= \frac{1}{\omega} \int_\kappa^{\kappa+\omega} b(t, x) \Delta t, \\ \tilde{B}(x) &= \frac{1}{\omega} \int_\kappa^{\kappa+\omega} |b(t, x)| \Delta t, \end{aligned} \quad (7)$$

where $a \in C_{\text{rd}}(\mathbb{T})$ is an ω -periodic function, $b : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is rd-continuous and ω -periodic in its first variable.

In order to achieve the prior estimation in the case of dynamic equations on a time scale $\mathbb{T}$, we also require the following inequality which is proved in [11, Theorem 2.1].

Lemma 2. Let $t_1, t_2 \in I_\omega$, and $t \in \mathbb{T}$. If $\varphi \in C_{\text{rd}}^1(\mathbb{T})$ is an ω -periodic real function, then

$$\begin{aligned}\varphi(t) &\leq \varphi(t_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |\varphi^\Delta(t)| \Delta t, \\ \varphi(t) &\geq \varphi(t_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |\varphi^\Delta(t)| \Delta t.\end{aligned}\quad (8)$$

3. Existence of Periodic Solutions

In this section, we study the existence of periodic solutions of system (1). For the sake of generality, we make the following fundamental assumptions for system (1):

- (H₁) $\sigma, \tau : \mathbb{T} \rightarrow \mathbb{R}^+$ are rd-continuous and ω -periodic such that $t - \sigma(t), t - \tau(t) \in \mathbb{T}$ for all $t \in \mathbb{T}$.
- (H₂) $d : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous and ω -periodic such that $\hat{d} > 0$.
- (H₃) $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is rd-continuous and ω -periodic in the first variable and is continuously differentiable in the second variable such that $\tilde{f}(0) > 0$, $\lim_{x \rightarrow +\infty} \tilde{f}(x) < 0$, and $(\partial f / \partial x)(t, x) < 0$ for all $t \in \mathbb{T}, x > 0$.
- (H₄) $p : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}^+$ is rd-continuous and ω -periodic in the first variable and is continuously differentiable in the second variable.
- (H₅) $h : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}^+$ is rd-continuous and ω -periodic in the first variable and is continuously differentiable in the second variable such that $h(t, 0) = 0$ for all $t \in \mathbb{T}$.
- (H₆) $\tilde{f}(0) > \hat{c}$ and $\sup_{x \geq 0} \tilde{h}(x) > \hat{d}$, where $c(t) =: \sup_{x > 0} p(t, x)$ for all $t \in \mathbb{T}$.

From (H₃), we have

$$\tilde{f}'(x) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \frac{\partial f}{\partial x}(t, x) \Delta t < 0. \quad (9)$$

Thus $\tilde{f}(x)$ is strictly decreasing on $[0, +\infty)$. From this, (H₃), and (H₆), one can easily see that each of the equations

$$\tilde{f}(x) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \sup_{x > 0} p(t, x) \Delta t, \quad \tilde{f}(x) = 0, \quad (10)$$

has a unique positive solution. Let V_1, V_2 be their solutions, respectively, then $V_1 < V_2$.

3.1. Monotone Case. In this section, we study the existence of ω -periodic solution of system (1) when the numerical response function $h(t, x)$ satisfies the following monotone condition:

- (H₇) $(\partial h / \partial x)(t, x) > 0$ for $t \in \mathbb{T}, x > 0$.

From (H₅) and (H₇), we have

$$\tilde{h}'(x) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \frac{\partial h}{\partial x}(t, x) \Delta t > 0. \quad (11)$$

Then $\tilde{h}(x)$ is strictly increasing on $[0, +\infty)$. From this, (H₅), and (H₆), one can easily see that the equation $\tilde{h}(x) = \hat{d}$ has a unique positive solution. Let v_0 be its solution.

Theorem 3. Suppose that (H₁)–(H₇) hold. Then system (1) has at least one ω -periodic solution.

Proof. Take

$$\begin{aligned}X = Y &= \{u = (u_1(t), u_2(t))^T \mid u_i \in C(\mathbb{T}), \\ u_i(t + \omega) &= u_i(t), i = 1, 2\},\end{aligned}\quad (12)$$

$$\|u\| = \|(u_1(t), u_2(t))^T\| = \max_{t \in I_\omega} |u_1(t)| + \max_{t \in I_\omega} |u_2(t)|,$$

then X and Y are Banach spaces with the norm $\|\cdot\|$. Set

$$L : \text{Dom } L \subset X \rightarrow Y, \quad L \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} u_1^\Delta(t) \\ u_2^\Delta(t) \end{pmatrix}, \quad (13)$$

where $\text{Dom } L = \{u = (u_1(t), u_2(t))^T \in X \mid u_i \in C_{\text{rd}}^1(\mathbb{T}), i = 1, 2\}$ and

$$\begin{aligned}N : X &\rightarrow Y, \\ N \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} &= \begin{pmatrix} f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)}) \\ -d(t) + h(t, e^{u_1(t-\tau(t))-u_2(t-\tau(t))}) \end{pmatrix}.\end{aligned}\quad (14)$$

With these notations system (1) can be written in the form

$$Lu = Nu, \quad u \in X. \quad (15)$$

Obviously, $\text{Ker } L = \mathbb{R}^2$, $\text{Im } L = \{(u_1(t), u_2(t))^T \in Y : \hat{u}_i = 0, i = 1, 2\}$ is closed in Y , and $\dim \text{Ker } L = \text{codim Im } L = 2$. Therefore L is a Fredholm mapping of index zero. Now define two projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ as

$$P \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = Q \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, \quad \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in X = Y, \quad (16)$$

then P and Q are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im}(I - Q). \quad (17)$$

Furthermore, through an easy computation we find that the generalized inverse K_P of L_P has the form

$$\begin{aligned}K_P : \text{Im } L &\rightarrow \text{Dom } L \cap \text{Ker } P, \\ K_P(u) &= \int_{\kappa}^t u(s) \Delta s - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t u(s) \Delta s \Delta t.\end{aligned}\quad (18)$$

Then $QN : X \rightarrow Y$ and $K_P(I - Q)N : X \rightarrow X$ read as

$$\begin{aligned} QNu &= \frac{1}{\omega} \left(\int_{\kappa}^{\kappa+\omega} [f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)})] \Delta t \right. \\ &\quad \left. - \int_{\kappa}^{\kappa+\omega} [-d(t) + h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))})] \Delta t \right), \\ K_P(I - Q)Nu &= \int_{\kappa}^t Nu(s) \Delta s - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t Nu(s) \Delta s \Delta t \\ &\quad - \frac{1}{\omega} \left(t - \kappa - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (s - \kappa) \Delta s \right) \int_{\kappa}^{\kappa+\omega} Nu(s) \Delta s. \end{aligned} \quad (19)$$

Clearly, QN and $K_P(I - Q)N$ are continuous. By using the Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_P(I - Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Therefore N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

In order to apply Lemma 1, we need to find an appropriate open, bounded subset in X . Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} u_1^\Delta(t) &= \lambda [f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)})], \\ u_2^\Delta(t) &= \lambda [-d(t) + h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))})]. \end{aligned} \quad (20)$$

Suppose that $(u_1(t), u_2(t))^\top \in X$ is a solution of (20) for a certain $\lambda \in (0, 1)$. Integrating (20) on both sides from κ to $\kappa + \omega$ leads to

$$\begin{aligned} \int_{\kappa}^{\kappa+\omega} \lambda [f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)})] \Delta t &= 0, \\ \int_{\kappa}^{\kappa+\omega} \lambda [-d(t) + h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))})] \Delta t &= 0. \end{aligned} \quad (21)$$

That is

$$\int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(t-\sigma(t))}) \Delta t = \int_{\kappa}^{\kappa+\omega} p(t, e^{u_1(t)-u_2(t)}) \Delta t, \quad (22)$$

$$\int_{\kappa}^{\kappa+\omega} h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))}) \Delta t = \int_{\kappa}^{\kappa+\omega} d(t) \Delta t = \hat{d}\omega. \quad (23)$$

From (22), we have

$$\begin{aligned} \tilde{f}(0)\omega &= \int_{\kappa}^{\kappa+\omega} [f(t, 0) - f(t, e^{u_1(t-\sigma(t))}) \\ &\quad + p(t, e^{u_1(t)-u_2(t)})] \Delta t. \end{aligned} \quad (24)$$

It follows from (20), (22), (23), (24), and (H_3) – (H_5) that

$$\begin{aligned} &\int_{\kappa}^{\kappa+\omega} |u_1^\Delta(t)| \Delta t \\ &\leq \lambda \int_{\kappa}^{\kappa+\omega} |f(t, e^{u_1(t-\sigma(t))}) - p(t, e^{u_1(t)-u_2(t)})| \Delta t \\ &< \int_{\kappa}^{\kappa+\omega} \{ [f(t, 0) - f(t, e^{u_1(t-\sigma(t))})] + p(t, e^{u_1(t)-u_2(t)}) \} \Delta t \\ &\quad + \int_{\kappa}^{\kappa+\omega} |f(t, 0)| \Delta t = [\tilde{F}(0) + \tilde{f}(0)]\omega, \end{aligned} \quad (25)$$

$$\begin{aligned} &\int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &\leq \lambda \int_{\kappa}^{\kappa+\omega} |-d(t) + h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))})| \Delta t \\ &< \int_{\kappa}^{\kappa+\omega} |d(t)| \Delta t + \int_{\kappa}^{\kappa+\omega} h(t, e^{u_1(t-\tau(t)) - u_2(t-\tau(t))}) \Delta t \\ &= (\hat{D} + \hat{d})\omega. \end{aligned} \quad (26)$$

Since $(u_1(t), u_2(t))^\top \in X$, there exist $\xi_i, \eta_i \in I_\omega$ such that

$$u_i(\xi_i) = \max_{t \in I_\omega} u_i(t), \quad u_i(\eta_i) = \min_{t \in I_\omega} u_i(t), \quad i = 1, 2. \quad (27)$$

Then from (22), (27), (H_3) , (H_4) , and the monotonicity of function f , we have

$$\begin{aligned} \tilde{f}(e^{u_1(\xi_1)}) &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(\xi_1)}) \Delta t \\ &\leq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(t-\sigma(t))}) \Delta t \\ &\leq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \sup_{x>0} p(t, x) \Delta t, \end{aligned} \quad (28)$$

which, together with the monotonicity of function $\tilde{f}$, leads to

$$u_1(\xi_1) \geq \ln V_1. \quad (29)$$

By Lemma 2, we obtain from (25) and (29) that for all $t \in I_\omega$

$$\begin{aligned} u_1(t) &\geq u_1(\xi_1) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1^\Delta(t)| \Delta t \\ &> \ln V_1 - \frac{\tilde{F}(0) + \tilde{f}(0)}{2} \omega := B_1. \end{aligned} \quad (30)$$

From (22), (27), (H_3) , and the monotonicity of function f , we have

$$\begin{aligned} \tilde{f}(e^{u_1(\eta_1)}) &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(\eta_1)}) \Delta t \\ &\geq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(t-\sigma(t))}) \Delta t > 0, \end{aligned} \quad (31)$$

which, together with the monotonicity of function $\tilde{f}$, leads to

$$u_1(\eta_1) \leq \ln V_2. \quad (32)$$

By Lemma 2, we get from (25) and (32) that for all $t \in I_\omega$

$$\begin{aligned} u_1(t) &\leq u_1(\eta_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1^\Delta(t)| \Delta t \\ &< \ln V_2 + \frac{\tilde{F}(0) + \tilde{f}(0)}{2} \omega := B_2. \end{aligned} \quad (33)$$

From (23), (27), (H_5) , and the monotonicity of function h , we have

$$\begin{aligned} \tilde{h}(e^{u_1(\xi_1)-u_2(\eta_2)}) &\geq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h(t, e^{u_1(t-\tau(t))-u_2(t-\tau(t))}) \Delta t = \hat{d}, \\ \tilde{h}(e^{u_1(\eta_1)-u_2(\xi_2)}) &\leq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h(t, e^{u_1(t-\tau(t))-u_2(t-\tau(t))}) \Delta t = \hat{d}, \end{aligned} \quad (34)$$

which, together with the monotonicity of function $\tilde{h}$, lead to

$$e^{u_1(\xi_1)-u_2(\eta_2)} \geq v_0, \quad e^{u_1(\eta_1)-u_2(\xi_2)} \leq v_0. \quad (35)$$

That is

$$\begin{aligned} u_2(\eta_2) &\leq u_1(\xi_1) - \ln v_0, \\ u_2(\xi_2) &\geq u_1(\eta_1) - \ln v_0. \end{aligned} \quad (36)$$

By Lemma 2, we get from (26), (30), (33), and (36) that for all $t \in I_\omega$

$$\begin{aligned} u_2(t) &\leq u_2(\eta_2) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &< B_1 - \ln v_0 + \frac{\hat{D} + \hat{d}}{2} \omega := B_3, \\ u_2(t) &\geq u_2(\xi_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &> B_2 - \ln v_0 - \frac{\hat{D} + \hat{d}}{2} \omega := B_4. \end{aligned} \quad (37)$$

In view of (30), (33), (37), we obtain

$$\max_{t \in I_\omega} |u_1(t)| \leq \max\{|B_1|, |B_2|\} := B_5, \quad (38)$$

$$\max_{t \in I_\omega} |u_2(t)| \leq \max\{|B_3|, |B_4|\} := B_6. \quad (39)$$

Clearly, B_5 and B_6 are independent of λ .

From (H_3) , (H_6) , and the monotonicity of functions $\tilde{f}$ and $\tilde{h}$, it is easy to show that the algebraic system

$$\begin{aligned} \tilde{f}(e^{u_1}) - \tilde{p}(e^{u_1-u_2}) &= 0, \\ -\hat{d} + \tilde{h}(e^{u_1-u_2}) &= 0, \end{aligned} \quad (40)$$

has a unique solution $(u_1^*, u_2^*)^T \in \mathbb{R}^2$.

We now take $\Omega = \{(u_1(t), u_2(t))^T \in X : \|(u_1(t), u_2(t))^T\| < B_5 + B_6 + B_0\}$, where B_0 is taken sufficiently large such that $\|(u_1^*, u_2^*)^T\| = |u_1^*| + |u_2^*| < B_0$. With the help of (38) and (39), it is easy to see that Ω satisfies condition (a) in Lemma 1. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^2$, $(u_1(t), u_2(t))^T$ is a constant vector in $\mathbb{R}^2$ with $|u_1| + |u_2| = B_5 + B_6 + B_0$. Thus, we have

$$QN \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \tilde{f}(e^{u_1}) - \tilde{p}(e^{u_1-u_2}) \\ -\hat{d} + \tilde{h}(e^{u_1-u_2}) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (41)$$

This proves that condition (b) in Lemma 1 is satisfied.

Taking $J = I : \text{Im } Q \rightarrow \text{Ker } L$, $(u_1, u_2)^T \rightarrow (u_1, u_2)^T$, a direct calculation shows that

$$\begin{aligned} \deg(JQN, \Omega \cap \text{Ker } L, 0) \\ = \text{sign} \left\{ -\tilde{f}'(e^{u_1^*}) \tilde{h}'(e^{u_1^*-u_2^*}) e^{2u_1^*-u_2^*} \right\} \\ = 1 \neq 0. \end{aligned} \quad (42)$$

By now we have proved that Ω satisfies all the requirements in Lemma 1. Hence, (1) has at least one ω -periodic solution. This completes the proof. $\square$

Noticing that both systems (2) and (3) are special cases of system (1), by Theorem 3, we can obtain the following results.

Theorem 4. Suppose that (H_1) – (H_7) hold. Then system (2) has at least one positive ω -periodic solution.

Theorem 5. Suppose that (H_1) – (H_7) hold. Then system (3) has at least one positive ω -periodic solution.

3.2. Nonmonotone Case. In this section, we study the existence of ω -periodic solution of system (1) when the numerical response function $h(t, x)$ satisfies the following nonmonotone condition:

(H_8) $\lim_{v \rightarrow +\infty} h(t, x) = 0$ for $t \in \mathbb{R}$, and there exists a positive constant $\underline{v}$ such that

$$(x - \underline{v}) \frac{\partial h}{\partial x}(t, x) < 0, \quad \text{for } t \in \mathbb{T}, x > 0, x \neq \underline{v}. \quad (43)$$

By (H_5) and (H_8) , we have

$$(x - \underline{v}) \tilde{h}'(x) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (x - \underline{v}) \frac{\partial h}{\partial x}(t, x) \Delta t < 0, \quad (44)$$

$$\lim_{v \rightarrow +\infty} \tilde{h}(x) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \lim_{x \rightarrow +\infty} h(t, x) \Delta t = 0,$$

then $\tilde{h}(x)$ is strictly increasing on $[0, \underline{v}]$ and strictly decreasing on $[\underline{v}, +\infty)$. By these, (H_5) and (H_6) , one can easily see that the equation

$$\tilde{h}(x) = \hat{d}, \quad (45)$$

has two distinct positive solutions, namely, v_1, v_2 . Without loss of generality, we suppose that $v_1 < v_2$, then $v_1 < \underline{v} < v_2$.

Theorem 6. In addition to (H_1) – (H_6) and (H_8) , suppose further that the following condition holds:

$$(H_9) \quad (V_2/V_1)e^{[\tilde{F}(0)+\tilde{f}(0)+\widehat{D}+\widehat{d}]\omega} < v_2/v_1.$$

Then system (1) has at least two ω -periodic solutions.

Proof. The proof is similar to that of Theorem 3. To complete the proof, we need to find two disjoint open bounded subsets in X . Under condition (H_9) , (34) is no longer valid. So, we need to make a corresponding change.

Designating $z(t) = u_1(t) - u_2(t)$, there also exist $\xi_0, \eta_0 \in I_\omega$ such that

$$z(\xi_0) = \max_{t \in I_\omega} z(t), \quad z(\eta_0) = \min_{t \in I_\omega} z(t). \quad (46)$$

From (23), (27), and the monotonicity of functions h and $\tilde{h}$, we will show that $z(\xi_0)$ and $z(\eta_0)$ can not simultaneously lie in $(-\infty, \ln v_1)$, $(\ln v_1, \ln v_2)$ or $(\ln v_2, +\infty)$. In fact, if $z(\eta_0) \leq z(\xi_0) < \ln v_1$, then

$$\begin{aligned} \widehat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h(t, e^{z(t-\tau(t))}) \Delta t \\ &\leq \tilde{h}(e^{z(\xi_0)}) < \tilde{h}(v_1) = \widehat{d}. \end{aligned} \quad (47)$$

This is a contradiction. If $\ln v_2 < z(\eta_0) \leq z(\xi_0)$, then

$$\begin{aligned} \widehat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h(t, e^{z(t-\tau(t))}) \Delta t \\ &\leq \tilde{h}(e^{z(\eta_0)}) < \tilde{h}(v_2) = \widehat{d}. \end{aligned} \quad (48)$$

This is also a contradiction. If $\ln v_1 < z(\eta_0) \leq z(\xi_0) < \ln v_2$, then

$$\begin{aligned} \widehat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h(t, e^{z(t-\tau(t))}) \Delta t \\ &\geq \min\{\tilde{h}(e^{z(\xi_0)}), \tilde{h}(e^{z(\eta_0)})\} \\ &> \tilde{h}(v_1) = \tilde{h}(v_2) = \widehat{d}. \end{aligned} \quad (49)$$

This is also a contradiction. Consequently, the distribution of $z(\xi_0)$ and $z(\eta_0)$ only have the following two cases.

Case 1. $z(\eta_0) \leq \ln v_2 \leq z(\xi_0)$. Noticing that

$$\begin{aligned} z(\eta_0) &\geq u_1(\eta_1) - u_2(\xi_2), \\ z(\xi_0) &\leq u_1(\xi_1) - u_2(\eta_2), \end{aligned} \quad (50)$$

from (27), (30) and (33), we obtain

$$\begin{aligned} u_2(\xi_2) &\geq u_1(\eta_1) - z(\eta_0) \geq u_1(\eta_1) - \ln v_2 > B_1 - \ln v_2, \\ u_2(\eta_2) &\leq u_1(\xi_1) - z(\xi_0) \leq u_1(\xi_1) - \ln v_2 < B_2 - \ln v_2. \end{aligned} \quad (51)$$

By Lemma 2, we find from (26) and (51) that for all $t \in I_\omega$

$$\begin{aligned} u_2(t) &\geq u_2(\xi_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &> B_1 - \ln v_2 - \frac{\widehat{D} + \widehat{d}}{2} \omega := B_7, \\ u_2(t) &\leq u_2(\eta_2) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &< B_2 - \ln v_2 + \frac{\widehat{D} + \widehat{d}}{2} \omega := B_8. \end{aligned} \quad (52)$$

Case 2. $z(\eta_0) \leq \ln v_1 \leq z(\xi_0)$. From (27), (30) and (33), we also obtain

$$\begin{aligned} u_2(\xi_2) &\geq u_1(\eta_1) - z(\eta_0) \geq u_1(\eta_1) - \ln v_1 > B_1 - \ln v_1, \\ u_2(\eta_2) &\leq u_1(\xi_1) - z(\xi_0) \leq u_1(\xi_1) - \ln v_1 < B_2 - \ln v_1. \end{aligned} \quad (53)$$

By Lemma 2, we find from (26) and (53) that for all $t \in I_\omega$

$$\begin{aligned} u_2(t) &\geq u_2(\xi_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &> B_1 - \ln v_1 - \frac{\widehat{D} + \widehat{d}}{2} \omega := B_9, \\ u_2(t) &\leq u_2(\eta_2) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2^\Delta(t)| \Delta t \\ &< B_2 - \ln v_1 + \frac{\widehat{D} + \widehat{d}}{2} \omega := B_{10}. \end{aligned} \quad (54)$$

By (H_9) , we know that

$$B_7 < B_8 < B_9 < B_{10}. \quad (55)$$

Clearly, B_7, B_8, B_9 , and B_{10} are independent of λ .

By the monotonicity of functions $\tilde{f}$ and $\tilde{h}$, (H_3) , (H_5) , and (H_6) , it is easy to show that algebraic system (40) has two distinct solutions:

$$u^{(i)} = (\ln V_0^{(i)}, \ln V_0^{(i)} - \ln v_i)^T, \quad i = 1, 2, \quad (56)$$

where $V_0^{(i)}$ is the unique solution of equation $\tilde{f}(v) - \tilde{p}(v_i) = 0$. Obviously,

$$V_1 \leq V_0^{(i)} < V_2. \quad (57)$$

We now take

$$\begin{aligned} \Omega_1 &= \{(u_1(t), u_2(t))^T \in X : |u_1(t)| \\ &< B_5 + \beta_0, u_2(t) \in (B_7, B_8)\}, \\ \Omega_2 &= \{(u_1(t), u_2(t))^T \in X : |u_1(t)| \\ &< B_5 + \beta_0, u_2(t) \in (B_9, B_{10})\}, \end{aligned} \quad (58)$$

where β_0 is taken sufficiently large such that

$$\beta_0 > \max \{ |\ln V_0^{(1)}|, |\ln V_0^{(2)}| \}, \quad (59)$$

Then both Ω_1 and Ω_2 are bounded open subsets of X . It follows from (55), (57), and (59) that $u^{(1)} \in \Omega_1$, $u^{(2)} \in \Omega_2$, and $\Omega_1 \cap \Omega_2 = \emptyset$. With the help of (38), (52), (54), and (59), it is easy to see that Ω_1 and Ω_2 satisfy condition (a) in Lemma 1. When $(u_1(t), u_2(t))^T \in \partial\Omega_i \cap \text{Ker } L = \partial\Omega_i \cap \mathbb{R}^2$ ($i = 1, 2$), $(u_1(t), u_2(t))^T$ is a constant vector in $\mathbb{R}^2$. Thus, we have

$$QN \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \tilde{f}(e^{u_1}) - \tilde{p}(e^{u_1-u_2}) \\ -\tilde{d} + \tilde{h}(e^{u_1-u_2}) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (60)$$

This proves that condition (b) in Lemma 1 is satisfied. A direct calculation shows that

$$\begin{aligned} \deg(JQN, \Omega_i \cap \text{Ker } L, 0) \\ = \text{sign} \{ -v_i V_0^{(i)} \tilde{f}'(V_0^{(i)}) \tilde{h}'(v_i) \} \\ = (-1)^{i-1} \neq 0, \quad i = 1, 2. \end{aligned} \quad (61)$$

By now we have proved that Ω_1 and Ω_2 satisfy all the requirements in Lemma 1. Hence, system (1) has at least two ω -periodic solutions $u^*(t)$ and $u^\dagger(t)$ in $\text{Dom } L \cap \Omega_1$ and $\text{Dom } L \cap \Omega_2$, respectively. This completes the proof. $\square$

Noticing that both systems (2) and (3) are special cases of system (1), by Theorem 6, we can obtain the following results.

Theorem 7. Suppose that (H_1) – (H_6) and (H_8) – (H_9) hold. Then system (2) has at least two positive ω -periodic solutions.

Theorem 8. Suppose that (H_1) – (H_6) and (H_8) – (H_9) hold. Then system (3) has at least two positive ω -periodic solutions.

Remark 9. The proof of Theorems 3 and 6 shows that all the above results also remain valid if some delay terms are replaced by terms with distributed or state-dependent delays.

Remark 10. In their Theorem 2.2, Ding and Jiang [14] proved that (2) has at least two positive ω -periodic solutions if (H_1) – (H_6) , (H_8) , and

$$(H_9') \quad (V_2/V_1)e^{2[\tilde{F}(0)+\tilde{F}(0)+\tilde{D}+\tilde{d}]\omega} < v_2/v_1,$$

hold. Obviously, the condition (H_9') implies (H_9) . Hence, our Theorem 7 improves Theorem 2.2 of [14].

Remark 11. Shao [10] studied a special case of system (1) for $f(t, x) = a(t) - b(t)x$. Therefore, our Theorem 3 generalizes Theorem 1 of [10].

Remark 12. Fan and Wang [17], Wang and Li [21], and Fan et al. [16, 18] studied some special cases of system (2) for $f(t, x) = a(t) - b(t)x$. Therefore, our Theorem 4 generalizes Theorem 2.1 of [17], Theorem 3.1 of [16], Theorem 3.5 of [18], and Theorem 2.1 of [21].

Remark 13. Fan and Wang [23] and Xia et al. [24] studied some special cases of system (3) for $f(t, x) = a(t) - b(t)x$. Therefore, our Theorem 5 generalizes Theorem 2.1 of [23], and Theorem 3.1 of [24].

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