Research Article

Existence of Nonoscillatory Solutions for System of Higher-Order Neutral Differential Equations with Distributed Deviating Arguments

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In this paper, we consider the existence of nonoscillatory solutions for system of variable coefficients higher-order neutral differential equations with distributed deviating arguments. We use the *Banach* contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions.

1. Introduction and Preliminary

In this paper, we consider the system of higher-order neutral differential equations with distributed deviating arguments:

$$\begin{aligned} & \left[r(t)(\mathbf{x}(t) + P(t)\mathbf{x}(t - \theta))^{(n-1)} \right]' \\ & + (-1)^n \int_c^d Q_1(t, \tau) \mathbf{x}(t - \tau) d\tau \\ & - \int_e^f Q_2(t, \sigma) \mathbf{x}(t - \sigma) d\sigma = \mathbf{0}, \quad t \geq t_0, \end{aligned} \quad (1)$$

(1) where n is a positive integer, $n > 1$, $\theta > 0$, $0 < c < d$, and $0 < e < f$;

(2) $r \in C([t_0, \infty), \mathbb{R}^+)$, $r(t) > 0$, and $P \in C([t_0, \infty), \mathbb{R})$;

(3) $\mathbf{x} \in \mathbb{R}^n$, Q_i is continuous $n \times n$ matrix on $[t_0, \infty)$, $i = 1, 2$, and matrix coefficients system of higher

order neutral differential equations with distributed deviating arguments:

$$\begin{aligned} & \left[r(t)(\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t - \theta))^{(n-1)} \right]' \\ & + (-1)^n \int_c^d Q_1(t, \tau) \mathbf{x}(t - \tau) d\tau \\ & - \int_e^f Q_2(t, \sigma) \mathbf{x}(t - \sigma) d\sigma = \mathbf{0}, \quad t \geq t_0, \end{aligned} \quad (2)$$

(1) where n is a positive integer, $n > 1$, $\theta > 0$, $0 < c < d$, and $0 < e < f$;

(2) $r \in C([t_0, \infty), \mathbb{R}^+)$, $r(t) > 0$, and $\mathbf{B}$ is nonsingular constant $n \times n$ matrix;

(3) $\mathbf{x} \in \mathbb{R}^n$, and Q_i is continuous $n \times n$ matrix on $[t_0, \infty)$, $i = 1, 2$.

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations with positive and negative coefficients. In 2013,

Candan [1] has investigated existence of nonoscillatory solutions for system of higher-order nonlinear neutral differential equations:

$$\begin{aligned} & [\mathbf{x}(t) + P(t)\mathbf{x}(t-\theta)]^{(n)} \\ & + (-1)^{n+1} [Q_1(t)\mathbf{x}(t-\sigma_1) - Q_2(t)\mathbf{x}(t-\sigma_2)] = \mathbf{0} \end{aligned} \quad (3)$$

and matrix coefficient system of higher order neutral functional differential equation:

$$\begin{aligned} & [\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\theta)]^{(n)} \\ & + (-1)^{n+1} [Q_1(t)\mathbf{x}(t-\sigma_1) - Q_2(t)\mathbf{x}(t-\sigma_2)] = \mathbf{0}. \end{aligned} \quad (4)$$

In 2012, Candan [2] studies higher-order nonlinear differential equation:

$$\begin{aligned} & [r(t)[x(t) + P(t)x(t-\tau)]^{(n-1)}]' \\ & + (-1)^n \times [Q_1(t)g_1(x(t-\sigma_1)) \\ & - Q_2(t)g_2(x(t-\mu)) - f(t)] = 0. \end{aligned} \quad (5)$$

has obtained sufficient conditions for the existence of nonoscillatory solutions. For related work, we refer the reader to the books [3–12].

A solution of system of (1) and (2) is a continuous function $\mathbf{x}(t)$ defined on $([t_1 - \mu, \infty), \mathbf{R}^n)$, for some $t_1 > t_0$, such that $\mathbf{x}(t) - P(t)\mathbf{x}(t-\theta)$ and $\mathbf{x}(t) - \mathbf{B}\mathbf{x}(t-\theta)$ are $n-1$ times continuously differentiable, and $r(t)(\mathbf{x}(t) - P(t)\mathbf{x}(t-\theta))^{(n-1)}$ and $r(t)(\mathbf{x}(t) - \mathbf{B}\mathbf{x}(t-\theta))^{(n-1)}$ are continuously differentiable, and system of (1) and (2) holds for all $n > 1$. Here, $\mu = \max\{\theta, d, f\}$.

2. The Main Results

Theorem 1. Assume that $0 \leq P(t) \leq p < 1/2$ and

$$\int_t^\infty \frac{s^{n-2}}{r(s)} \int_{t_0}^s \left\| \int_c^d Q_i(u, \tau) d\tau \right\| du ds < \infty, \quad i = 1, 2. \quad (6)$$

Then, (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{x \in \Lambda, M_1 \leq \|\mathbf{x}(t)\| \leq M_2, t \geq t_0\}$, where M_1 and M_2 are positive constants and $\mathbf{b}$ is a constant vector, such that $pM_2 + M_1 < \|\mathbf{b}\| < 2\|\mathbf{b}\| < M_2 + M_1$. From (6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \frac{1}{(n-2)!} \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\ & \leq \frac{\|\mathbf{b}\| - pM_2 - M_1}{M_2}, \quad t > t_1, \end{aligned} \quad (7)$$

and define an operator T on A as follows:

$$\begin{aligned} & (T\mathbf{x})(t) \\ & = \begin{cases} \left\{ \mathbf{b} - P(t)\mathbf{x}(t-\theta) + \frac{1}{(n-2)!} \right. \\ \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau)\mathbf{x}(u-\tau) d\tau \right. \\ \quad \left. \left. - \int_e^f Q_2(u, \sigma)\mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (8)$$

It is easy to see that $T\mathbf{x}$ is continuous, for $t \geq t_1$, $\mathbf{x} \in A$; by using (7), we have

$$\begin{aligned} (T\mathbf{x})(t) & = \left\| \mathbf{b} - P(t)\mathbf{x}(t-\theta) + \frac{1}{(n-2)!} \right. \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau)\mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma)\mathbf{x}(u-\sigma) d\sigma \right] du ds \right\| \\ & \leq \|\mathbf{b}\| - p\|\mathbf{x}(t-\theta)\| + \frac{1}{(n-2)!} \\ & \quad \times \left\| \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \right. \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau)\mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma)\mathbf{x}(u-\sigma) d\sigma \right] du ds \right\| \\ & \leq \|\mathbf{b}\| - p\|\mathbf{x}(t-\theta)\| + \frac{1}{(n-2)!} \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| \|\mathbf{x}(u-\tau)\| d\tau \right. \\ & \quad \left. + \int_e^f \|Q_2(u, \sigma)\| \|\mathbf{x}(u-\sigma)\| d\sigma \right] du ds \\ & \leq \|\mathbf{b}\| - pM_2 + \frac{M_2}{(n-2)!} \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \end{aligned}$$

$$\begin{aligned} & \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ & \quad \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \leq M_2; \end{aligned} \quad (9)$$

and taking (7) into account, we have

$$\begin{aligned} (Tx)(t) &= \left\| \mathbf{b} - P(t) \mathbf{x}(t - \theta) + \frac{1}{(n-2)!} \right. \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u - \tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u - \sigma) d\sigma \right] du ds \right\| \\ &\geq \|\mathbf{b}\| \|p\mathbf{x}(t - \theta)\| - \frac{1}{(n-2)!} \\ & \quad \times \left\| \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \right. \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u - \tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u - \sigma) d\sigma \right] du ds \right\| \\ &\geq \|\mathbf{b}\| - \|p\mathbf{x}(t - \theta)\| - \frac{1}{(n-2)!} \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) \mathbf{x}(u - \tau) d\tau \right\| \right. \\ & \quad \left. + \left\| \int_e^f Q_2(u, \sigma) \mathbf{x}(u - \sigma) d\sigma \right\| \right] du ds \\ &\geq \|\mathbf{b}\| - pM_2 - \frac{M_2}{(n-2)!} \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ & \quad \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \geq M_1. \end{aligned} \quad (10)$$

These show that $TA \subset A$. Since A is a bounded, close, convex subset of Λ , in order to apply the contraction principle, we have to show that T is a contraction mapping on A . For all $\mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} & |(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)| \\ & \leq P(t) \|\mathbf{x}_1(t - \theta) - \mathbf{x}_2(t - \theta)\| + \frac{1}{(n-2)!} \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) \|\mathbf{x}_1(u - \tau) - \mathbf{x}_2(u - \tau)\| d\tau \right\| \right. \\ & \quad \left. + \left\| \int_e^f Q_2(u, \sigma) \|\mathbf{x}_1(u - \sigma) - \mathbf{x}_2(u - \sigma)\| d\sigma \right\| \right] du ds. \end{aligned} \quad (11)$$

Using (7),

$$\begin{aligned} & |(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)| \\ & \leq \|\mathbf{x}_1 - \mathbf{x}_2\| \\ & \quad \times \left(p + \frac{1}{(n-2)!} \right. \\ & \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ & \quad \left. \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \right) \\ & < q_1 \|\mathbf{x}_1 - \mathbf{x}_2\|, \quad (0 < q_1 < 1). \end{aligned} \quad (12)$$

This implies, with the sup norm, that

$$\|T\mathbf{x}_1 - T\mathbf{x}_2\| < q_1 \|\mathbf{x}_1 - \mathbf{x}_2\|, \quad (0 < q_1 < 1), \quad (13)$$

which shows that T is a contraction mapping on A , and therefore there exists a unique solution. Consequently there exists a unique solution of (1) $\mathbf{x} \in A$ of $T\mathbf{x} = \mathbf{x}$. The proof is complete. $\square$

Theorem 2. Assume that $2 < p_1 \leq P(t) \leq p_0 < \infty$ and that (6) holds.

Then, (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{\mathbf{x} \in \Lambda, M_3 \leq \|\mathbf{x}(t)\| \leq M_4, t \geq t_0\}$, where M_3 and M_4 are positive constants such that $p_0 M_3 + M_4 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leq p_1 M_4 + p_0 M_3$. From

(6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \frac{1}{(n-2)!} \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\ & \leq \frac{\|\mathbf{b}\| - M_4 - p_0 M_3}{M_4}, \quad t > t_1, \end{aligned} \quad (14)$$

and define an operator T on A as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{P(t+\theta)} \times \left\{ \mathbf{b} - \mathbf{x}(t+\theta) + \frac{1}{(n-2)!} \right. \\ \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases} \quad (15)$$

It is easy to see that $T\mathbf{x}$ is continuous, for $t \geq t_1$, $\mathbf{x} \in A$; by using (14), we have

$$\begin{aligned} & (T\mathbf{x})(t) \\ & \leq \frac{1}{p_1} \left\{ \left\| \mathbf{b} - \mathbf{x}(t-\theta) + \frac{1}{(n-2)!} \right. \right. \\ & \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} \\ & \leq \frac{1}{p_1} \left\{ \|\mathbf{b}\| + \|\mathbf{x}(t-\theta)\| + \frac{1}{(n-2)!} \right. \\ & \quad \times \left\| \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \right. \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} \end{aligned}$$

$$\begin{aligned} & \leq \frac{1}{p_1} \left\{ \|\mathbf{b}\| + \|\mathbf{x}(t-\theta)\| + \frac{1}{(n-2)!} \right. \\ & \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| \|\mathbf{x}(u-\tau)\| d\tau \right. \\ & \quad \left. \left. + \int_e^f \|Q_2(u, \sigma)\| \|\mathbf{x}(u-\sigma)\| d\sigma \right] du ds \right\} \\ & \leq \frac{1}{p_1} \left\{ \|\mathbf{b}\| + M_4 + \frac{M_4}{(n-2)!} \right. \\ & \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ & \quad \left. \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \right\} \leq M_4; \end{aligned} \quad (16)$$

and taking (14) into account, we have

$$\begin{aligned} & (T\mathbf{x})(t) \\ & \geq \frac{1}{p_0} \left\{ \left\| \mathbf{b} - \mathbf{x}(t-\theta) + \frac{1}{(n-2)!} \right. \right. \\ & \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} \\ & \geq \frac{1}{p_0} \left\{ \|\mathbf{b}\| - \|\mathbf{x}(t-\theta)\| - \frac{1}{(n-2)!} \right. \\ & \quad \times \left\| \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \right. \\ & \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ & \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} \\ & \geq \frac{1}{p_0} \left\{ \|\mathbf{b}\| - \|\mathbf{x}(t-\theta)\| - \frac{1}{(n-2)!} \right. \\ & \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \end{aligned}$$

$$\begin{aligned}
& \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| \|\mathbf{x}(u - \tau)\| d\tau \right. \\
& \quad \left. + \int_e^f \|Q_2(u, \sigma)\| \|\mathbf{x}(u - \sigma)\| d\sigma \right] du ds \Big\} \\
& \geq \frac{1}{p_0} \left\{ \|\mathbf{b}\| - M_4 - \frac{M_4}{(n-2)!} \right. \\
& \quad \times \int_{t+\theta}^\infty \frac{(s-t-\theta)^{n-2}}{r(s)} \\
& \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| d\tau \right. \\
& \quad \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \Big\} \geq M_3.
\end{aligned} \tag{17}$$

These show that $TA \subset A$. Since A is a bounded, close, convex subset of Λ , in order to apply the contraction principle, we have to show that T is a contraction mapping on A . For all $x_1, x_2 \in A$, and $t \geq t_1$,

$$\begin{aligned}
& |(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)| \\
& \leq \frac{1}{p_1} \left\{ \|\mathbf{x}_1(t + \theta) - \mathbf{x}_2(t + \theta)\| + \frac{1}{(n-2)!} \right. \\
& \quad \times \int_{t+\theta}^\infty \frac{(s-t-\theta)^{n-2}}{r(s)} \\
& \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| \|\mathbf{x}_1(u - \tau) - \mathbf{x}_2(u - \tau)\| d\tau \right. \\
& \quad \left. + \left\| \int_e^f Q_2(u, \sigma) \|\mathbf{x}_1(u - \sigma) \right. \right. \\
& \quad \left. \left. - \mathbf{x}_2(u - \sigma)\| d\sigma \right\| \right] du ds \Big\},
\end{aligned} \tag{18}$$

or using (14),

$$\begin{aligned}
& |(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)| \\
& \leq \frac{\|\mathbf{x}_1 - \mathbf{x}_2\|}{p_1} \\
& \quad \times \left\{ 1 + \frac{1}{(n-2)!} \right. \\
& \quad \times \int_{t+\theta}^\infty \frac{(s-t-\theta)^{n-2}}{r(s)} \\
& \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| d\tau \right. \\
& \quad \left. + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \Big\} \\
& < \|\mathbf{x}_1 - \mathbf{x}_2\|.
\end{aligned} \tag{19}$$

This implies, with the sup norm, that

$$\|T\mathbf{x}_1 - T\mathbf{x}_2\| < \|\mathbf{x}_1 - \mathbf{x}_2\|, \tag{20}$$

which shows that T is a contraction mapping on A , and therefore there exists a unique solution. Consequently there exists a unique solution of (1) $\mathbf{x} \in A$ of $T\mathbf{x} = \mathbf{x}$. The proof is complete. $\square$

Theorem 3. Assume that $-1/2 < p_2 \leq P(t) \leq 0$ and that (6) holds.

Then, (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{x \in \Lambda, M_5 \leq \|\mathbf{x}(t)\| \leq M_6, t \geq t_0\}$, where M_5 and M_6 are positive constants such that $-p_2 M_6 + M_5 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leq M_6 + M_5$. From (6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned}
& \frac{1}{(n-2)!} \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\
& \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| d\tau + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\
& \leq \frac{\|\mathbf{b}\| + p_2 M_6 - M_5}{M_6}, \quad t > t_1,
\end{aligned} \tag{21}$$

and define an operator T on A as follows

$$\begin{aligned}
& (T\mathbf{x})(t) \\
& = \begin{cases} \mathbf{b} - P(t)\mathbf{x}(t - \theta) + \frac{1}{(n-2)!} \\ \quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d \|Q_1(u, \tau)\| \mathbf{x}(u - \tau) d\tau \right. \\ \quad \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u - \sigma) d\sigma \right] du ds & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases}
\end{aligned} \tag{22}$$

It is easy to see that $T\mathbf{x}$ is continuous. Since the proof is similar to that of Theorem 1, we omit the remaining part of the proof. The proof is complete. $\square$

Theorem 4. Assume that $-\infty < p_3 \leq P(t) \leq p_4 < -2$ and that (6) holds.

Then, (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{x \in \Lambda, M_7 \leq \|\mathbf{x}(t)\| \leq M_8, t \geq t_0\}$, where M_7 and M_8 are positive constants such that $-p_3 M_7 + M_8 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leq -p_4 M_8 - p_3 M_7$.

From (6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \frac{1}{(n-2)!} \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-1}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\ & \leq \frac{\|\mathbf{b}\| + p_3 M_7 + M_8}{M_8}, \quad t > t_1, \end{aligned} \quad (23)$$

and define an operator T on A as follows:

$$\begin{aligned} & (T\mathbf{x})(t) \\ & = \begin{cases} \frac{1}{P(t+\theta)} \\ \quad \times \left\{ \mathbf{b} - \mathbf{x}(t+\theta) + \frac{1}{(n-2)!} \right. \\ \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ \quad \left. \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (24)$$

It is easy to see that $T\mathbf{x}$ is continuous. The remaining part of the proof is similar to that of Theorem 2; therefore, it is omitted. The proof is complete. $\square$

Theorem 5. Assume that $0 < \|\mathbf{B}\| < 1/2$ and that (6) holds. Then, (2) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{x \in \Lambda, N_1 \leq \|\mathbf{x}(t)\| \leq N_2, t \geq t_0\}$, where N_1 and N_2 are positive constants such that $\|\mathbf{B}\|N_2 + N_1 < \|\mathbf{b}\| < 2\|\mathbf{b}\| \leq N_2 + N_1$. From (6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \frac{1}{(n-2)!} \\ & \quad \times \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\ & \leq \frac{\|\mathbf{b}\| - \|\mathbf{B}\|N_2 - N_1}{N_2}, \quad t > t_1, \end{aligned} \quad (25)$$

and define an operator T on A as follows:

$$\begin{aligned} & (T\mathbf{x})(t) \\ & = \begin{cases} \mathbf{b} - \mathbf{B}\mathbf{x}(t-\theta) + \frac{1}{(n-2)!} \\ \quad \times \int_t^{\infty} \frac{(s-t)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ \quad \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] du ds & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (26)$$

It is easy to see that $T\mathbf{x}$ is continuous. Since the proof is similar to that of Theorem 1, we omit the remaining part of the proof. The proof is complete. $\square$

Theorem 6. Assume that $0 < \|\mathbf{B}^{-1}\| < 1/2$ and that (6) holds. Then, (2) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ and the sup norm. Set $A = \{x \in \Lambda, N_3 \leq \|\mathbf{x}(t)\| \leq N_4, t \geq t_0\}$, where N_3 and N_4 are positive constants such that $\|\mathbf{B}^{-1}\|N_4 + N_3 < \|\mathbf{B}^{-1}\|\mathbf{b} < 2\|\mathbf{B}^{-1}\|\mathbf{b} \leq N_4 - N_3$. From (6), one can choose a $t_1 \geq t_0$, $t_1 \geq t_0 + \mu$, sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \frac{1}{(n-2)!} \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ & \quad \times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| + \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \\ & \leq \frac{\|\mathbf{B}^{-1}\|\mathbf{b} - N_3 - N_4 \|\mathbf{B}^{-1}\|}{N_4 \|\mathbf{B}^{-1}\|}, \quad t > t_1, \end{aligned} \quad (27)$$

and define an operator T on A as follows:

$$\begin{aligned} & (T\mathbf{x})(t) \\ & = \begin{cases} \mathbf{B}^{-1} \left\{ \mathbf{b} - \mathbf{x}(t+\theta) + \frac{1}{(n-2)!} \right. \\ \quad \times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ \quad \times \int_{t_1}^s \left[\int_c^d Q_1(u, \tau) \mathbf{x}(u-\tau) d\tau \right. \\ \quad \left. - \int_e^f Q_2(u, \sigma) \mathbf{x}(u-\sigma) d\sigma \right] ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (28)$$

It is easy to see that $T\mathbf{x}$ is continuous, for $t \geq t_1$, $\mathbf{x} \in A$; by using (27), we have

$$\begin{aligned} (T\mathbf{x})(t) &\leq \|\mathbf{B}^{-1}\mathbf{b}\| + \|\mathbf{B}^{-1}\| \\ &\times \left\{ N_4 + \frac{N_4}{(n-2)!} \right. \\ &\times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ &\times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ &\left. \left. - \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \right\} \leq N_4, \end{aligned} \quad (29)$$

and taking (27) into account, we have

$$\begin{aligned} (T\mathbf{x})(t) &\geq \|\mathbf{B}^{-1}\mathbf{b}\| - \|\mathbf{B}^{-1}\| \\ &\times \left\{ N_4 - \frac{N_4}{(n-2)!} \right. \\ &\times \int_{t+\theta}^{\infty} \frac{(s-t-\theta)^{n-2}}{r(s)} \\ &\times \int_{t_1}^s \left[\left\| \int_c^d Q_1(u, \tau) d\tau \right\| \right. \\ &\left. \left. - \left\| \int_e^f Q_2(u, \sigma) d\sigma \right\| \right] du ds \right\} \geq N_3. \end{aligned} \quad (30)$$

It is easy to see that $T\mathbf{x}$ is continuous. The remaining part of the proof is similar to that of Theorem 2; therefore, it is omitted. The proof is complete. $\square$

3. Example

Example 1. Consider high-order neutral differential equation with distributed deviating arguments:

$$\begin{aligned} &\left(e^{-t}(\mathbf{x}(t) + e^{-1}\mathbf{x}(t-1)) \right)^{(3)'} \\ &+ \frac{2}{3} \int_1^2 \tau e^{-2t} \left(\frac{3}{2} \frac{1}{2} \right) \mathbf{x}(t-\tau) d\tau \\ &- \frac{2}{3} \int_1^2 \sigma e^{-2t} \left(\frac{2}{5} \frac{1}{10} \right) \mathbf{x}(t-\sigma) d\sigma = \mathbf{0}. \end{aligned} \quad (31)$$

Here, $n = 4$, $c = e = 1$, $d = f = 2$, $\theta = 1$, $r(t) = e^{-t}$, $P(t) = e^{-1}$, $Q_1(t, \tau) = (2/3)\tau e^{-2t} \left(\frac{3}{1/2} \frac{1}{7/2} \right)$, and $Q_2(t, \sigma) = (2/3)\sigma e^{-2t} \left(\frac{2}{5/3} \frac{1}{10/3} \right)$.

It is easy to see that

$$\begin{aligned} 0 < P(t) &= e^{-1} < \frac{1}{2}, \\ \int_t^{\infty} e^t s^2 \int_{t_0}^s \int_1^2 \left\| \frac{2}{3} \tau e^{-2u} \left(\frac{2}{5} \frac{1}{10} \right) d\tau \right\| du ds &< \infty, \\ \int_t^{\infty} e^t s^2 \int_{t_0}^s \int_1^2 \left\| \frac{2}{3} \sigma e^{-2u} \left(\frac{3}{2} \frac{1}{7} \right) d\sigma \right\| du ds &< \infty; \end{aligned} \quad (32)$$

thus Theorem 1 holds. In fact, $\mathbf{x}(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ is a nonoscillatory solution of (31).

Example 2. Consider high-order neutral differential equation with distributed deviating arguments:

$$\begin{aligned} &\left(\frac{1}{|\sin t|} \left(\mathbf{x}(t) + e^{-\pi} \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \end{pmatrix} \mathbf{x}(t-\pi) \right)^{(2)} \right)' \\ &+ \frac{3}{8} \int_1^3 \tau^2 e^{-4t} \left(\frac{5}{2} \frac{7}{2} \right) \mathbf{x}(t-\tau) d\tau \\ &- \frac{3}{8} \int_1^3 \sigma^2 e^{-4t} \left(\frac{2}{7} \frac{4}{11} \right) \mathbf{x}(t-\sigma) d\sigma = \mathbf{0}. \end{aligned} \quad (33)$$

Here, $n = 3$, $r(t) = 1/|\sin t|$, $\theta = \pi$, $c = e = 1$, $d = f = 3$, $\mathbf{B} = e^{-\pi} \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \end{pmatrix}$, $Q_1(t, \tau) = (3/8)\tau^2 e^{-4t} \left(\frac{5/2}{5} \frac{7/2}{1} \right)$, and $Q_2(t, \sigma) = (3/8)\sigma^2 e^{-4t} \left(\frac{2/7}{7/3} \frac{4}{11/3} \right)$.

It is easy to see that

$$\begin{aligned} 0 < \|\mathbf{B}\| &= \left\| e^{-\pi} \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \end{pmatrix} \right\| < \frac{1}{2}, \\ \int_t^{\infty} |\sin s| s \int_{t_0}^s \frac{3}{8} \int_1^3 \left\| \tau^2 e^{-4u} \left(\frac{5}{2} \frac{7}{2} \right) d\tau \right\| du ds &< \infty, \\ \int_t^{\infty} |\sin s| s \int_{t_0}^s \frac{3}{8} \int_1^3 \left\| \sigma^2 e^{-4u} \left(\frac{2}{7} \frac{4}{11} \right) d\sigma \right\| du ds &< \infty; \end{aligned} \quad (34)$$

thus Theorem 5 holds. In fact, $\mathbf{x}(t) = \begin{pmatrix} 2+\sin t \\ 2+\sin t \end{pmatrix}$ is a nonoscillatory solution of (33).

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