Research Article

Permanence in a Discrete Mutualism Model with Infinite Deviating Arguments and Feedback Controls

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We propose and deal with a discrete mutualism model with infinite deviating arguments and feedback controls. Sufficient conditions which guarantee the permanence of the system are obtained by using the difference inequality theory. The paper ends with brief conclusions.

1. Introduction

It is well known that the long-term coexistence of species in mathematical ecology is an important and ubiquitous problem. Several mathematical concepts of coexistence of species are developed to deal with this aspect. Permanence is one important topic in these concepts. In recent years, permanence has received great attention and has been investigated in a number of notable studies. For example, Fan and Li [1] analyzed permanence of a delayed ratio-dependent predator-prey model with Holling type functional response. Mukherjee [2] addressed the permanence and global attractivity for facultative mutualism system with delay. Zhao and Jiang [3] focused on the permanence and extinction for nonautonomous Lotka-Volterra system. Chen [4] made a theoretical discussion on the permanence and global attractivity of Lotka-Volterra competition system with feedback control. Teng et al. [5] established the permanence criteria for a delayed discrete nonautonomous-species Kolmogorov systems. For more research on the permanence behavior of predator-prey models, one can see [6–19].

In 2007, Chen and You [20] investigated the permanence of the following two species integrodifferential model of mutualism:

$$\frac{dN_1(t)}{dt} = r_1(t) N_1(t)$$

$$\times \left[\frac{K_1(t) + \alpha_1(t) \int_0^\infty J_2(s) N_2(t-s) ds}{1 + \int_0^\infty J_2(s) N_2(t-s) ds} - N_1(t - \sigma_1(t)) \right],$$

$$\frac{dN_2(t)}{dt} = r_2(t) N_2(t)$$

$$\times \left[\frac{K_2(t) + \alpha_2(t) \int_0^\infty J_1(s) N_1(t-s) ds}{1 + \int_0^\infty J_1(s) N_1(t-s) ds} - N_2(t - \sigma_2(t)) \right],$$

(1)

where r_i , K_i , α_i , and σ_i ($i = 1, 2$) are continuous functions bounded above and below by positive constants. $\alpha_i > K_i$, $i = 1, 2$; $J_i \in C([0, +\infty), [0, +\infty))$ and $\int_0^\infty J_i(s) ds = 1$, $i = 1, 2$. Using the differential inequality theory, they obtained a set of sufficient conditions to ensure the permanence of system (1). For more background and biological adjustments of system (1), one could refer to [20–24] and the references cited therein.

Many authors [24–33] have argued that discrete time models governed by difference equations are more appropriate to describe the dynamics relationship among populations than continuous ones when the populations have nonoverlapping generations. Moreover, discrete time models can also provide efficient models of continuous ones for numerical simulations. Motivated by the above viewpoint, Li and Yang [34] considered the permanence of the following discrete model of mutualism with infinite deviating arguments:

$$\begin{aligned}
 x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) \left[\frac{K_1(n) + \alpha_1(n) \sum_{s=0}^{\infty} J_2(s) x_2(n-s)}{1 + \sum_{s=0}^{\infty} J_2(s) x_2(n-s)} \right. \right. \\
 &\quad \left. \left. - x_1(n - \sigma_1(n)) \right] \right\}, \\
 x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) \left[\frac{K_2(n) + \alpha_2(n) \sum_{s=0}^{\infty} J_1(s) x_1(n-s)}{1 + \sum_{s=0}^{\infty} J_1(s) x_1(n-s)} \right. \right. \\
 &\quad \left. \left. - x_2(n - \sigma_2(n)) \right] \right\}, \tag{2}
 \end{aligned}$$

where $x_i(n)$ ($i = 1, 2$) is the density of mutualism species i at the generation, $\{r_i(n)\}$, $\{K_i(n)\}$, $\{\alpha_i(n)\}$, $\{\sigma_i(n)\}$ ($i = 1, 2$), $J_1(n)$, and $J_2(n)$ are bounded nonnegative sequences. Applying the comparison theorem of difference equation and some lemmas, they derived some sufficient conditions which guarantee the permanence of system (2).

It is well known that ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates [35]. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. To the authors' knowledge, it is the first time to deal with system (2) with feedback control.

The main objective of this paper is to investigate the following discrete mutualism model with infinite deviating arguments and feedback controls:

$$\begin{aligned}
 x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) \right. \\
 &\quad \times \left[\frac{K_1(n) + \alpha_1(n) \sum_{s=0}^{\infty} J_2(s) x_2(n-s)}{1 + \sum_{s=0}^{\infty} J_2(s) x_2(n-s)} \right. \\
 &\quad \left. \left. - x_1(n - \sigma_1(n)) - \beta_1(n) u_1(n) \right] \right\},
 \end{aligned}$$

$$\begin{aligned}
 x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) \right. \\
 &\quad \times \left[\frac{K_2(n) + \alpha_2(n) \sum_{s=0}^{\infty} J_1(s) x_1(n-s)}{1 + \sum_{s=0}^{\infty} J_1(s) x_1(n-s)} \right. \\
 &\quad \left. \left. - x_2(n - \sigma_2(n)) - \beta_2(n) u_2(n) \right] \right\}, \\
 \Delta u_1(n) &= -\gamma_1(n) u_1(n) + \eta_1(n) x_1(n), \\
 \Delta u_2(n) &= -\gamma_2(n) u_2(n) + \eta_2(n) x_2(n), \tag{3}
 \end{aligned}$$

where $x_i(n)$ ($i = 1, 2$) is the density of mutualism species i at the generation and $u_i(n)$ ($i = 1, 2$) is the control variable. $r_1(n)$, $r_2(n)$, $K_1(n)$, $K_2(n)$, $\alpha_1(n)$, $\alpha_2(n)$, $J_1(n)$, $J_2(n)$, $\sigma_1(n)$, and $\sigma_2(n)$ are bounded nonnegative sequences.

Throughout this paper, we assume that

$$\text{(H)} \quad 0 < r_i^l \leq r_i^u, \quad 0 < K_i^l \leq K_i^u, \quad 0 < \alpha_i^l \leq \alpha_i^u, \quad 0 < \beta_i^l \leq \beta_i^u, \quad 0 < \sigma_i^l \leq \sigma_i^u, \quad \sum_{n=0}^{\infty} J_i(n) = 1, \quad (i = 1, 2).$$

Here, for any bounded sequence $\{f(n)\}$, $f^u = \sup_{n \in \mathbb{N}} \{f(n)\}$ and $f^l = \inf_{n \in \mathbb{N}} \{f(n)\}$.

Let $\sigma = \sup_{n \in \mathbb{Z}} \{\sigma_i(n)\}$, $\tilde{\sigma} = \inf_{n \in \mathbb{Z}} \{\sigma_i(n)\}$, $i = 1, 2$. We consider (1) together with the following initial conditions:

$$x_i(\theta) = \varphi_i(\theta) \geq 0, \tag{4}$$

$$\theta \in N[-\tau, 0] = \{-\tau, -\tau+1, \dots, 0\}, \quad \varphi_i(0) > 0.$$

It is not difficult to see that solutions of (1) and (4) are well defined for all $n \geq 0$ and satisfy

$$x_i(n) > 0, \quad \text{for } n \in \mathbb{Z}, \quad i = 1, 2. \tag{5}$$

The remainder of the paper is organized as follows. In Section 2, basic definitions and lemmas are given, some sufficient conditions for the permanence of system (1) are established. Brief conclusions are presented in Section 3.

2. Permanence

In order to obtain the main result of this paper, we will first state the definition of permanence and several lemmas which will be useful in the proving of the main result.

Definition 1. We say that system (1) is permanence if there are positive constants M and m such that for each positive solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ of system (1) satisfies

$$\begin{aligned}
 m &\leq \lim_{n \rightarrow +\infty} \inf x_i(n) \leq \lim_{n \rightarrow +\infty} \sup x_i(n) \leq M, \\
 &\quad (i = 1, 2), \\
 m &\leq \lim_{n \rightarrow +\infty} \inf u_i(n) \leq \lim_{n \rightarrow +\infty} \sup u_i(n) \leq M, \\
 &\quad (i = 1, 2). \tag{6}
 \end{aligned}$$

Let us consider the following single species discrete model:

$$N(n+1) = N(n) \exp(a(n) - b(n)N(n)), \quad (7)$$

where $\{a(n)\}$ and $\{b(n)\}$ are strictly positive sequences of real numbers defined for $n \in N = \{0, 1, 2, \dots\}$ and $0 < a^l \leq a^u$, $0 < b^l \leq b^u$. Similarly to the proofs Propositions 1 and 3 in [36], we can obtain the following Lemma 2.

Lemma 2. Any solution of system (7) with initial condition $N(0) > 0$ satisfies

$$m \leq \lim_{n \rightarrow +\infty} \inf N(n) \leq \lim_{n \rightarrow +\infty} \sup N(n) \leq M, \quad (8)$$

where

$$M = \frac{1}{b^l} \exp(a^u - 1), \quad m = \frac{a^l}{b^u} \exp(a^l - b^u M). \quad (9)$$

Let us consider the first order difference equation:

$$y(n+1) = Ay(n) + B, \quad n = 1, 2, \dots, \quad (10)$$

where A and B are positive constants. Following Theorem 6.2 of L. Wang and M. Q. Wang [37, page 125], we have the following Lemma 3.

Lemma 3 (see [37]). Assume that $|A| < 1$, for any initial value $y(0)$, there exists a unique solution $y(n)$ of (10) which can be expressed as follows:

$$y(n) = A^n (y(0) - y^*) + y^*, \quad (11)$$

where $y^* = B/(1 - A)$. Thus, for any solution $\{y(n)\}$ of system (10), $\lim_{n \rightarrow +\infty} y(n) = y^*$.

Lemma 4 (see [37]). Let $n \in N_{n_0}^+ = \{n_0, n_0 + 1, \dots, n_0 + l, \dots\}$, $r \geq 0$. For any fixed n , $g(n, r)$ is a nondecreasing function with respect to r , and for $n \geq n_0$, the following inequalities hold:

$$y(n+1) \leq g(n, y(n)), \quad u(n+1) \geq g(n, u(n)). \quad (12)$$

If $y(n_0) \leq u(n_0)$, then $y(n) \leq u(n)$ for all $n \geq n_0$.

Proposition 5. Assume that the condition (H) holds, then

$$\lim_{n \rightarrow +\infty} \sup x_i(n) \leq M_i, \quad \lim_{n \rightarrow +\infty} \sup u_i(n) \leq U_i, \quad i = 1, 2, \quad (13)$$

where

$$M_i = \frac{1}{r_i^l} \exp\{r_i^u (K_i^u + \alpha_i^u) (\sigma + 1) - 1\}, \quad (14)$$

$$U_i = \frac{\eta_i^u M_i}{\gamma_i^l}, \quad (i = 1, 2).$$

Proof. Let $(x_1(n), x_2(n), u_1(n), u_2(n))$ be any positive solution of system (1) with the initial condition $(x_1(0), x_2(0),$

$u_1(0), u_2(0))$. It follows from the first equation of system (1) that

$$x_1(n+1) \leq x_1(n) \exp\{r_1(n)K_1(n) + r_1(n)\alpha_1(n)\}. \quad (15)$$

Let $x_1(n) = \exp\{y_1(n)\}$ then (15) is equivalent to

$$y_1(n+1) - y_1(n) \leq r_1(n)K_1(n) + r_1(n)\alpha_1(n). \quad (16)$$

Summing both sides of (16) from $n - \sigma_1(n)$ to $n - 1$, we have

$$\begin{aligned} & \sum_{j=n-\sigma_1(n)}^{n-1} (y_1(j+1) - y_1(j)) \\ & \leq \sum_{j=n-\sigma_1(n)}^{n-1} (r_1(j)K_1(j) + r_1(j)\alpha_1(j)) \\ & \leq r_1^u (K_1^u + \alpha_1^u) \sigma, \end{aligned} \quad (17)$$

which leads to

$$y_1(n - \sigma_1(n)) \geq y_1(n) - r_1^u (K_1^u + \alpha_1^u) \sigma. \quad (18)$$

Then

$$x_1(n - \sigma_1(n)) \geq x_1(n) \exp\{-r_1^u (K_1^u + \alpha_1^u) \sigma\}. \quad (19)$$

Substituting (19) into the first equation of system (1), it follows that

$$\begin{aligned} & x_1(n+1) \\ & \leq x_1(n) \exp\{r_1(n)(K_1(n) + \alpha_1(n)) \\ & \quad - r_1(n) \exp\{-r_1^u (K_1^u + \alpha_1^u) \sigma\} x_1(n)\}. \end{aligned} \quad (20)$$

It follows from (20) and Lemma 2 that

$$\lim_{n \rightarrow +\infty} \sup x_1(n) \leq \frac{1}{r_1^l} \exp\{r_1^u (K_1^u + \alpha_1^u) (\sigma + 1) - 1\} := M_1. \quad (21)$$

For any positive constant $\varepsilon > 0$, it follows (21) that there exists a $N_1 > 0$ such that for all $n > N_1 + \sigma$

$$x_1(n) \leq M_1 + \varepsilon. \quad (22)$$

From the second equation of system (1), we get

$$x_2(n+1) \leq x_2(n) \exp\{r_2(n)K_2(n) + r_2(n)\alpha_2(n)\}. \quad (23)$$

Let $x_2(n) = \exp\{y_2(n)\}$ then (23) is equivalent to

$$y_2(n+1) - y_2(n) \leq r_2(n)K_2(n) + r_2(n)\alpha_2(n). \quad (24)$$

Summing both sides of (24) from $n - \sigma_2(n)$ to $n - 1$, we have

$$\begin{aligned} & \sum_{j=n-\sigma_2(n)}^{n-1} (y_2(j+1) - y_2(j)) \\ & \leq \sum_{j=n-\sigma_2(n)}^{n-1} (r_2(j)K_2(j) + r_2(j)\alpha_2(j)) \\ & \leq r_2^u (K_2^u + \alpha_2^u) \sigma, \end{aligned} \quad (25)$$

which leads to

$$y_2(n - \sigma_2(n)) \geq y_2(n) - r_2^u(K_2^u + \alpha_2^u)\sigma. \quad (26)$$

Then

$$x_2(n - \sigma_2(n)) \geq x_2(n) \exp\{-r_2^u(K_2^u + \alpha_2^u)\sigma\}. \quad (27)$$

Substituting (27) into the second equation of system (1), it follows that

$$\begin{aligned} x_2(n+1) &\leq x_2(n) \exp\{r_2(n)(k_2(n) + \alpha_2(n)) \\ &\quad - r_2(n) \exp\{-r_2^u(K_2^u + \alpha_2^u)\sigma\} x_2(n)\}. \end{aligned} \quad (28)$$

It follows from (28) and Lemma 2 that

$$\lim_{n \rightarrow +\infty} \sup x_2(n) \leq \frac{1}{r_2^l} \exp\{r_1^u(K_2^u + \alpha_2^u)(\sigma + 1) - 1\} := M_2. \quad (29)$$

For any positive constant $\varepsilon > 0$, it follows (29) that there exists a $N_2 > N_1 + \sigma$ such that for all $n > N_2 + \sigma$

$$x_2(n) \leq M_2 + \varepsilon. \quad (30)$$

In view of the third and fourth equations of system (1), we can obtain

$$\begin{aligned} \Delta u_1(n) &\leq -\gamma_1(n)u_1(n) + \eta_1(n)(M_1 + \varepsilon), \\ \Delta u_2(n) &\leq -\gamma_2(n)u_2(n) + \eta_2(n)(M_2 + \varepsilon). \end{aligned} \quad (31)$$

Then

$$\begin{aligned} u_1(n+1) &\leq (1 - \gamma_1^l)u_1(n) + \eta_1^u(M_1 + \varepsilon), \\ u_2(n+1) &\leq (1 - \gamma_2^l)u_2(n) + \eta_2^u(M_2 + \varepsilon). \end{aligned} \quad (32)$$

Applying Lemmas 3 and 4, it immediately follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup u_1(n) &\leq \frac{\eta_1^u(M_1 + \varepsilon)}{\gamma_1^l}, \\ \lim_{n \rightarrow +\infty} \sup u_2(n) &\leq \frac{\eta_2^u(M_2 + \varepsilon)}{\gamma_2^l}. \end{aligned} \quad (33)$$

Setting $\varepsilon \rightarrow 0$, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup u_1(n) &\leq \frac{\eta_1^u M_1}{\gamma_1^l} := U_1, \\ \lim_{n \rightarrow +\infty} \sup u_2(n) &\leq \frac{\eta_2^u M_2}{\gamma_2^l} := U_2. \end{aligned} \quad (34)$$

This completes the proof of Proposition 5. $\square$

Theorem 6. Assume that (H) holds, then system (1) is permanent.

Proof. By applying Proposition 5, we can easily see that to end the proof of Theorem 6, it is enough to show that under the conditions of Theorem 6,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \inf x_2(n) &\geq m_1, & \lim_{n \rightarrow +\infty} \inf x_2(n) &\geq m_2, \\ \lim_{n \rightarrow +\infty} \inf u_1(n) &\geq v_1, & \lim_{n \rightarrow +\infty} \inf u_2(n) &\geq v_2. \end{aligned} \quad (35)$$

In view of Proposition 5, for all $\varepsilon > 0$, there exists a $N_3 > 0$, $N_3 \in N$, for all $n > N_3$:

$$\begin{aligned} x_1(n) &\leq M_1 + \varepsilon, & x_2(n) &\leq M_2 + \varepsilon, \\ u_1(n) &\leq U_1 + \varepsilon, & u_2(n) &\leq U_2 + \varepsilon. \end{aligned} \quad (36)$$

It follows from the first equation of systems (1) and (36) that

$$\begin{aligned} x_1(n+1) &\geq x_1(n) \exp\left\{r_1(n) \left[\frac{K_1(n)}{1 + \sum_{s=0}^{\infty} J_2(s)x_2(n-s)} \right. \right. \\ &\quad \left. \left. - x_1(n - \sigma_1(n)) \right] \right\} \\ &\geq x_1(n) \exp\left\{ \frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right\} \end{aligned} \quad (37)$$

for all $n > N_3 + \sigma$.

Let $x_1(n) = \exp\{y_1(n)\}$; then (37) is equivalent to

$$y_1(n+1) - y_1(n) \geq \frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon). \quad (38)$$

Summing both sides of (38) from $n - \delta_1(n)$ to $n - 1$ leads to

$$\begin{aligned} \sum_{j=n-\delta_1(n)}^{n-1} (y_1(j+1) - y_1(j)) &\geq \sum_{j=n-\delta_1(n)}^{n-1} \left[\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right] \\ &\geq \left[\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right] \tilde{\sigma}. \end{aligned} \quad (39)$$

Then

$$y_1(n - \delta_1(n)) \leq y_1(n) - \left[\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right] \tilde{\sigma}. \quad (40)$$

Thus

$$\begin{aligned} x_1(n - \delta_1(n)) &\leq x_1(n) \exp\left\{ - \left[\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right] \tilde{\sigma} \right\}. \end{aligned} \quad (41)$$

Substituting (36) and (41) into the first equation of (1), we have

$$\begin{aligned}
 x_1(n+1) &\geq x_1(n) \exp \left\{ r_1(n) \left[\frac{K_1(n)}{1 + \sum_{s=0}^{\infty} J_2(s) x_2(n-s)} \right. \right. \\
 &\quad \left. \left. - x_1(n - \sigma_1(n)) \right] \right\} \\
 &\geq x_1(n) \exp \left\{ \frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - x_1(n) \right. \\
 &\quad \left. \times \exp \left\{ - \left[\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right] \tilde{\sigma} \right\} \right\} \quad (42)
 \end{aligned}$$

for all $n > N_3 + \sigma$.

By applying Lemmas 2 and 4, it immediately follows that

$$\lim_{n \rightarrow +\infty} \inf x_1(n) \geq m_1^\varepsilon, \quad (43)$$

where

$$\begin{aligned}
 m_1^\varepsilon &= \frac{r_1^l K_1^l}{r_1^u (1 + M_1 + \varepsilon)} \\
 &\times \exp \left\{ \frac{r_1^l K_1^l}{1 + M_1 + \varepsilon} - r_1^u \right. \\
 &\quad \left. \times \exp \left[- \left(\frac{r_1^l K_1^l}{1 + M_2 + \varepsilon} - (M_1 + \varepsilon) \right) \tilde{\sigma} \right] M_1 \right\}. \quad (44)
 \end{aligned}$$

Setting $\varepsilon \rightarrow 0$ in (43), then

$$\lim_{n \rightarrow +\infty} \inf x_1(n) \geq m_1, \quad (45)$$

where

$$\begin{aligned}
 m_1 &= \frac{r_1^l K_1^l}{r_1^u (1 + M_1)} \\
 &\times \exp \left\{ \frac{r_1^l K_1^l}{1 + M_1} - r_1^u \right. \\
 &\quad \left. \times \exp \left[- \left(\frac{r_1^l K_1^l}{1 + M_2} - M_1 \right) \tilde{\sigma} \right] M_1 \right\}. \quad (46)
 \end{aligned}$$

By the second equation of systems (1) and (36), we can obtain

$$\begin{aligned}
 x_2(n+1) &\geq x_2(n) \\
 &\times \exp \left\{ r_2(n) \left[\frac{K_2(n)}{1 + \sum_{s=0}^{\infty} J_2(s) x_2(n-s)} \right. \right. \\
 &\quad \left. \left. - x_2(n - \sigma_2(n)) \right] \right\} \quad (47) \\
 &\geq x_2(n) \exp \left\{ \frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right\}
 \end{aligned}$$

for all $n > N_3 + \sigma$.

Let $x_2(n) = \exp\{y_2(n)\}$; then (47) is equivalent to

$$y_2(n+1) - y_2(n) \geq \frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon). \quad (48)$$

Summing both sides of (48) from $n - \delta_2(n)$ to $n - 1$ leads to

$$\begin{aligned}
 &\sum_{j=n-\delta_2(n)}^{n-1} (y_2(n+1) - y_2(n)) \\
 &\geq \sum_{j=n-\delta_2(n)}^{n-1} \left[\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right] \quad (49) \\
 &\geq \left[\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right] \tilde{\sigma}.
 \end{aligned}$$

Then

$$y_2(n - \delta_2(n)) \leq y_2(n) - \left[\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right] \tilde{\sigma}. \quad (50)$$

Thus

$$\begin{aligned}
 x_2(n - \delta_2(n)) &\leq x_2(n) \exp \left\{ - \left[\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right] \tilde{\sigma} \right\}. \quad (51)
 \end{aligned}$$

Substituting (36) and (51) into the second equation of (1), we have

$$\begin{aligned}
 x_2(n+1) &\geq x_2(n) \exp \left\{ r_2(n) \left[\frac{K_2(n)}{1 + \sum_{s=0}^{\infty} J_1(s) x_1(n-s)} \right. \right. \\
 &\quad \left. \left. - x_2(n - \sigma_2(n)) \right] \right\} \\
 &\geq x_2(n) \exp \left\{ \frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - x_2(n) \right. \\
 &\quad \left. \times \exp \left\{ - \left[\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right] \tilde{\sigma} \right\} \right\} \quad (52)
 \end{aligned}$$

for all $n > N_3 + \sigma$.

By applying Lemmas 2 and 4, it immediately follows that

$$\lim_{n \rightarrow +\infty} \inf x_2(n) \geq m_2^\varepsilon, \quad (53)$$

where

$$\begin{aligned}
 m_2^\varepsilon &= \frac{r_2^l K_2^l}{r_2^u (1 + M_2 + \varepsilon)} \\
 &\times \exp \left\{ \frac{r_2^l K_2^l}{1 + M_2 + \varepsilon} - r_2^u \right. \\
 &\quad \left. \times \exp \left[- \left(\frac{r_2^l K_2^l}{1 + M_1 + \varepsilon} - (M_2 + \varepsilon) \right) \tilde{\sigma} \right] M_2 \right\}. \quad (54)
 \end{aligned}$$

Setting $\varepsilon \rightarrow 0$ in (53), then

$$\lim_{n \rightarrow +\infty} \inf x_2(n) \geq m_2, \quad (55)$$

where

$$\begin{aligned} m_2 &= \frac{r_2^l K_2^l}{r_2^u (1 + M_2)} \\ &\times \exp \left\{ \frac{r_2^l K_2^l}{1 + M_2} - r_2^u \right. \\ &\quad \left. \times \exp \left[- \left(\frac{r_2^l K_2^l}{1 + M_1} - M_2 \right) \tilde{\sigma} \right] M_2 \right\}. \end{aligned} \quad (56)$$

Without loss of generality, we assume that $\varepsilon < (1/2) \min\{m_1, m_2\}$. For any positive constant ε small enough, it follows from (45) and (55) that there exists enough large $N_4 > N_3 + \sigma$ such that

$$x_1(n) \geq m_1 - \varepsilon, \quad x_2(n) \geq m_2 - \varepsilon \quad (57)$$

for any $n \geq N_4$.

From the third and fourth equations of systems (1) and (57), we can derive that

$$\begin{aligned} \Delta u_1(n) &\geq -\gamma_1(n) u_1(n) + \eta_1(n) (m_1 - \varepsilon), \\ \Delta u_2(n) &\geq -\gamma_2(n) u_2(n) + \eta_2(n) (m_2 - \varepsilon). \end{aligned} \quad (58)$$

Hence

$$\begin{aligned} u_1(n+1) &\geq (1 - \gamma_1^u) u_1(n) + \eta_1^l (m_1 - \varepsilon), \\ u_2(n+1) &\geq (1 - \gamma_2^u) u_2(n) + \eta_2^l (m_2 - \varepsilon). \end{aligned} \quad (59)$$

By applying Lemmas 2 and 3, it immediately follows that

$$\lim_{n \rightarrow +\infty} \inf u_1(n) \geq \frac{\eta_1^l (m_1 - \varepsilon)}{\gamma_1^u}, \quad (60)$$

$$\lim_{n \rightarrow +\infty} \inf u_2(n) \geq \frac{\eta_2^l (m_2 - \varepsilon)}{\gamma_2^u}. \quad (61)$$

Setting $\varepsilon \rightarrow 0$ in the previous inequality leads to

$$\begin{aligned} \lim_{n \rightarrow +\infty} \inf u_1(n) &\geq \frac{\eta_1^l m_1}{\gamma_1^u}, \\ \lim_{n \rightarrow +\infty} \inf u_2(n) &\geq \frac{\eta_2^l m_2}{\gamma_2^u}. \end{aligned} \quad (62)$$

This completes the proof of Theorem 6. $\square$

3. Conclusions

In the present paper, we have investigated the permanence of a discrete mutualism model with infinite deviating arguments and feedback controls. Sufficient conditions which ensure the permanence of the system are established. We have shown the effect of delay to the permanence of system and concluded that delay is an important factor to decide the permanence of the system.

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