Research Article

On Fibonacci Functions with Period $k$

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A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function if $f(x + 2) = f(x + 1) + f(x)$ for all $x \in \mathbb{R}$. In 2012, some properties on the Fibonacci functions were presented. In this paper, for any positive integer $k$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period $k$ if $f(x + 2k) = f(x + k) + f(x)$ for all $x \in \mathbb{R}$; we present some properties on the Fibonacci functions with period $k$.

1. Introduction

Presently, there are many research articles about Fibonacci numbers (see [1]). Fibonacci numbers are also involved in the golden ratio (see [2]). In 2008, Kim and Neggers [3] studied Fibonacci means. In 2009, Jung [4] studied Hyers-Ulam stability of Fibonacci functional equation. In 2010, Han et al. [5] studied a Fibonacci norm of positive integers. In 2012, Han et al. [6] studied Fibonacci sequences in groupoids. Moreover, they [7] gave some properties on Fibonacci functions; a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function if $f(x + 2) = f(x + 1) + f(x)$, for all $x \in \mathbb{R}$, using the concept of $f$-even and $f$-odd functions. They also showed that if $f$ is a Fibonacci function, then $\lim_{x \to \infty} f(x+1)/f(x) = (1+\sqrt{5})/2$.

In this paper, for any positive integer $k$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period $k$ if $f(x + 2k) = f(x + k) + f(x)$ for all $x \in \mathbb{R}$; we present some properties on the Fibonacci functions with period $k$. Moreover, we also present some properties on the odd Fibonacci functions with period $k$.

2. Fibonacci Functions with Period $k$

Definition 1. Let $k$ be a positive integer. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a Fibonacci function with period $k$ if $f(x + 2k) = f(x + k) + f(x)$ for all $x \in \mathbb{R}$.

Example 2. Let $f(x) = a^{x/k}$ be a Fibonacci function with period $k \in \mathbb{N}$, where $a > 0$. It follows that $a^{x/(k+2)} = a^{x/(k+1)} + a^{x/k}$ for all $x \in \mathbb{R}$, so $a^2 = a + 1$. Then $a = (1 + \sqrt{5})/2$. Thus, $f(x) = ((1 + \sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$.

Proposition 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$. Assume that $f$ is differentiable. Then $f'$ is also a Fibonacci function with period $k$.

Proof. Let $x \in \mathbb{R}$. Since $f(x + 2k) = f(x + k) + f(x)$, it follows that $f'(x + 2k) = f'(x + k) + f'(x)$.

Proposition 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$, and define $g_t(x) = f(x + t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then $g_t$ is a Fibonacci function with period $k$.

Proof. Let $x \in \mathbb{R}$. Then $g_t(x + 2k) = f(x + 2k + t) = f(x + t + k) + f(x + t) = g_t(x + k) + g_t(x)$.

Example 5. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by $g_t(x) = ((1 + \sqrt{5})/2)^{(x+t)/k}$ for all $x \in \mathbb{R}$. Then $g_t$ is a Fibonacci function with period $k$.

Theorem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Fibonacci function with period $k \in \mathbb{N}$, and let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of Fibonacci numbers with $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(x + nk) = F_n f(x + k) + F_{n-1} f(x)$.
Proof. Let $x \in \mathbb{R}$. We note that $f(x + k) = F_1 f(x + k) + F_0 f(x)$ and $f(x + 2k) = F_2 f(x + k) + F_1 f(x)$. Now, we assume that $f(x + nk) = F_n f(x + k) + F_{n-1} f(x)$ and $f(x + (n + 1)k) = F_{n+1} f(x + k) + F_n f(x)$, where $n \in \mathbb{N}$. Then

\[
\begin{align*}
  f(x + (n + 2)k) &= f(x + (n + 1)k) + f(x + nk) \\
  &= F_{n+1} f(x + k) + F_n f(x) + F_n f(x + k) + F_{n-1} f(x) \\
  &= (F_{n+1} + F_n) f(x + k) + (F_n + F_{n-1}) f(x) \\
  &= F_{n+2} f(x + k) + F_{n+1} f(x).
\end{align*}
\]

This proof is completed. \(\square\)

3. Odd Fibonacci Functions with Period $k$

Definition 7. Let $k$ be a positive integer. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be an odd Fibonacci function with period $k$ if $f(x + 2k) = -f(x + k) + f(x)$ for all $x \in \mathbb{R}$.

Example 8. Let $f(x) = a^{x/k}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, where $a > 0$. It follows that $a^{(x/k)+2} = -a^{x/k} + a^{x/k}$ for all $x \in \mathbb{R}$, so $a^2 = a + 1$. Then $a = (1 + \sqrt{5})/2$. Thus, $f(x) = ((1 + \sqrt{5})/2)^{x/k}$ for all $x \in \mathbb{R}$.

Proposition 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$. Assume that $f$ is differentiable. Then $f'$ is also an odd Fibonacci function with period $k$.

Proof. Let $x \in \mathbb{R}$. Since $f(x + 2k) = -f(x + k) + f(x)$, it follows that $f'(x + 2k) = -f'(x + k) + f'(x)$.

\(\square\)

Proposition 10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, and define $g_t(x) = f(x + t)$ for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then $g_t$ is also an odd Fibonacci function with period $k$.

Proof. Let $x \in \mathbb{R}$. Then $g_t(x + 2k) = f(x + 2k + t) = -f(x + t + k) + f(x + t) = -g_t(x + k) + g_t(x)$.

\(\square\)

Example 11. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \rightarrow \mathbb{R}$ by $g_t(x) = ((1 + \sqrt{5})/2)^{(x+t)/k}$ for all $x \in \mathbb{R}$. Then $g_t$ is an odd Fibonacci function with period $k$.

Theorem 12. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd Fibonacci function with period $k \in \mathbb{N}$, and let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of Fibonacci numbers with $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = -F_n + F_{n-1}$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(x + nk) = F_n f(x + k) + F_{n-1} f(x)$.

Proof. Let $x \in \mathbb{R}$. We note that $f(x + k) = F_1 f(x + k) + F_0 f(x)$ and $f(x + 2k) = F_2 f(x + k) + F_1 f(x)$. Now, we assume that $f(x + nk) = F_n f(x + k) + F_{n-1} f(x)$ and $f(x + (n + 1)k) = F_{n+1} f(x + k) + F_n f(x)$, where $n \in \mathbb{N}$. Then

\[
\begin{align*}
f(x + nk) &= f(x + (n - 1)k) + f(x + nk) \\
&= F_{n-1} f(x + k) + F_{n-2} f(x) + F_n f(x + k) + F_{n-1} f(x) \\
&= (F_{n-1} + F_n) f(x + k) + (F_{n-2} + F_{n-1}) f(x) \\
&= F_{n+1} f(x + k) + F_n f(x).
\end{align*}
\]

This proof is completed. \(\square\)

4. $f$-Even Functions with Period $k$

Definition 13. Let $k$ be a positive integer and let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be such that if $ah = 0$, where $h : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $\alpha h = 0$. The function $\alpha$ is said to be an $f$-even function with period $k$ if $\alpha(x + k) = \alpha(x)$ for all $x \in \mathbb{R}$.

Example 14. Define $\alpha(x) = x - [x]$ for all $x \in \mathbb{R}$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $ah = 0$. For any $x \notin \mathbb{Z}$, we have $\alpha(x) \neq 0$, so $h(x) = 0$. Since $\mathbb{R} \setminus \mathbb{Z}$ is dense in $\mathbb{R}$ and $h$ is continuous, it follows that $h = 0$. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\alpha(x + k) = x + k - [x + k] = x + k - [x] = x - [x] = \alpha(x)$. Hence, $\alpha$ is an $f$-even function with period $k$.

Theorem 15. Let $k \in \mathbb{N}$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be an $f$-even function with period $k$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then $g$ is a Fibonacci function with period $k$ if and only if $\alpha g$ is a Fibonacci function with period $k$.

Proof. First, we assume that $g$ is a Fibonacci function with period $k$. For any $x \in \mathbb{R}$, we have

\[
\begin{align*}
(\alpha g)(x + 2k) &= \alpha (x + 2k) g(x + 2k) \\
&= \alpha (x + k) (g(x + k) + g(x)) \\
&= \alpha (x + k) g(x) + \alpha (x + k) g(x) \\
&= \alpha (x + k) g(x) + \alpha (x + k) g(x) \\
&= (\alpha g)(x + k)\end{align*}
\]

Hence, $\alpha g$ is a Fibonacci function with period $k$. Next, we assume that $\alpha g$ is a Fibonacci function with period $k$. Let $x \in \mathbb{R}$. Then \[\alpha (x + k) g(x + 2k) = \alpha (x + 2k) g(x + 2k) = \alpha (x + k) (g(x + k) + g(x)) = \alpha g(x) + (\alpha g)(x)\]
\[ = \alpha (x + k) g (x + k) + \alpha (x) g (x) \]
\[ = \alpha (x + k) g (x + k) + \alpha (x + k) g (x) \]
\[ = \alpha (x + k) (g (x + k) + g (x)). \]  

(4)

By the assumption of \( \alpha \), we obtain that \( g(x + 2k) = g(x + k) + g(x) \). Hence, \( g \) is a Fibonacci function with period \( k \).

**Theorem 17.** Let \( k \in \mathbb{N} \) and \( \alpha : \mathbb{R} \to \mathbb{R} \) be an \( f \)-even function with period \( k \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function. Then \( g \) is an odd Fibonacci function with period \( k \) if and only if \( ag \) is an odd Fibonacci function with period \( k \).

**Proof.** First, we assume that \( g \) is an odd Fibonacci function with period \( k \). For any \( x \in \mathbb{R} \), we have

\[ (ag) (x + 2k) \]
\[ = \alpha (x + 2k) g (x + 2k) \]
\[ = \alpha (x + 2k) (-g (x + k) + g (x)) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x + k) g (x) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x) g (x) \]
\[ = -\alpha (x + k) g (x + k) + (ag) (x). \]

Hence, \( ag \) is an odd Fibonacci function with period \( k \).

Next, we assume that \( ag \) is an odd Fibonacci function with period \( k \). Let \( x \in \mathbb{R} \). Then

\[ \alpha (x + k) g (x + 2k) \]
\[ = \alpha (x + 2k) g (x + 2k) \]
\[ = (ag) (x + 2k) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x + k) g (x) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x) g (x) \]
\[ = -\alpha (x + k) g (x + k) + (ag) (x). \]

(5)

By the assumption of \( \alpha \), we obtain that \( g(x + 2k) = -g(x + k) + g(x) \). Hence, \( g \) is an odd Fibonacci function with period \( k \).

**Example 16.** Let \( k \in \mathbb{N} \). Define \( \alpha(x) = x - \lfloor x \rfloor \) and \( g(x) = ((1 + \sqrt{5})/2)^x/k \) for all \( x \in \mathbb{R} \). For all \( x \in \mathbb{R} \), we have \( ag(x) = (x - \lfloor x \rfloor)((1 + \sqrt{5})/2)^x/k \). We recall that \( \alpha \) is an \( f \)-even function with period \( k \) and \( g \) is an odd Fibonacci function with period \( k \). Hence, \( ag \) is an odd Fibonacci function with period \( k \).

**5. \( f \)-Odd Functions with Period \( k \)**

**Definition 19.** Let \( k \) be a positive integer and let \( \alpha : \mathbb{R} \to \mathbb{R} \) be such that if \( ah = 0 \) where \( h : \mathbb{R} \to \mathbb{R} \) is continuous, then \( h = 0 \). The function \( \alpha \) is said to be an \( f \)-odd function with period \( k \) if \( \alpha(x + k) = -\alpha(x) \) for all \( x \in \mathbb{R} \).

**Example 20.** Define \( \alpha(x) = \sin(\pi x) \) for all \( x \in \mathbb{R} \). Let \( h : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( ah \neq 0 \). For any \( x \in \pi \mathbb{Z} \), we have \( \alpha(x) = 0 \). Since \( \mathbb{R} \setminus \pi \mathbb{Z} \) is dense in \( \mathbb{R} \) and \( h \) is continuous, it follows that \( h = 0 \). Let \( k \) be a positive odd integer and \( x \in \mathbb{R} \). Then \( \alpha(x + k) = \sin(\pi x + \pi k) = -\sin(\pi x) = -\alpha(x) \). Hence, \( \alpha \) is an \( f \)-even function with period \( k \).

**Theorem 21.** Let \( k \in \mathbb{N} \) and \( \alpha : \mathbb{R} \to \mathbb{R} \) be an \( f \)-odd function with period \( k \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function. Then \( g \) is a Fibonacci function with period \( k \) if and only if \( ag \) is an odd Fibonacci function with period \( k \).

**Proof.** First, we assume that \( g \) is a Fibonacci function with period \( k \). For any \( x \in \mathbb{R} \), we have

\[ (ag) (x + 2k) \]
\[ = \alpha (x + 2k) g (x + 2k) \]
\[ = -\alpha (x + k) (g (x + k) + g (x)) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x + k) g (x) \]
\[ = -\alpha (x + k) g (x + k) + (ag) (x). \]

Hence, \( ag \) is an odd Fibonacci function with period \( k \).

Next, we assume that \( ag \) is an odd Fibonacci function with period \( k \). Let \( x \in \mathbb{R} \). Then

\[ \alpha (x + k) g (x + 2k) \]
\[ = -\alpha (x + k) g (x + 2k) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x) g (x) \]
\[ = -\alpha (x + k) g (x + k) + (ag) (x) \]
\[ = (ag) (x + k) - (ag) (x). \]

Hence, \( ag \) is an odd Fibonacci function with period \( k \).

Next, we assume that \( ag \) is an odd Fibonacci function with period \( k \). Let \( x \in \mathbb{R} \). Then

\[ \alpha (x + k) g (x + 2k) \]
\[ = -\alpha (x + k) g (x + 2k) \]
\[ = -\alpha (x + k) g (x + k) + \alpha (x) g (x) \]
\[ = -\alpha (x + k) g (x + k) + (ag) (x) \]
\[ = (ag) (x + k) - (ag) (x). \]

(7)

(8)

By the assumption of \( \alpha \), we obtain that \( g(x + 2k) = g(x + k) + g(x) \). Hence, \( g \) is a Fibonacci function with period \( k \).
Example 22. Let \( k \) be a positive odd integer. Define \( \alpha(x) = \sin(\pi x) \) and \( g(x) = ((1 + \sqrt{5})/2)^{x/k} \) for all \( x \in \mathbb{R} \). We have \( \alpha g(x) = (\sin(\pi x))(1 + \sqrt{5}/2)^{x/k} \) for all \( x \in \mathbb{R} \). We recall that \( \alpha \) is an \( f \)-odd function with period \( k \) and \( g \) is a Fibonacci function with period \( k \). Hence, \( \alpha g \) is an odd Fibonacci function with period \( k \).

\[ \alpha g(x) = \frac{1 + \sqrt{5}}{2} \]

Theorem 23. Let \( k \in \mathbb{N} \) and \( \alpha : \mathbb{R} \to \mathbb{R} \) be an \( f \)-odd function with period \( k \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function. Then \( g \) is an odd Fibonacci function with period \( k \) if and only if \( \alpha g \) is a Fibonacci function with period \( k \).

Proof. First, we assume that \( g \) is an odd Fibonacci function with period \( k \). For any \( x \in \mathbb{R} \), we have

\[ (\alpha g)(x + 2k) = \alpha(x + 2k)g(x + 2k) \]
\[ = -\alpha(x + k)(-g(x + k) + g(x)) \]
\[ = \alpha(x + k)g(x + k) - \alpha(x + k)g(x) \]
\[ = \alpha(x + k)g(x + k) + \alpha(x)g(x) \]
\[ = (\alpha g)(x + k) + (\alpha g)(x). \]

Hence, \( \alpha g \) is a Fibonacci function with period \( k \).

Next, we assume that \( \alpha g \) is a Fibonacci function with period \( k \). Let \( x \in \mathbb{R} \). Then

\[ \alpha(x + k)g(x + 2k) \]
\[ = -\alpha(x + 2k)g(x + 2k) \]
\[ = -(\alpha g)(x + 2k) \]
\[ = -((\alpha g)(x + k) + (\alpha g)(x)) \]
\[ = -(\alpha g)(x + k) - (\alpha g)(x) \]
\[ = -\alpha(x + k)g(x + k) - \alpha(x)g(x) \]
\[ = -\alpha(x + k)g(x + k) + \alpha(x + k)g(x) \]
\[ = \alpha(x + k)(-g(x + k) + g(x)). \]

By the assumption of \( \alpha \), we obtain that \( g(x + 2k) = -g(x + k) + g(x) \). Hence, \( g \) is an odd Fibonacci function with period \( k \).

Example 24. Let \( k \) be a positive odd integer. Define \( \alpha(x) = \sin(\pi x) \) and \( g(x) = ((-1 + \sqrt{5})/2)^{x/k} \) for all \( x \in \mathbb{R} \). We have \( \alpha g(x) = (\sin(\pi x))((-1 + \sqrt{5}/2)^{x/k} \) for all \( x \in \mathbb{R} \). We recall that \( \alpha \) is an \( f \)-odd function with period \( k \) and \( g \) is an odd Fibonacci function with period \( k \). Hence, \( \alpha g \) is a Fibonacci function with period \( k \).

6. Open Problems

Conjecture 25. If \( f \) is a Fibonacci function with period \( k \in \mathbb{N} \), then

\[ \lim_{x \to \infty} \frac{f(x + k)}{f(x)} = \frac{1 + \sqrt{5}}{2}. \]

Conjecture 26. If \( f \) is an odd Fibonacci function with period \( k \in \mathbb{N} \), then

\[ \lim_{x \to \infty} \frac{f(x + k)}{f(x)} = -\frac{1 - \sqrt{5}}{2}. \]

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References
