Research Article

General Solution of Linear Fractional Neutral Differential Difference Equations

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Received 2 April 2013; Accepted 27 May 2013

Academic Editor: Pedro R. S. Antunes

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This paper is concerned with the general solution of linear fractional neutral differential difference equations. The exponential estimates of the solution and the variation of constant formula for linear fractional neutral differential difference equations are derived by using the Gronwall integral inequality and the Laplace transform method, respectively. The obtained results extend the corresponding ones of integer order linear ordinary differential equations and delay differential equations.

1. Introduction

Fractional differential equations have been proved to be an excellent tool in the modelling of many phenomena in various fields of engineering, physics, and economics. Many practical systems can be represented more accurately through fractional derivative formulation. For more details on fractional calculus theory, one can see the monographs of Miller and Ross [1], Podlubny [2], Magin [3], Diethelm [4], and Kilbas et al. [5]. Fractional differential equations without delay involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention in [6–20] and references therein. In [5], Kilbas et al. have discussed the explicit solutions of linear fractional ordinary differential equations based on the method of successive approximations. In [6, 7], the theory of inequalities, local existence, extremal solutions, comparison results, and global existence of the solutions of fractional differential equations are established. In [8], Li et al. have developed an operator theory to study fractional Cauchy problems with the Riemann-Liouville fractional derivatives in infinite-dimensional Banach spaces. In [9], Bonilla et al. have considered the explicit solution for linear fractional ordinary differential equations employing the exponential

matrix function and the fractional Green function. In [10], Odibat has derived the exact solution for the initial value problems of linear fractional ordinary differential systems by analytical approaches.

On the other hand, time delays are present inherently in many interconnected real systems due to transportation of energy and materials. For example, feedback control systems containing time delays and fractional processes and controllers lead to fractional delay systems. While delay differential systems with integer order have been thoroughly investigated during the past decades (see [21–23] and references therein), the research of fractional delay differential systems is still in the initial and developing stage [24–29].

Motivated and inspired by the mentioned works, in this paper, we investigate the representation of the general solution to linear fractional neutral differential difference system with the form

$$\begin{aligned} {}^c D^\alpha [x(t) - Cx(t - \tau)] &= Ax(t) + Bx(t - \tau) + f(t), \quad t \geq 0, \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \quad (1)$$

where ${}^c D^\alpha x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, $0 < \alpha \leq 1$, A, B are $n \times n$ constant matrices,

τ is a constant with $\tau > 0$, $f(t)$ is a n -dimensional continuous vector-valued function, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, and $C^1([-\tau, 0], \mathbb{R}^n)$ denotes space of continuously differentiable functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$.

As we all know, the Laplace transform method is an effective and convenient method for solving linear fractional differential equations. The exponential estimate of the solution is an indispensable tool, which guarantees the rationality of solving fractional differential equations by the Laplace transform method. In [11], Lin failed to take into consideration the exponential estimates of the solution, but the existence of the Laplace transform was taken for granted to solve fractional differential equations. The earlier studies concerning the Laplace transform of fractional differential equations can be found in [1–5, 11, 16], which focused especially on the nondelayed case.

The purpose of this paper is to construct the representation of the general solution for system (1) by using the Laplace transform method. The exponential estimate for system (1) is presented by using the Gronwall integral inequality, which is basic to apply the Laplace transform. Moreover, the expression of the general solution for the homogeneous system and the variation of constant formula for system (1) are derived. The obtained results extend the corresponding ones of integer order linear ordinary differential equations and delay differential equations.

This paper is organized as follows. In the next section, we present some definitions and preliminary facts used in the paper. In Section 3, we derive the exponential estimates of the solutions for system (1), the expression of the general solution for the homogeneous system, and the variation of constant formula for system (1).

2. Preliminaries

In this section, we recall some definitions and preliminary facts which are used throughout this paper. For more details, one can see [1–5].

Definition 1. The Riemann-Liouville's fractional integral of order $\alpha > 0$ for a function $f: R^+ \rightarrow R^n$ is defined as

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \quad (2)$$

Definition 2. The Caputo's fractional derivative of α ($0 \leq m \leq \alpha < m+1$) for a function $f: R^+ \rightarrow R^n$ is defined as

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha+1)} \int_0^t (t-s)^{m-\alpha} f^{(m+1)}(s) ds. \quad (3)$$

Definition 3. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \quad (4)$$

where $f(t)$ is n -dimensional vector-valued function.

From the above definitions, we know that if the integral (4) is convergent at the point $s_0 \in \mathbb{C}$, then it converges

absolutely for $s \in \mathbb{C}$ such that $\Re(s) > \Re(s_0)$. Moreover, for $\alpha \in (0, 1]$, we have

$$\begin{aligned} \mathcal{L}[{}^c D^\alpha f(t)] &= s^\alpha \mathcal{L}[f(t)] - s^{\alpha-1} f(0), \\ \mathcal{L}[{}^c D^{-\alpha} f(t)] &= s^{-\alpha} \mathcal{L}[f(t)]. \end{aligned} \quad (5)$$

Lemma 4 (Gronwall integral inequality [21]). *If $x(t)$ and $u(t)$ are real valued continuous functions on $[a, b]$, and $u(t)$ is a nondecreasing function on $[a, b]$. In addition, $g(t) \geq 0$ is integrable on $[a, b]$ with*

$$x(t) \leq u(t) + \int_a^t g(s) x(s) ds, \quad t \in [a, b], \quad (6)$$

then

$$x(t) \leq u(t) e^{\int_a^t g(s) ds}, \quad t \in [a, b]. \quad (7)$$

3. Main Results

In this section, we derive the exponential estimation of the solution $x(t)$ for system (1) which depends on φ and f based on Gronwall integral inequality. These estimates are basic to the applications of the Laplace transform, and we obtain the representation of the general solution for linear homogeneous fractional neutral system and the variation of constant formula for linear nonhomogeneous fractional neutral system.

Theorem 5. *Assume system (1) has a unique continuous solution $x(t)$, if $f(t)$ is continuous on $[0, +\infty)$, then there exist positive constants $M > 0$ and $\sigma > 0$, such that*

$$\|x(t)\| \leq \left[M \|\varphi\| + \frac{\sigma \tau^\alpha t^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t)\| \right] e^{\sigma t}, \quad (8)$$

where $\|F(t)\| = \sup_{s \in [0, t]} \|f(s)\|$, $N = (\|A\| + \|B\|)/\Gamma(\alpha+1)$.

Proof. For $t \geq 0$, system (1) is equivalent to the following Volterra integral equation:

$$\begin{aligned} x(t) &= \varphi(0) - C\varphi(-\tau) + Cx(t-\tau) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Ax(s) + Bx(s-\tau)] ds, \quad t \geq 0. \end{aligned} \quad (9)$$

Let $\|\varphi\| = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$. Applying the appropriate property of the norm, it follows that

$$\begin{aligned} \|x(t)\| &\leq \|\varphi(0)\| + \|C\| \|\varphi(-\tau)\| + \|C\| \|x(t-\tau)\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times [\|A\| \|x(s)\| + \|B\| \|x(s-\tau)\|] ds. \end{aligned} \quad (10)$$

Let $\|y(t)\| = \sup_{\theta \in [-\tau, 0]} \|x(t + \theta)\|$. For $0 \leq t \leq \tau$, then

$$\begin{aligned} \|y(t)\| &\leq (1 + 2\|C\|) \|\varphi\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times [\|A\| \|y(s)\| + \|B\| \|y(s)\| + \|f(s)\|] ds \\ &= (1 + 2\|C\|) \|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \\ &\quad + \frac{\|A\| + \|B\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\| ds. \end{aligned} \quad (11)$$

An application of Lemma 4 yields that

$$\begin{aligned} \|y(t)\| &\leq \left[(1 + 2\|C\|) \|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \right] \\ &\quad \times e^{((\|A\| + \|B\|)/\Gamma(\alpha+1))t^\alpha}, \quad t \in [0, \tau]. \end{aligned} \quad (12)$$

Let $M = 1 + 2\|C\|$ and $N = (\|A\| + \|B\|)/\Gamma(\alpha+1)$, then (12) can be estimated as

$$\|y(t)\| \leq \left[M \|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \right] e^{Nt^\alpha}, \quad t \in [0, \tau]. \quad (13)$$

Moreover, the same argument implies the following estimation:

$$\begin{aligned} \|y(t)\| &\leq \left[M \|y(t_0)\| + \frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \right] e^{N(t-t_0)^\alpha}, \\ &\quad t \in [t_0, t_0 + \tau]. \end{aligned} \quad (14)$$

Next, we need to prove that

$$\begin{aligned} \|y(t)\| &\leq \left[M \|\varphi\| + \frac{\sigma \tau^\alpha t^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t)\| \right] e^{\sigma t^\alpha}, \\ &\quad t \in [0, k\tau], \end{aligned} \quad (15)$$

where $N = (\|A\| + \|B\|)/\Gamma(\alpha+1)$, $\sigma > N + \tau^{-\alpha}$. According to (13), we know that (15) is true for $t \in [0, \tau]$. Assume that (15) is true for $t \in [0, k\tau]$, $k \geq 1$. By this hypothesis, we need to

prove that (15) is true for $t \in [0, (k+1)\tau]$. For $t \in [\tau, (k+1)\tau]$, we denote $t_1 = t - \tau \in [0, k\tau]$, then it yields that

$$\begin{aligned} \|y(t)\| &\leq \left[M \|y(t_1)\| + \frac{(t-t_1)^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \right] e^{N(t-t_1)^\alpha} \\ &= \left[M \|y(t_1)\| + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \|F(t)\| \right] e^{N\tau^\alpha} \\ &\leq \left\{ M \left[M \|\varphi\| + \frac{\sigma \tau^\alpha (t-\tau)^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t-\tau)\| \right] \right. \\ &\quad \times e^{\sigma(t-\tau)^\alpha} + \frac{\tau^\alpha \|F(t)\|}{\Gamma(\alpha+1)} \left. \right\} e^{N\tau^\alpha} \\ &\leq \left\{ \left[M \|\varphi\| + \frac{\sigma \tau^\alpha (t-\tau)^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t-\tau)\| \right] \right. \\ &\quad \times e^{\sigma(t-\tau)^\alpha + N\tau^\alpha + \ln M} + \frac{\tau^\alpha \|F(t)\|}{\Gamma(\alpha+1)} \left. \right\} e^{N\tau^\alpha} \\ &\leq \left\{ \left[M \|\varphi\| + \frac{\sigma \tau^\alpha (t-\tau)^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t)\| \right] \right. \\ &\quad \times e^{\sigma(t-\tau)^\alpha + N\tau^\alpha + \ln M} + \frac{\tau^\alpha \|F(t)\|}{\Gamma(\alpha+1)} \left. \right\} e^{N\tau^\alpha} \\ &\leq \left[M \|\varphi\| + \frac{\sigma \tau^\alpha (t-\tau)^\alpha + \tau^\alpha (\ln M + N^\alpha)}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t)\| \right] e^{\sigma t^\alpha} \\ &\leq \left[M \|\varphi\| + \frac{\sigma \tau^\alpha t^\alpha}{\Gamma(\alpha+1)(\ln M + N^\alpha)} \|F(t)\| \right] e^{\sigma t^\alpha}. \end{aligned} \quad (16)$$

Thus, the proof is completed. $\square$

Remark 6. From the proof of Theorem 5, we know that the exponential estimation of the solution $x(t)$ for system (1) depends on φ and f , then we denote solution $x(t)$ for system (1) as $x(t, \varphi, f)$.

Remark 7. From Theorem 5, if $f(t)$ is continuous on $[0, +\infty)$ and exponentially bounded, then $x(t)$ is exponentially bounded, which is basic to apply the Laplace transform.

Next, we consider the general solution of linear homogeneous equation of the form

$${}^c D^\alpha [x(t) - Cx(t-\tau)] = Ax(t) + Bx(t-\tau). \quad (17)$$

The fundamental solution of the homogeneous system (17) is defined as follows:

Let $X(t) \in \mathbb{R}^{n \times n}$, and satisfy

$${}^c D^\alpha [X(t) - CX(t-\tau)] = AX(t) + BX(t-\tau),$$

$$X(t) = \begin{cases} I, & t = 0, \\ 0, & t \in [-\tau, 0), \end{cases} \quad (18)$$

then $X(t)$ is called the corresponding fundamental solution of system (17).

Let

$$H(s) = s^\alpha I - A - e^{-s\tau} B - s^\alpha e^{-s\tau} C \quad (19)$$

and $\mathcal{E}^{-1}$ denote inverse transform of the Laplace transform. Applying the Laplace transform to system (17), from (18), we have

$$X(t) = \mathcal{E}^{-1} [s^{\alpha-1} H^{-1}(s)]. \quad (20)$$

In terms of the fundamental solution $X(t)$ of system (17), the general solution $x(t, \varphi, 0)$ for system (17) can be represented in the following theorem.

Theorem 8. *If $X(t)$ is the fundamental solution of system (17), then the general solution $x(t, \varphi, 0)$ of system (17) can be represented in the following form:*

$$\begin{aligned} x(t, \varphi, 0) = & X(t) [\varphi(0) - C\varphi(-\tau)] \\ & + B \int_{-\tau}^0 \left[({}^c D^{1-\alpha} X)(t - \tau - \theta) \right] \varphi(\theta) d\theta \\ & + B \int_{-\tau}^0 \frac{(t - \tau - \theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\ & + C \int_{-\tau}^0 \dot{X}(t - \tau - \theta) \varphi(\theta) d\theta \\ & - C \int_{-\tau}^0 [dX(t - \tau - \theta)] \varphi(\theta) d\theta. \end{aligned} \quad (21)$$

Proof. Applying the Laplace transform to system (17), we have

$$\begin{aligned} s^\alpha \mathcal{E}[x(t) - Cx(t - \tau)] - s^{\alpha-1} \varphi(0) + s^{\alpha-1} C\varphi(-\tau) \\ = A\mathcal{E}[x(t)] + B\mathcal{E}[x(t - \tau)]. \end{aligned} \quad (22)$$

It follows from the properties of integral that

$$\begin{aligned} \mathcal{E}[x(t - \tau)] &= \int_0^{+\infty} e^{-st} x(t - \tau) dt \\ &= e^{-s\tau} \mathcal{E}[x(t)] + e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt. \end{aligned} \quad (23)$$

Taking into account (22) and (23), it yields that

$$\begin{aligned} H(s) \mathcal{E}[x(t)] &= s^{\alpha-1} \varphi(0) - s^{\alpha-1} C\varphi(-\tau) \\ &+ B e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) \\ &+ s^\alpha C e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t). \end{aligned} \quad (24)$$

Note that $H(s) = s^\alpha I - A - e^{-s\tau} B - s^\alpha e^{-s\tau} C$ and $s^{\alpha-1} H^{-1}(s) = \mathcal{E}[X(t)]$, we have

$$\begin{aligned} \mathcal{E}[x(t)] &= \mathcal{E}[X(t)] [\varphi(0) - C\varphi(-\tau)] \\ &+ \mathcal{E}[X(t)] B s^{1-\alpha} e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt \\ &+ \mathcal{E}[X(t)] C s e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt. \end{aligned} \quad (25)$$

In order to apply the convolution theorem to the terms of the right in (25), we define a function $\omega(\cdot)$ and extend the initial function $\varphi(\cdot)$ as follows:

$$\begin{aligned} \omega(t) &= \begin{cases} 0, & t \geq 0, \\ 1, & t \in [-\tau, 0), \end{cases} \\ \hat{\varphi}(t) &= \begin{cases} \varphi(0), & t \geq 0, \\ \varphi(t), & t \in [-\tau, 0). \end{cases} \end{aligned} \quad (26)$$

Therefor, we have

$$\begin{aligned} \mathcal{E}[X(t)] B s^{1-\alpha} e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt \\ = B s^{1-\alpha} \mathcal{E}[X(t)] e^{-s\tau} \int_{-\tau}^{+\infty} e^{-st} \hat{\varphi}(t) \omega(t) dt \\ = B s^{1-\alpha} \mathcal{E}[X(t)] \int_0^{+\infty} e^{-st} \hat{\varphi}(t - \tau) \omega(t - \tau) dt \\ = B s^{1-\alpha} \mathcal{E}[X(t)] \mathcal{E}[\hat{\varphi}(t - \tau) \omega(t - \tau)]. \end{aligned} \quad (27)$$

Moreover, the same argument implies the following equality:

$$\begin{aligned} \mathcal{E}[X(t)] C s e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt \\ = C s \mathcal{E}[X(t)] \mathcal{E}[\hat{\varphi}(t - \tau) \omega(t - \tau)]. \end{aligned} \quad (28)$$

Since

$$\begin{aligned} \mathcal{E}[({}^c D^{1-\alpha} X)(t)] &= s^{1-\alpha} \mathcal{E}[X(t)] - s^{-\alpha} X(0), \\ X(0) = I, \quad s^{-\alpha} \mathcal{E}[x(t)] &= \mathcal{E}[D^{-\alpha} x(t)], \end{aligned} \quad (29)$$

then

$$\begin{aligned} \mathcal{E}[x(t)] &= \mathcal{E}[X(t)] [\varphi(0) - C\varphi(-\tau)] \\ &+ B \mathcal{E}[({}^c D^{1-\alpha} X)(t)] \mathcal{E}[\hat{\varphi}(t - \tau) \omega(t - \tau)] \\ &+ B \mathcal{E}[D^{-\alpha} [\hat{\varphi}(t - \tau) \omega(t - \tau)]] \\ &+ C \mathcal{E}[\dot{X}(t)] \mathcal{E}[\hat{\varphi}(t - \tau) \omega(t - \tau)] \\ &+ C \mathcal{E}[\hat{\varphi}(t - \tau) \omega(t - \tau)]. \end{aligned} \quad (30)$$

The convolution theorem and inverse theorem of the Laplace transform applied to (30) yields the form

$$\begin{aligned}
 x(t) &= X(t) [\varphi(0) - C\varphi(-\tau)] + B \left[({}^c D^{1-\alpha} X)(t) \right] \\
 &\quad * [\widehat{\varphi}(t-\tau) \omega(t-\tau)] \\
 &\quad + BD^{-\alpha} [\widehat{\varphi}(t-\tau) \omega(t-\tau)] \\
 &\quad + C [\dot{X}(t)] * [\widehat{\varphi}(t-\tau) \omega(t-\tau)] \\
 &\quad + C\widehat{\varphi}(t-\tau) \omega(t-\tau) \\
 &= X(t) [\varphi(0) - C\varphi(-\tau)] \\
 &\quad + B \int_0^t [({}^c D^{1-\alpha} X)(t-s)] \widehat{\varphi}(s-\tau) \omega(s-\tau) ds \\
 &\quad + B \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \widehat{\varphi}(s-\tau) \omega(s-\tau) ds \\
 &\quad + C \int_0^t \dot{X}(t-s) \widehat{\varphi}(s-\tau) \omega(s-\tau) ds \\
 &\quad + C\widehat{\varphi}(t-\tau) \omega(t-\tau) \\
 &= X(t) [\varphi(0) - C\varphi(-\tau)] \\
 &\quad + B \int_0^\tau [({}^c D^{1-\alpha} X)(t-s)] \varphi(s-\tau) ds \\
 &\quad + B \int_0^\tau \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s-\tau) ds \\
 &\quad + C \int_0^\tau \dot{X}(t-s) \varphi(s-\tau) ds \\
 &\quad + C\varphi(t-\tau) \omega(t-\tau).
 \end{aligned} \tag{31}$$

Let $s = \tau + \theta$, then it follows from (31) that

$$\begin{aligned}
 x(t) &= X(t) [\varphi(0) - C\varphi(-\tau)] \\
 &\quad + B \int_{-\tau}^0 [({}^c D^{1-\alpha} X)(t-\tau-\theta)] \varphi(\theta) d\theta \\
 &\quad + B \int_{-\tau}^0 \frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\
 &\quad + C \int_{-\tau}^0 \dot{X}(t-\tau-\theta) \varphi(\theta) d\theta \\
 &\quad + C\varphi(t-\tau) \omega(t-\tau),
 \end{aligned} \tag{32}$$

where $\omega(\theta) = 1$ for $\theta < 0$ and $\omega(\theta) = 0$ for $\theta \geq 0$. If $t \geq \tau$, the term involving ω is not present. If $t < \tau$, then this term is $C\varphi(t-\tau)$ which is precisely the value of the Stieltjes integral

$-\int_{-\tau}^{t-\tau} [dX(t-\tau-\theta)] \varphi(\theta) d\theta$. Therefore, if we make use of the Stieltjes integral, the relation for $x(t)$ can be written as

$$\begin{aligned}
 x(t) &= X(t) [\varphi(0) - C\varphi(-\tau)] \\
 &\quad + B \int_{-\tau}^0 [({}^c D^{1-\alpha} X)(t-\tau-\theta)] \varphi(\theta) d\theta \\
 &\quad + B \int_{-\tau}^0 \frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\
 &\quad + C \int_{-\tau}^0 \dot{X}(t-\tau-\theta) \varphi(\theta) d\theta \\
 &\quad - C \int_{-\tau}^0 [dX(t-\tau-\theta)] \varphi(\theta) d\theta.
 \end{aligned} \tag{33}$$

The proof is thus completed. $\square$

Remark 9. For the particular case $\alpha = 1$, $B = C = 0$, then formula (21) becomes

$$x(t) = e^{At} x(0), \tag{34}$$

which is consistent with the classical result of the first-order linear differential equation with constant coefficient.

Remark 10. For the particular case $\alpha = 1$, $C = 0$, from Definitions 1 and 2, we have

$$x(t) = X(t) \varphi(0) + B \int_{-\tau}^0 X(t-\tau-\theta) \varphi(\theta) d\theta, \tag{35}$$

which is consistent with the general solution of the first-order linear homogeneous delay differential equation [21].

Based on the Laplace transform method, we derive the variation of constant formula for the nonhomogeneous system (1).

Theorem 11. *If $f(t)$ is continuous on $[0, +\infty)$ and exponentially bounded, and $X(t)$ is the fundamental solution of system (17), then the general solution $x^*(t, \varphi, f)$ of system (1) can be represented in the following form:*

$$\begin{aligned}
 x^*(t, \varphi, f) &= X(t) [\varphi(0) - C\varphi(-\tau)] \\
 &\quad + B \int_{-\tau}^0 [({}^c D^{1-\alpha} X)(t-\tau-\theta)] \varphi(\theta) d\theta \\
 &\quad + B \int_{-\tau}^0 \frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\
 &\quad + C \int_{-\tau}^0 \dot{X}(t-\tau-\theta) \varphi(\theta) d\theta + D^{-\alpha} f(t) \\
 &\quad - C \int_{-\tau}^0 [dX(t-\tau-\theta)] \varphi(\theta) d\theta \\
 &\quad + \int_0^t ({}^c D^{1-\alpha} X)(t-\theta) f(\theta) d\theta.
 \end{aligned} \tag{36}$$

Proof. Applying the Laplace transform to system (1), it yields that

$$\begin{aligned} s^\alpha \mathcal{E} [x^*(t) - Cx^*(t - \tau)] - s^{\alpha-1} [\varphi(0) - C\varphi(-\tau)] \\ = A\mathcal{E} [x^*(t)] + B\mathcal{E} [x^*(t - \tau)] + \mathcal{E} [f(t)]. \end{aligned} \quad (37)$$

Note that $H(s) = s^\alpha I - A - e^{-s\tau}B - s^\alpha e^{-s\tau}C$ and $s^{\alpha-1}H^{-1}(s) = \mathcal{E}[X(t)]$, then we have

$$\begin{aligned} \mathcal{E} [x^*(t)] &= \mathcal{E} [X(t)] [\varphi(0) - C\varphi(-\tau)] \\ &\quad + \mathcal{E} [X(t)] B s^{1-\alpha} e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt \\ &\quad + \mathcal{E} [X(t)] C s e^{-s\tau} \int_{-\tau}^0 e^{-st} \varphi(t) dt \\ &\quad + s^{1-\alpha} \mathcal{E} [X(t)] \mathcal{E} [f(t)]. \end{aligned} \quad (38)$$

According to the proof of Theorem 8, we have

$$\mathcal{E} [x^*(t)] = \mathcal{E} [x(t)] + s^{1-\alpha} \mathcal{E} [X(t)] \mathcal{E} [f(t)]. \quad (39)$$

Since

$$\begin{aligned} \mathcal{E} \left[\left({}^c D^{1-\alpha} X \right) (t) \right] &= s^{1-\alpha} \mathcal{E} [X(t)] - s^{-\alpha} X(0), \\ X(0) &= I, \quad s^{-\alpha} \mathcal{E} [f(t)] = \mathcal{E} [D^{-\alpha} f(t)], \end{aligned} \quad (40)$$

then (39) can be represented in the following manner:

$$\begin{aligned} \mathcal{E} [x^*(t)] &= \mathcal{E} [x(t)] + \mathcal{E} \left[\left({}^c D^{1-\alpha} X \right) (t) \right] \mathcal{E} [f(t)] \\ &\quad + \mathcal{E} [D^{-\alpha} f(t)]. \end{aligned} \quad (41)$$

The convolution theorem and inverse theorem of Laplace transform applied to (41) yields the form

$$\begin{aligned} x^*(t, \varphi, f) &= x(t, \varphi, 0) + D^{-\alpha} f(t) \\ &\quad + \int_0^t \left({}^c D^{1-\alpha} X \right) (t-s) f(s) ds. \end{aligned} \quad (42)$$

From Theorem 8, we know that the solution $x(t, \varphi, 0)$ can be represented as

$$\begin{aligned} x(t, \varphi, 0) &= X(t) [\varphi(0) - C\varphi(-\tau)] \\ &\quad + B \int_{-\tau}^0 \left[\left({}^c D^{1-\alpha} X \right) (t - \tau - \theta) \right] \varphi(\theta) d\theta \\ &\quad + B \int_{-\tau}^0 \frac{(t - \tau - \theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\ &\quad + C \int_{-\tau}^0 \dot{X}(t - \tau - \theta) \varphi(\theta) d\theta \\ &\quad - C \int_{-\tau}^0 [dX(t - \tau - \theta)] \varphi(\theta) d\theta, \end{aligned} \quad (43)$$

hence, the general solution $x^*(t, \varphi, f)$ of system (1) can be represented as

$$\begin{aligned} x^*(t, \varphi, f) &= X(t) [\varphi(0) - C\varphi(-\tau)] \\ &\quad + B \int_{-\tau}^0 \left[\left({}^c D^{1-\alpha} X \right) (t - \tau - \theta) \right] \varphi(\theta) d\theta \\ &\quad + B \int_{-\tau}^0 \frac{(t - \tau - \theta)^{\alpha-1}}{\Gamma(\alpha)} \varphi(\theta) d\theta \\ &\quad + C \int_{-\tau}^0 \dot{X}(t - \tau - \theta) \varphi(\theta) d\theta + D^{-\alpha} f(t) \\ &\quad - C \int_{-\tau}^0 [dX(t - \tau - \theta)] \varphi(\theta) d\theta \\ &\quad + \int_0^t \left({}^c D^{1-\alpha} X \right) (t - \theta) f(\theta) d\theta. \end{aligned} \quad (44)$$

The proof is thus completed. $\square$

Remark 12. For the particular case $\alpha = 1$, $C = 0$, from Definitions 1 and 2, we have

$$\begin{aligned} x^*(t, \varphi, f) &= X(t) \varphi(0) + B \int_{-\tau}^0 X(t - \tau - \theta) \varphi(\theta) d\theta \\ &\quad + \int_0^t X(t - \theta) f(\theta) d\theta, \end{aligned} \quad (45)$$

which is consistent with the general solution of the first-order linear nonhomogeneous delay differential equation [21].

Acknowledgments

The authors are very grateful to the Editor, Professor Pedro R. S. Antunes, and the two anonymous reviewers for their helpful and valuable comments and suggestions, which significantly contributed to improving the quality of the paper. This work is jointly supported by the National Natural Science Foundation of China under Grant nos. 61272530, 11072059, and 11071001, the Doctoral Fund of Ministry of Education of China under Grant no. 20093401110001, the Natural Science Foundation of Jiangsu Province of China under Grant no. BK2012741, and the Key Programs of Educational Commission of Anhui Province of China under Grant nos. KJ2010ZD02 and KJ2011A197.

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