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Research Article

The Kirchhoff Index of Hypercubes and Related Complex Networks

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The resistance distance between any two vertices of G is defined as the network effective resistance between them if each edge of G is replaced by a unit resistor. The Kirchhoff index $\mathrm{Kf}(G)$ is the sum of resistance distances between all the pairs of vertices in G. We firstly provided an exact formula for the Kirchhoff index of the hypercubes networks Q_n by utilizing spectral graph theory. Moreover, we obtained the relationship of Kirchhoff index between hypercubes networks Q_n and its three variant networks $l(Q_n)$, $s(Q_n)$, $t(Q_n)$ by deducing the characteristic polynomial of the Laplacian matrix related networks. Finally, the special formulae for the Kirchhoff indexes of $l(Q_n)$, $s(Q_n)$, and $t(Q_n)$ were proposed, respectively.

1. Introduction

In this work we are concerned with finite undirected connected simple graphs. For the graph theoretical definitions and notations we follow [1]. A network is usually modelled by connected graphs $G = (V, E)$, where V denotes the set of processors and E denotes the set of communication links between processors. It is well known that the standard distance between two vertices of G, denoted by d_{ij}, is the shortest path connecting the two vertices. The Wiener index [2], denoted by $W(G)$, is a famous distance-based topological index and is defined as the sum of distances between all the pairs of vertices in G:

$$W(G) = \sum_{i<j} d_{ij}(G). \tag{1}$$

As an analogue to the Wiener index $W(G)$, another novel distance function named resistance distance was firstly proposed by Klein and Randić [3]. The resistance distance between two arbitrary in an electrical networks, many properties over resistance distances have been actually proved [2, 4–9]. The resistance distance between any two vertices of G is defined as the networks effective resistance between them if each edge of G is replaced by a unit resistor. They also defined the Kirchhoff index $\mathrm{Kf}(G)$ of G as the sum of resistance distances between all pairs of vertices in G; that is,

$$\mathrm{Kf}(G) = \sum_{i<j} r_{ij}(G). \tag{2}$$

Klein and Randić [3] proved that $r_{ij} \le d_{ij}$ and $\mathrm{Kf}(G) \le W(G)$ with equality if and only if G is a tree; it is shown that the Kirchhoff index has very nice purely mathematical and physical interpretations.

The Kirchhoff index has wide applications in physical interpretations, electric circuit, graph theory, and chemistry [10–15]. For example, Zhu et al. [16] and Gutman and Mohar [17] proved that the Kirchhoff index of a graph or networks is the sum of reciprocal nonzero Laplacian eigenvalues of the graph or networks multiplied by the number of the vertices. The Kirchhoff index also is a structure descriptor like the Wiener index [9]. The Kirchhoff index has been computed for cycles [4], geodetic graphs [5], and some fullerenes including buckminsterfullerenes [6]. The Kirchhoff index of some product graphs, join graphs, and corona graphs was

studied [8]. More results of the applications on Kirchhoff index were explored in [2, 7, 10, 14].

The hypercubes network Q_n obtained considerable attention due to its perfect properties, such as symmetry, regular structure, strong connectivity, and small diameter [18, 19]. For more results on the hypercubes network and its applications, see [18–25]. As the importance the hypercubes networks Q_n, many variants of it were presented, among which, for instance, are generalized hypercubes, folded hypercubes, the line graphs of hypercubes $l(Q_n)$, the subdivision graphs of hypercubes $s(Q_n)$, and the total graphs of hypercubes $t(Q_n)$ [19, 20].

The hypercubes networks Q_n may be constructed from the family of subsets of a set with a binary string of length n, by making a vertex for each possible subset and joining two vertices by an edge whenever the corresponding subsets differ in a single binary string. The hypercubes networks Q_n admits several definitions of which two are stated as follows [26].

Definition 1 (see [26]). The hypercubes networks Q_n has 2^n vertices each labelled with a binary string of length n. Two vertices $X = x_1x_2\cdots x_n$ and $Y = y_1y_2\cdots y_n$ are adjacent if and only if there exists an i, $1 \le i \le n$, such that $x_i = \overline{y_i}$, where $\overline{y_i}$ denoted the complement of binary digit y_i and $x_j = y_j$, for all $j \ne i$, and $1 \le j \le n$.

Definition 2 (recursive construction [26]). The hypercubes network Q_n is recursively constructed by taking two copies of Q_{n-1}, denoted by Q^0_{n-1} and Q^1_{n-1}, and adding 2^{n-1} edges as follows: let $V(Q^0_{n-1}) = \{0U = 0u_2u_3\cdots u_n : u_i = 0 \text{ or } 1\}$ and $V(Q^1_{n-1}) = \{1V = 1v_2v_3\cdots v_n : v_i = 0 \text{ or } 1\}$. A vertex $0U = 0u_2u_3\cdots u_n$ of Q^0_{n-1} is joined to a vertex $1V = 1v_2v_3\cdots v_n$ of Q^1_{n-1} if and only if $u_i = v_i$ for every $i, 2 \le i \le n$.

At the end of [10], the authors presented a problem to consider the Kirchhoff index derived from a single graph, such as the line graph, the subdivision graph and the total graph Gao et al. [27] obtained special formulae for the Kirchhoff index of $l(G)$, $s(G)$, and $t(G)$, where G is a regular graph. Motivated by the previous results, we present the corresponding formulae for the Kirchhoff index of the hypercubes network Q_n and its three variant networks $l(Q_n)$, $s(Q_n)$, and $t(Q_n)$ in this paper.

The remainder of this paper is organized as follows. Section 2 gives some basic notations and some preliminaries in our discussion. The proofs of our main results are in Section 3. Finally, some conclusions are given in Section 4.

2. Notations and Some Preliminaries

In this section, we recall some basic notations and results in graphs theory. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix with the (i, j)-entry equal to 1 if vertices i and j are adjacent and 0 otherwise. Suppose that $D(G) = \operatorname{diag}(d_1(G), d_2(G), \ldots, d_n(G))$ is the degree diagonal matrix of G, where $d_i(G)$ is the degree of the vertex i, $i = 1, 2, \ldots, n$. Let $L(G) = D(G) - A(G)$ be called the Laplacian matrix of G. Then the eigenvalues of $A(G)$ and $L(G)$ are called eigenvalues and Laplacian eigenvalues of G, respectively. For more details the readers may refer to [1].

Yin and Wang [28] proved the following Lemma.

Lemma 3 (see [28]). *For the hypercubes networks Q_n with $n \ge 2$,*

$$\operatorname{Spec}(Q_n) = \begin{pmatrix} 0 & 2 & \cdots & 2i & \cdots & 2n \\ C_n^0 & C_n^1 & \cdots & C_n^i & \cdots & C_n^n \end{pmatrix}, \tag{3}$$

where the $2i$ $(i = 0, 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks, and C_n^i are the multiplicities of the eigenvalues $2i$.

Gutman and Mohar [17] and Zhu et al. [16] presented the Kirchhoff index of a graph in terms of Laplacian eigenvalues as follows.

Lemma 4 (see [16, 17]). *Let G be a connected graph with $n \ge 2$ vertices; then*

$$\operatorname{Kf}(G) = n\sum_{i=1}^{n-1}\frac{1}{\lambda_i}. \tag{4}$$

Let $P_{(G)}(x)$ be the characteristic polynomial of the Laplacian matrix of a graph G, the following results were shown in [27].

Lemma 5 (see [27]). *Let G be an r-regular connected graph with n vertices and m edges; then*

$$\begin{aligned} P_{l(G)}(x) &= (x-2r)^{m-n}P_G(x), \\ P_{s(G)}(x) &= (-1)^m(2-x)^{m-n}P_G(x(r+2-x)), \\ P_{t(G)}(x) &= (-1)^m(r+1-x)^n \\ &\quad \times (2r+2-x)^{m-n}P_G\left(\frac{x(r+2-x)}{r-x+1}\right), \end{aligned} \tag{5}$$

where $P_{l(G)}(x)$, $P_{s(G)}(x)$, and $P_{t(G)}(x)$ are the characteristic polynomials for the Laplacian matrix of graphs $l(G)$, $s(G)$, and $t(G)$, respectively.

It is worthwhile to note that the conclusion of Lemma 5 is not completely correct, the authors [29] recently show the Laplacian characteristic polynomial of t(G), where G is a regular graph, which correct the Lemma 5 in Gao et al. [27] (2012) as follows.

Lemma 6 (see [29]). *Let G be a r-regular connected graph with n vertices and m edges, then*

$$\begin{aligned} P_{s(G)}(x) &= (-1)^n(2-x)^{m-n}P_G(x(r+2-x)), \\ P_{t(G)}(x) &= x(x-r-2)(x-2r-2)^{m-n} \\ &\quad \times \prod_{i=1}^{n-1}\left[\left(x^2-2x-rx\right)+(3-2x+r)\mu_i+\mu_i^2\right]. \end{aligned} \tag{6}$$

where $P_{s(G)}(x)$, $P_{t(G)}(x)$ are the characteristic polynomial for the Laplacian matrix of graphs $s(G)$ and $t(G)$, respectively.

The following Lemma give an expression on $\mathrm{Kf}(t(G))$ and $\mathrm{Kf}(G)$ of a regular graph G.

Lemma 7 (see [29]). *Let G be a r-regular connected graph with n vertices and m edges, and $r \geq 2$, then*

$$\mathrm{Kf}(t(G)) = \frac{(r+2)^2}{2(r+3)}\mathrm{Kf}(G) + \frac{n^2(r^2-4)}{8(r+1)} + \frac{n}{2} + \frac{n(r+2)(r+4)}{2(r+3)}\sum_{i=1}^{n-1}\frac{1}{\mu_i+3+r}. \tag{7}$$

For proving the formula of the Kirchhoff index on the subdivision graph of hypercubes, we prove the following Lemma by utilizing Vieta's Theorem; in our proof, some techniques in [27] are referred to.

Lemma 8. *Let $P_{Q_n}(x)$ be the characteristic polynomial of the Laplacian matrix of the hypercubes networks Q_n with $n \geq 2$ and*

$$P_{Q_n}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \cdots + a_{2^n-1}x; \tag{8}$$

then

$$\frac{\mathrm{Kf}(Q_n)}{2^n} = -\frac{a_{2^n-2}}{a_{2^n-1}}, \tag{9}$$

where a_{2^n-1}, a_{2^n-2} are the coefficients of x and x^2 in the characteristic polynomial, respectively.

Proof. Let $\mathrm{Spec}(Q_n) = (\lambda_0, \lambda_1, \lambda_2, \ldots \lambda_n, \lambda_{n+1}, \ldots, \lambda_{2^n-1})$. Then $\lambda_i, i = 1, 2, \ldots, 2^n - 1$ satisfy the following equation:

$$x^{2^n-1} + a_1 x^{2^n-2} + \cdots + a_{2^n-1} = 0; \tag{10}$$

it is not difficult to check that $1/\lambda_i, i = 1, 2, \ldots, 2^n - 1$ are the roots of equation

$$a_{2^n-1}x^{2^n-1} + a_{2^n-2}x^{2^n-2} + \cdots + a_1 x + 1 = 0. \tag{11}$$

Note that Q_n is connected graph and the multiplicity of 0 as an eigenvalue of $L(Q_n)$ is equal to the number of the connected components in Q_n. So, $a_{2^n-1} \neq 0$; by Lemma 4 and Vieta's Theorem

$$\frac{\mathrm{Kf}(Q_n)}{2^n} = \sum_{i=1}^{2^n-1}\frac{1}{\lambda_i} = -\frac{a_{2^n-2}}{a_{2^n-1}}, \tag{12}$$

where a_{2^n-1}, a_{2^n-2} are the coefficients of x and x^2 in the characteristic polynomial of the Laplacian matrix of the hypercubes networks Q_n. □

3. Main Results

3.1. The Kirchhoff Index in Hypercubes Networks Q_n. In this section, we firstly give formula for the Kirchhoff index in the hypercubes Q_n with $n \geq 2$.

Theorem 9. *For the hypercubes networks Q_n with $n \geq 2$,*

$$\mathrm{Kf}(Q_n) = 2^n \sum_{i=1}^{n}\frac{C_n^i}{2i}, \tag{13}$$

where the $2i$ $(i = 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks and the binomial coefficients C_n^i are the multiplicities of the eigenvalues $2i$.

Proof. Since the hypercubes networks Q_n have 2^n vertices and $n2^{n-1}$ edges, then by Lemma 3,

$$\mathrm{Spec}(Q_n) = \begin{pmatrix} 0 & 2 & \cdots & 2i & \cdots & 2n \\ C_n^0 & C_n^1 & \cdots & C_n^i & \cdots & C_n^n \end{pmatrix}; \tag{14}$$

then

$$\begin{aligned}\mathrm{Kf}(Q_n) &= 2^n \sum_{i=1}^{2^n-1}\frac{1}{\lambda_i} \\ &= 2^n\left(\frac{C_n^1}{2\times 1} + \frac{C_n^2}{2\times 2} + \cdots + \frac{C_n^i}{2\times n-1} + \cdots + \frac{C_n^n}{2\times n}\right) \\ &= 2^n \sum_{i=1}^{n}\frac{C_n^i}{2i}.\end{aligned} \tag{15}$$

The proof of Theorem 9 is completed. □

Remark 10. Palacios and Renom studied the Kirchhoff index of the d-dimensional hypercube in [5] by using probabilistic tools, where they obtained a closed-form formula for the Kirchhoff index and found the asymptotic value $2^{2d}/d$. We present the formula for Kirchhoff index by directly calculating the eigenvalues of the Laplacian matrix of hypercubes networks, which is different from their technique and results.

3.2. The Kirchhoff Index in the Line Graph of Hypercubes Networks $l(Q_n)$. The line graph of a graph G, denoted by $l(G)$, is the graph whose vertices correspond to the edges of G, in which two vertices are adjacent if and only if the corresponding edges of G share a common vertex. In the following theorem, we proposed a formula for the Kirchhoff index in the line graph of hypercubes networks $l(Q_n)$, denoted by $\mathrm{Kf}(l(Q_n))$, where the eigenvalues of Q_n are $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots \leq \lambda_{2^n-1}$.

Theorem 11. *Let $l(Q_n)$ be line graphs of hypercubes Q_n with $n \geq 2$; then*

$$\mathrm{Kf}(l(Q_n)) = n2^{n-1}\sum_{i=1}^{n}\frac{C_n^i}{2i} + +n2^{2n-3} - 2^{2n-2}, \tag{16}$$

where the $2i$ $(i = 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks and the binomial coefficients C_n^i are the multiplicities of the eigenvalues $2i$.

Proof. Now, for convenience, we denoted the numbers of vertices and edges in the hypercubes networks Q_n by p and q, respectively. Obviously, $p = 2^n, q = n2^{n-1}$, so the line graphs of hypercubes $l(Q_n)$ have $q = n2^{n-1}$ vertices.

By Lemma 5,

$$P_{l(Q_n)}(x) = (x-2r)^{q-p} P_G(x). \tag{17}$$

Comparing the spectrum of Q_n,

$$\operatorname{Spec}(Q_n) = \begin{pmatrix} 0 & 2 & \cdots & 2i & \cdots & 2n \\ C_n^0 & C_n^1 & \cdots & C_n^i & \cdots & C_n^n \end{pmatrix}. \tag{18}$$

We can easily obtain the spectrum of $l(Q_n)$ as follows:

$$\operatorname{Spec}(l(Q_n)) = (2n \;\; 2n \;\; \cdots \;\; 2n \;\; \lambda_0 \;\; \cdots \;\; \lambda_{2_n-1}), \tag{19}$$

where $\lambda_0, \lambda_1, \ldots \lambda_n, \lambda_{n+1}, \ldots, \lambda_{2^n-1}$ are the eigenvalues of Q_n and $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \lambda_n \le \lambda_{n+1} \le \cdots \le \lambda_{2^n-1}$.

Notice that the line graphs of hypercubes $l(Q_n)$ have $q = n2^{n-1}$ vertices; by Lemma 4,

$$\operatorname{Kf}(l(Q_n)) = q\left(\sum_{i=1}^{2^n-1} \frac{1}{\lambda_i} + \frac{q-p}{2n}\right) \tag{20}$$

$$= \frac{n2^{n-1}}{2^n}\operatorname{Kf}(Q_n) + \frac{q^2-pq}{2n} \tag{21}$$

$$= \frac{n}{2}\operatorname{Kf}(Q_n) + n2^{2n-3} - 2^{2n-2}. \tag{22}$$

Substituting the results of Theorem 9 (15) into (22), we can get the formula for the Kirchhoff index on the line graph of hypercubes $\operatorname{Kf}(l(Q_n))$ as follows:

$$\operatorname{Kf}(l(Q_n)) = n2^{n-1}\sum_{i=1}^{n} \frac{C_n^i}{2i} + n2^{2n-3} - 2^{2n-2}. \tag{23}$$

This completes the proof. □

3.3. The Kirchhoff Index in the Subdivision Graph of Hypercubes Networks $s(Q_n)$. The subdivision graph of a graph G, denoted by $s(G)$, is the graph obtained by replacing every edge in G with a copy of P_2 (path of length two). In an almost identical way as Theorem 11, we required the formula for the Kirchhoff index in the subdivision graph of hypercubes $s(Q_n)$, denoted by $\operatorname{Kf}(s(Q_n))$.

Theorem 12. *Let $s(Q_n)$ be subdivision graphs of the hypercubes Q_n with $n \ge 2$; then*

$$\operatorname{Kf}(s(Q_n)) = (n+2)\left(2^n + n2^{n-1}\right) \times \sum_{i=1}^{n} \frac{C_n^i}{2i} + 2^{n-1} + n^2 2^{2n-3} - 2^{2n-1}, \tag{24}$$

where the $2i$ $(i = 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks and the binomial coefficients C_n^i are the multiplicities of the eigenvalues $2i$.

Proof. Now supposing that $n \ge 2$, $P_{(Q_n)}(x)$ is the characteristic polynomial of the Laplacian matrix of the hypercubes Q_n, $P_{Q_n}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \cdots + a_{2^n-1}x$, and

$$\operatorname{Spec}(Q_n) = \{\lambda_0, \lambda_1, \lambda_2, \ldots \lambda_n, \lambda_{n+1}, \ldots, \lambda_{2^n-1}\}, \tag{25}$$

where $\lambda_i, i = 0, 1, 2, \ldots, 2^n - 1$ are the Laplacian eigenvalues of Q_n Then by Lemma 8,

$$\frac{\operatorname{Kf}(Q_n)}{2^n} = -\frac{a_{2^n-2}}{a_{2^n-1}}. \tag{26}$$

By Lemma 5, we have

$$\begin{aligned} P_{s(Q_n)}(x) = {} & (-1)^p (2-x)^{q-p} \\ & \times \left[x^{2^n}(n+2-x)^{2^n} + a_1 x^{2^n-1}(n+2-x)^{2^n-1}\right. \\ & + \cdots + a_{2^n-2}x^2(n+2-x)^2 \\ & \left. + a_{2^n-1}x(n+2-x)\right]. \end{aligned} \tag{27}$$

Consequently, the coefficient of x^2 in $P_{s(Q_n)}(x)$ is

$$\begin{aligned} a_{2^n-2} = (-1)^p 2^{q-p} & \left[(n+2)^2 a_{2^n-2} - 2^{q-p} a_{2^n-1}\right. \\ & \left. - (q-p)\, 2^{q-p-1}(n+2)\, a_{2^n-1}\right], \end{aligned} \tag{28}$$

and the coefficient of x in $P_{s(Q_n)}(x)$ is

$$(-1)^p 2^{q-p}(n+2)\, a_{2^n-1}. \tag{29}$$

Note that $s(Q_n)$ has $2^n + n2^{n-1}$ vertices. By Lemma 6 and substituting the coefficients into (26),

$$\begin{aligned} & \frac{\operatorname{Kf}(s(G))}{2^n + n2^{n-1}} \\ & = -\left(2^{q-p}(n+2)^2 a_{2^n-2} - 2^{q-p} a_{2^n-1}\right. \\ & \quad \left. - (q-p)\, 2^{q-p-1} a_{2^n-1}(n+2)\right) \\ & \quad \times \left(2^{q-p} a_{2^n-1}(n+2)\right)^{-1} \end{aligned} \tag{30}$$

$$= -\frac{a_{2^n-2}(n+2)}{a_{2^n-1}} + \frac{1}{n+2} + \frac{q-p}{2}. \tag{31}$$

By substituting $q = n2^{n-1}$, $p = 2^n$ and the results of Lemma 5 into (31), we have

$$\begin{aligned} & \frac{\operatorname{Kf}(s(Q_n))}{2^n + n2^{n-1}} \\ & = -\frac{a_{2^n-2}(n+2)}{a_{2^n-1}} + \frac{1}{n+2} + \frac{n2^{n-1} - 2^n}{2} \end{aligned} \tag{32}$$

$$= \frac{(n+2)\operatorname{Kf}(Q_n)}{2^n} + \frac{1}{n+2} + \frac{n2^{n-1} - 2^n}{2}. \tag{33}$$

Simplifying (33) and substituting the results of Theorem 9 (15), we can get the formula of the Kirchhoff index on the subdivision graph of the hypercubes $\mathrm{Kf}(s(Q_n))$ as follows:

$$\begin{aligned}\mathrm{Kf}(s(Q_n)) &= \frac{(n+2)\left(2^n+n2^{n-1}\right)}{2^n}\mathrm{Kf}(Q_n) \\ &\quad + \frac{2^n+n2^{n-1}}{n+2} + \frac{\left(2^n+n2^{n-1}\right)\left(n2^{n-1}-2^n\right)}{2} \\ &= \frac{(n+2)\left(2^n+n2^{n-1}\right)}{2^n}\mathrm{Kf}(Q_n) \\ &\quad + 2^{n-1}+n^2 2^{2n-3}-2^{2n-1} \\ &= (n+2)\left(2^n+n2^{n-1}\right) \\ &\quad \times \sum_{i=1}^{n}\frac{C_n^i}{2i} + 2^{n-1}+n^2 2^{2n-3}-2^{2n-1}.\end{aligned} \tag{34}$$

Thus, the results of the Theorem 12 hold. □

3.4. The Kirchhoff Index in the Total Graph of Hypercubes Networks $t(Q_n)$. The total graph of a graph G, denoted by $t(G)$, is the graph whose vertices correspond to the union of the set of vertices and edges of G, with two vertices of $t(G)$ being adjacent if and only if the corresponding elements are adjacent or incident in G. We now proved the formula of the Kirchhoff index in the total graph of the hypercubes networks $t(Q_n)$, denoted by $\mathrm{Kf}(t(Q_n))$.

Theorem 13. *Let $t(Q_n)$ be the total graphs of the hypercubes networks Q_n with $n \geq 2$; then*

$$\begin{aligned}&\mathrm{Kf}(t(Q_n)) \\ &\quad = \frac{2^{n-1}(n+2)^2}{n+3}\sum_{i=1}^{n}\frac{C_n^i}{2i} + \frac{n^4-4n^2}{8(n+1)} \\ &\qquad + \frac{n}{2} + \frac{\left(n^2+2n\right)(n+4)}{2(n+3)}\sum_{i=1}^{n}\frac{C_n^i}{2i+3+n},\end{aligned} \tag{35}$$

where the $2i$ $(i = 0, 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks and the binomial coefficients C_n^i are the multiplicities of the eigenvalues $2i$.

Proof. Let

$$P_{t(Q_n)}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \ldots + a_{2^n-1}x. \tag{36}$$

Then by Lemma 8,

$$\frac{\mathrm{Kf}(t(Q_n))}{2^n} = -\frac{a_{2^n-2}}{a_{2^n-1}}. \tag{37}$$

Applying Lemma 6, the Laplacian characteristic polynomial of $t(Q_n)$ is

$$\begin{aligned}P_{t(Q_n)}(x) &= x(x-n-2)(x-2n-2)^{q-p} \\ &\quad \times \prod_{i=1}^{2^n-1}\left\{\left[x^2-2x-nx\right]+[3-2x+n]\lambda_i+\lambda_i^2\right\} \\ &= x(x-n-2)(x-2n-2)^{q-p} \\ &\quad \times \prod_{i=1}^{2^n-1}\left[x^2-(2+n+2\lambda_i)x+\lambda_i^2+(n+3)\lambda_i\right].\end{aligned} \tag{38}$$

Consider that Q_n has 2^n vertices and $r = n$, by Lemma 7, we can get the following equality,

$$\begin{aligned}\mathrm{Kf}(t(Q_n)) &= \frac{(r+2)^2}{2(r+3)}\mathrm{Kf}(Q_n) + \frac{n^2\left(r^2-4\right)}{8(r+1)} + \frac{n}{2} \\ &\quad + \frac{n(r+2)(r+4)}{2(r+3)}\sum_{i=1}^{2^n-1}\frac{1}{\lambda_i+3+r} \\ &= \frac{(n+2)^2}{2(n+3)}\mathrm{Kf}(Q_n) + \frac{n^4-4n^2}{8(n+1)} + \frac{n}{2} \\ &\quad + \frac{\left(n^2+2n\right)(n+4)}{2(n+3)}\sum_{i=1}^{2^n-1}\frac{1}{\lambda_i+3+n}.\end{aligned} \tag{39}$$

Consequently, the relationships between the hypercubes networks Q_n and its variant networks $t(Q_n)$ for Kirchhoff index is

$$\begin{aligned}\mathrm{Kf}(t(Q_n)) &= \frac{(n+2)^2}{2(n+3)}\mathrm{Kf}(Q_n) + \frac{n^4-4n^2}{8(n+1)} + \frac{n}{2} \\ &\quad + \frac{\left(n^2+2n\right)(n+4)}{2(n+3)}\sum_{i=1}^{2^n-1}\frac{1}{\lambda_i+3+n}.\end{aligned} \tag{40}$$

Substitutes the result of Theorem 9 and simplifies about equation, we can get the formula for the Kirchhoff index on the total graph of hypercubes networks $\mathrm{Kf}(t(Q_n))$ as follows.

$$\begin{aligned}\mathrm{Kf}(t(Q_n)) &= \frac{(n+2)^2}{2(n+3)}\mathrm{Kf}(Q_n) + \frac{n^4-4n^2}{8(n+1)} + \frac{n}{2} \\ &\quad + \frac{\left(n^2+2n\right)(n+4)}{2(n+3)}\sum_{i=1}^{2^n-1}\frac{1}{\lambda_i+3+n} \\ &= \frac{2^{n-1}(n+2)^2}{n+3}\sum_{i=1}^{n}\frac{C_n^i}{2i} + \frac{n^4-4n^2}{8(n+1)} + \frac{n}{2} \\ &\quad + \frac{\left(n^2+2n\right)(n+4)}{2(n+3)}\sum_{i=1}^{n}\frac{C_n^i}{2i+3+n}.\end{aligned} \tag{41}$$

This completes the proof of the Theorem. □

4. Conclusions

In this paper, we focused on the Kirchhoff index of the hypercubes networks and related networks, which are important networks topology indexes for parallel processing computer systems. We obtained some exact formulae for the Kirchhoff index of the hypercubes networks Q_n and related networks by utilizing spectral graph theory, such as $\mathrm{Kf}(Q_n) = 2^n \sum_{i=1}^{n} (C_n^i/2i)$, where the $2i$ $(i = 1, \ldots, n)$ are the eigenvalues of the Laplacian matrix of hypercubes networks and the binomial coefficients C_n^i are the multiplicities of the eigenvalues $2i$.

We also obtained the relationship for Kirchhoff index between hypercubes networks Q_n and its three variant networks $l(Q_n)$, $s(Q_n)$, and $t(Q_n)$, respectively, by deducing the characteristic polynomial of the Laplacian matrix related networks.

Finally, the special formulae for the Kirchhoff indexes of $l(Q_n)$, $s(Q_n)$, and $t(Q_n)$ were proposed, respectively, by making use of spectral graph theory and Vieta's Theorem.

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