Research Article

A Regularity Criterion for the Magneto-Micropolar Fluid Equations in $\dot{B}_{\infty,\infty}^{-1}$

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The paper is dedicated to study of the Cauchy problem for the magneto-micropolar fluid equations in three-dimensional spaces. A new logarithmically improved regularity criterion for the magneto-micropolar fluid equations is established in terms of the pressure in the homogeneous Besov space $\dot{B}_{\infty,\infty}^{-1}$.

1. Introduction

This paper concerns with the regularity of weak solutions to the magneto-micropolar fluid equations in three dimensions as

$$\begin{aligned} & \partial_t v - (\mu + \chi) \Delta v + v \cdot \nabla v - b \cdot \nabla b + \nabla (p + b^2) \\ & - \chi \nabla \times \omega = 0, \\ & \partial_t \omega - \gamma \Delta \omega - \kappa \nabla \operatorname{div} \omega + 2\chi \omega + v \cdot \nabla \omega - \chi \nabla \times v = 0, \\ & \partial_t b - \nu \Delta b + v \cdot \nabla b - b \cdot \nabla v = 0, \\ & \operatorname{div} v = \operatorname{div} b = 0, \\ & v(0, x) = v_0(x), \quad \omega(0, x) = \omega_0(x), \\ & b(0, x) = b_0(x), \end{aligned} \quad (1)$$

where $v(t, x) = (v_1(t, x), v_2(t, x), v_3(t, x)) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3$, $t \in [0, T)$, $\omega(t, x) \in \mathbb{R}^3$, $b(t, x) \in \mathbb{R}^3$, and $p(t, x) \in \mathbb{R}$ denote, respectively, the microrotational velocity, the magnetic field, and the hydrostatic pressure. μ , χ , κ , γ , ν are positive numbers associated to properties of the material: μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities, and $1/\nu$ is the magnetic Reynold. u_0 , ω_0 , b_0 are initial data for the velocity,

the angular velocity, and the magnetic field with properties $\operatorname{div} u_0 = 0$ and $\operatorname{div} b_0 = 0$. For more detailed background, we refer the readers to [1–3].

As we know, the problem of global regularity or finite time singularity for the weak solutions of the magneto-micropolar fluid equations model with large initial data still remains unsolved since (1) includes the 3D Navier-Stokes equations. It is of interest that the regularity of the weak solutions is under preassumption of certain growth conditions. There are a lot of lectures to study the regularity of weak solutions of the magneto-micropolar fluid equations (see, [4–6]). The purpose of this paper is to establish a new logarithmically improved regularity criterion for the micropolar fluid equations in terms of the pressure in Besov space $\dot{B}_{\infty,\infty}^{-1}$. Now we state the main results as follows.

Theorem 1. Let $(v_0(x), \omega_0(x), b_0(x)) \in H^1(\mathbb{R}^3)$. Let $T > 0$ and (v, ω, b) be a weak solution to the system (1). If the pressure filed P satisfies the following condition:

$$\int_0^T \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} dt < \infty, \quad (2)$$

then the weak solution (v, ω, b) is regular on $[0, T]$.

Remark 2. Since the space $\dot{B}_{\infty,\infty}^{-1}$ is wider than $\dot{\mathcal{M}}_{2,3}$, so our result resolves the limit case $r = 1$ in [7], which greatly improves the result in [7].

Remark 3. Since the space $\dot{B}_{\infty,\infty}^{-1}$ is wider than $L^{3/r,\infty}$, hence our result extends and improves the recent results given by [4].

2. Preliminaries and Lemmas

Throughout this paper, we introduce some function spaces, notations, and important inequalities.

Let $e^{t\Delta}$ denote the heat semigroup defined by

$$e^{t\Delta} f = K_t * f, \quad K_t = (4\pi t)^{-3/2} \exp\left(-\frac{|x|^2}{4t}\right) \quad (3)$$

for $t > 0$ and $x \in \mathbb{R}^3$, where $*$ denotes the convolution of functions defined on $\mathbb{R}^3$.

We now recall the definition of the homogeneous Besov space with negative indices $\dot{B}_{\infty,\infty}^{-\alpha}$ on $\mathbb{R}^n$ and the homogeneous Sobolev space $\dot{H}_q^\alpha$ of exponent $\alpha > 0$. It is known (p. 192 of [8]) that $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}_{\infty,\infty}^{-\alpha}$ if and only if $e^{t\Delta} f \in L^\infty$ for all $t > 0$ and $t^{\alpha/2} \|e^{t\Delta} f\|_\infty \in L^\infty(0, \infty; L^\infty)$. The norm of $\dot{B}_{\infty,\infty}^{-\alpha}$ is defined, up to equivalence, by

$$\|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}} = \sup_{t>0} (t^{\alpha/2} \|e^{t\Delta} f\|_\infty). \quad (4)$$

We introduce now the homogeneous Sobolev space $\dot{H}_q^\alpha(\mathbb{R}^3)$, which is defined by the set of functions $f \in L^r(\mathbb{R}^3)$, $1/r = (1/q) - (s/3)$ such that $(-\Delta)^{s/2} f \in L^q(\mathbb{R}^3)$. This space is endowed with the norm

$$\|f\|_{\dot{H}_q^\alpha} = \|(-\Delta)^{s/2} f\|_{L^q}, \quad (5)$$

and when $q = 2$, we just let $\dot{H}_2^\alpha(\mathbb{R}^3) = \dot{H}^\alpha(\mathbb{R}^3)$. Additionally, we have the following inclusion relations (see, e.g., [9]):

$$\begin{aligned} \dot{H}^{1/2}(\mathbb{R}^3) &\subset L^3(\mathbb{R}^3) \subset L^{3,\infty}(\mathbb{R}^3) \subset \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3), \\ \dot{H}^{1/2}(\mathbb{R}^3) &\subset L^3(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3) \subset \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3) \end{aligned} \quad (6)$$

with continuous injection.

Lemma 4 (see [10]). *Let $1 < p < q < \infty$ and $s = \alpha((q/p) - 1) > 0$. Then there exists a constant C depending only on α , p , and q such that for all $f \in \dot{H}_p^\alpha(\mathbb{R}^3) \cap \dot{B}_{\infty,\infty}^{-\alpha}(\mathbb{R}^3)$,*

$$\|f\|_{L^q} \leq C \|(-\Delta)^{s/2} f\|_{L^p}^{p/q} \|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{1-(p/q)}. \quad (7)$$

In particular, for $s = 1$, $p = 2$, and $q = 4$, we get $\alpha = 1$ and

$$\|f\|_{L^4} \leq C \|f\|_{\dot{H}^1}^{1/2} \|f\|_{\dot{B}_{\infty,\infty}^{-1}}^{1/2}. \quad (8)$$

Lemma 5 (see [11]). *Let $f \in W^{1,s}(\mathbb{R}^3)$ ($s \geq 1$), and $r \geq 1$, then there exists a positive constant C independent of f such that*

$$\|f\|_{L^r} \leq C \|f\|_{L^2}^{1-\alpha} \|\nabla f\|_{L^2}^\alpha, \quad (9)$$

where

$$\alpha = \frac{(1/r) - (1/\gamma)}{(1/3) - (1/s) - (1/r)}. \quad (10)$$

3. Proof of Theorem 1

For given initial data $(v_0, \omega_0, b_0) \in H^1(\mathbb{R}^3)$, the weak solution is the same as the local strong solution (v, ω, b) in a local interval $(0, T)$ as in the discussion of Navier-Stokes equations. For the uniqueness and existence of local strong solution, we refer to [1]. Thus, it proves that Theorem 1 is reduced to establish a priori estimates uniformly in $(0, T)$ for strong solutions. With the use of the a priori estimates, the local strong solution (v, ω, b) can be continuously extended to $t = T$ by a standard process to obtain global regularity of the weak solution. Therefore, we assume that the solution (v, ω, b) is sufficiently smooth on $(0, T)$.

Proof of Theorem 1. We show that Theorem 1 holds under condition (1). To prove the theorem, we need the L^4 -estimate. For this purpose, taking the inner product of the first equation of (1) with $|u|^2 u$ and integrating by parts, it can be deduced that

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} \|u\|_{L^4}^4 + (\mu + \chi) \|\nabla u\|_{L^2} \|u\|_{L^2}^2 \\ &\quad + \frac{1}{2} (\mu + \chi) \|\nabla |u|^2\|_{L^2}^2 \\ &\leq 2 \int_{\mathbb{R}^3} |P| |u|^2 |\nabla u| dx + 3\chi \int_{\mathbb{R}^3} |w| |u|^2 |\nabla u| dx \\ &\quad - \int_{\mathbb{R}^3} |b| |\nabla(|u|^2 u)| |b| dx, \end{aligned} \quad (11)$$

where we used the following relations by the divergence-free condition $\operatorname{div} u = 0$:

$$\begin{aligned} &\int_{\mathbb{R}^3} u \cdot \nabla u \cdot |u|^2 u dx = \frac{1}{2} \int_{\mathbb{R}^3} u \cdot \nabla |u|^4 dx = 0, \\ &\int_{\mathbb{R}^3} \Delta u \cdot |u|^2 u dx = - \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx, \\ &\int_{\mathbb{R}^3} \nabla \times \omega \cdot |u|^2 u dx \\ &\quad = - \int_{\mathbb{R}^3} |u|^2 \omega \cdot \nabla \times u dx - \int_{\mathbb{R}^3} \omega \cdot \nabla |u|^2 \times u dx, \\ &\quad |\nabla \times u| \leq |\nabla u|, \quad |\nabla |u|| \leq |\nabla u|. \end{aligned} \quad (12)$$

Similarly, taking the inner product of the second equation of (1) with $|\omega|^2 \omega$ and integrating by parts, it can be inferred that

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \|\omega\|_{L^4}^4 + \gamma \|\nabla \omega\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla |\omega|^2\|_{L^2}^2 + k \|\operatorname{div} \omega\|_{L^2}^2 \\ + 2\chi \|\omega\|_{L^4}^4 \leq 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| dx. \end{aligned} \quad (13)$$

Using an argument similar to that used in deriving the estimate (11)–(13), it can be obtained for the third equation of (1) that

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \|b\|_{L^4}^4 + \|\nabla b\|_{L^2}^2 + 2\|\nabla |b|\|_{L^2}^2 \\ \leq \int_{\mathbb{R}^3} |b| |\nabla (|b|^2 b)| |u| dx. \end{aligned} \quad (14)$$

Adding up (11), (13), and (14), then we obtain

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4 + \|b\|_{L^4}^4) + (\mu + \chi) \|\nabla u\|_{L^2}^2 \\ + \frac{1}{2} (\mu + \chi) \|\nabla |u|^2\|_{L^2}^2 + \gamma \|\nabla \omega\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla |\omega|^2\|_{L^2}^2 \\ + k \|\operatorname{div} \omega\|_{L^2}^2 + 2\chi \|\omega\|_{L^4}^4 + \|\nabla b\|_{L^2}^2 + 2\|\nabla |b|\|_{L^2}^2 \\ \leq 2 \int_{\mathbb{R}^3} |P| |u|^2 |\nabla u| dx + 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| dx \\ + 3\chi \int_{\mathbb{R}^3} |u| |\omega|^2 |\nabla \omega| dx - \int_{\mathbb{R}^3} |b| |\nabla (|u|^2 u)| |b| dx \\ + \int_{\mathbb{R}^3} |b| |\nabla (|b|^2 b)| |u| dx \\ \triangleq I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (15)$$

Applying the Hölder inequality and the Young inequality for I_2 , it follows that

$$I_2 \leq \frac{\chi + \mu}{2} \|\nabla u\|_{L^2}^2 + C (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4). \quad (16)$$

Arguing similarly to above, it can be derived for I_3 that

$$I_3 \leq \frac{\gamma}{2} \|\nabla \omega\|_{L^2}^2 + C (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4). \quad (17)$$

Considering the term I_1 , by virtue of the Cauchy inequality, we have

$$I_1 \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |v|^2|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |P|^2 |v|^2 dx. \quad (18)$$

Let us bound the integral $(1/2) \int_{\mathbb{R}^3} |P|^2 |v|^2 dx$. Applying the divergence operator div to the first equation of (1), one formally has $P = \sum_{i,j=1}^3 R_i R_j (u_i u_j - b_i b_j)$, where R_j denotes the j th Riesz operator. By the Calderon-Zygmund inequality, we have

$$\|\nabla P\|_{L^2} \leq C (\|\nabla |v|^2\|_{L^2} + \|\nabla |b|^2\|_{L^2}). \quad (19)$$

With the help of (8) and (19), by the Hölder inequality and the Young inequality, we deduce that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |P|^2 |v|^2 dx \\ \leq \frac{1}{2} \|P\|_{L^4}^2 \|v\|_{L^4}^2 \leq C \|\nabla P\|_{L^2} \|P\|_{\dot{B}_{\infty,\infty}^{-1}} \|v\|_{L^4}^2 \\ \leq C (\|\nabla |v|^2\|_{L^2} + \|\nabla |b|^2\|_{L^2}) \|P\|_{\dot{B}_{\infty,\infty}^{-1}} \|v\|_{L^4}^2 \\ = (\|\nabla |v|^2\|_{L^2} + \|\nabla |b|^2\|_{L^2}) \left(C \|P\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|v\|_{L^4}^4 \right)^{1/2} \\ \leq \frac{1}{4} (\|\nabla |v|^2\|_{L^2}^2 + \|\nabla |b|^2\|_{L^2}^2) + C \|P\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|v\|_{L^4}^4. \end{aligned} \quad (20)$$

So the term I_1 can be estimated as

$$\begin{aligned} I_1 \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |v|^2|^2 dx + \frac{1}{4} (\|\nabla |v|^2\|_{L^2}^2 + \|\nabla |b|^2\|_{L^2}^2) \\ + C \|P\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|v\|_{L^4}^4. \end{aligned} \quad (21)$$

Next we have the following estimate for the term I_4 :

$$I_4 \leq \int_{\mathbb{R}^3} |b|^2 |u| |\nabla |u|^2| dx. \quad (22)$$

Since $u \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ and using Cauchy inequality, generalized Hölder inequality, Gagliardo-Nirenberg inequality, and Sobolev imbedding theorem, we obtain

$$\begin{aligned} I_4 \leq C \| |b|^2 |u| \|_{L^2} \|\nabla |u|^2\|_{L^2} \leq C \| |b|^2 |u| \|_{L^2}^2 + \frac{\chi + \mu}{4} \|\nabla |u|^2\|_{L^2}^2 \\ \leq C \| |b|^2 \|_{L^6}^2 \|u\|_{L^3}^2 + \frac{\chi + \mu}{4} \|\nabla |u|^2\|_{L^2}^2 \\ \leq C \|\nabla |b|^2\|_{L^2}^2 \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \frac{\chi + \mu}{4} \|\nabla |u|^2\|_{L^2}^2 \\ \leq C \| |b| \nabla |b| \|_{L^2}^2 + \frac{\chi + \mu}{4} \|\nabla |u|^2\|_{L^2}^2. \end{aligned} \quad (23)$$

The last term of (15) can be treated in the same way as

$$\begin{aligned} I_5 \leq C \int_{\mathbb{R}^3} |b|^2 |u| |\nabla |b|^2| dx \leq C \| |b|^2 |u| \|_{L^2}^2 + \frac{1}{8} \|\nabla |b|^2\|_{L^2}^2 \\ \leq C \| |b| \nabla |b| \|_{L^2}^2. \end{aligned} \quad (24)$$

Inserting the estimates (15) and (21) into (14), it follows that

$$\begin{aligned}
& \frac{d}{dt} (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \|P\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|v\|_4^4 + C (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \|P\|_{\dot{B}_{\infty,\infty}^{-1}}^2 (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) + C (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \left(1 + \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} \right) \\
& \quad \times [1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})] (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \left(1 + \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} \right) \\
& \quad \times [1 + \ln(e + \|P(t, \cdot)\|_{L^3})] (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \left(1 + \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} \right) \\
& \quad \times [1 + \ln(e + \|v(t, \cdot)\|_{L^6}^2)] (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) \\
& \leq C \left(1 + \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} \right) \\
& \quad \times [1 + \ln(e + y(t))] (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4), \tag{25}
\end{aligned}$$

where $y(t)$ is defined by

$$y(t) =: \sup_{T_0 \leq s \leq t} (\|\Lambda^3 v\|_{L^2}^2 + \|\Lambda^3 \omega\|_{L^2}^2 + \|\Lambda^3 b\|_{L^2}^2). \tag{26}$$

Applying Gronwall's inequality on (25) for the interval $[T_0, t]$, one has

$$\begin{aligned}
\sup_{T_0 \leq s \leq t} (\|v\|_4^4 + \|\omega\|_4^4 + \|b\|_4^4) & \leq C_0 \exp(C\varepsilon (1 + \ln(e + y(t)))) \\
& \leq C_0 \exp(2C\varepsilon \ln(e + y(t))) \\
& \leq C_0 (e + y(t))^{2C\varepsilon} \tag{27}
\end{aligned}$$

provided that

$$\int_{T_0}^t \frac{\|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}}^2}{1 + \ln(e + \|P(t, \cdot)\|_{\dot{B}_{\infty,\infty}^{-1}})} ds < \varepsilon \ll 1, \tag{28}$$

where C_0 is a positive constant depending on T_0 .

Next we will estimate the L^2 -norm of ∇v , $\nabla \omega$, and ∇b . We multiply both sides of the first equation of (1) by $(-\Delta v)$, the

second equation of (1) by $(-\Delta \omega)$, and the third equation of (1) by $(-\Delta b)$, by integration by parts over $\mathbb{R}^3$, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 \\
& = \int_{\mathbb{R}^3} (v \cdot \nabla) v \cdot \Delta v \, dx \\
& \quad + \int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot \Delta v \, dx - \int_{\mathbb{R}^3} \operatorname{curl} \omega \Delta v \, dx \\
& \leq \|v\|_{L^4} \|\nabla v\|_{L^4} \|\Delta v\|_{L^2} \\
& \quad + \|b\|_{L^4} \|\nabla b\|_{L^4} \|\Delta v\|_{L^2} + \|\nabla \omega\|_{L^2} \|\Delta v\|_{L^2} \\
& \leq \|v\|_{L^4} \|v\|_{L^2}^{1/8} \|\Delta v\|_{L^2}^{7/8} \|\Delta v\|_{L^2} \\
& \quad + \|b\|_{L^4} \|b\|_{L^2}^{1/8} \|\Delta b\|_{L^2}^{7/8} \|\Delta v\|_{L^2} \\
& \quad + \frac{1}{16} \|\Delta v\|_{L^2}^2 + C \|\nabla \omega\|_{L^2}^2 \\
& \leq \frac{1}{8} \|\Delta v\|_{L^2}^2 + \frac{1}{8} \|\Delta \omega\|_{L^2}^2 + \frac{1}{8} \|\Delta b\|_{L^2}^2 \\
& \quad + C \|b\|_{L^4}^{16} \|b\|_{L^2}^2 + C \|v\|_{L^4}^{16} \|v\|_{L^2}^2 + C \|\omega\|_{L^2}^2, \tag{29}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \omega\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\nabla \operatorname{div} \omega\|_{L^2}^2 + 2 \|\nabla \omega\|_{L^2}^2 \\
& = \int_{\mathbb{R}^3} (v \cdot \nabla) \omega \cdot \Delta \omega \, dx - \int_{\mathbb{R}^3} \operatorname{curl} v \Delta \omega \, dx \\
& \leq \|v\|_{L^4} \|\nabla \omega\|_{L^4} \|\Delta \omega\|_{L^2} + \|\nabla v\|_{L^2} \|\Delta \omega\|_{L^2} \\
& \leq \|v\|_{L^4} \|\omega\|_{L^2}^{1/8} \|\Delta \omega\|_{L^2}^{7/8} \|\Delta \omega\|_{L^2} \\
& \quad + C \|v\|_{L^2}^{1/2} \|\Delta v\|_{L^2}^{1/2} \|\Delta \omega\|_{L^2} \\
& \leq \frac{1}{8} \|\Delta v\|_{L^2}^2 + \frac{1}{8} \|\Delta \omega\|_{L^2}^2 + C \|v\|_{L^4}^{16} \|\omega\|_{L^2}^2 + C \|v\|_{L^2}^2, \tag{30}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\
& = \int_{\mathbb{R}^3} (v \cdot \nabla b) \Delta b \, dx - \int_{\mathbb{R}^3} (b \cdot \nabla v) \Delta b \, dx \\
& \leq \|v\|_{L^4} \|\nabla b\|_{L^4} \|\Delta b\|_{L^2} + \|b\|_{L^4} \|\nabla v\|_{L^4} \|\Delta b\|_{L^2} \\
& \leq \|v\|_{L^4} \|b\|_{L^2}^{1/8} \|\Delta b\|_{L^2}^{7/8} \|\Delta v\|_{L^2} \\
& \quad + \|b\|_{L^4} \|v\|_{L^2}^{1/8} \|\Delta v\|_{L^2}^{7/8} \|\Delta b\|_{L^2} \\
& \leq \frac{1}{8} \|\Delta v\|_{L^2}^2 + \frac{1}{8} \|\Delta b\|_{L^2}^2 + C \|v\|_{L^4}^{16} \|b\|_{L^2}^2 + C \|b\|_{L^4}^{16} \|v\|_{L^2}^2, \tag{31}
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality:

$$\|\nabla f\|_{L^4} \leq C \|f\|_{L^2}^{1/8} \|\Delta f\|_{L^2}^{7/8}. \tag{32}$$

Combining (29), (30), and (31) and using the definition of the weak solution, we deduce that

$$\begin{aligned} & \|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \\ & \leq CC_0(e + y(t))^{6C\varepsilon} (t - T_0) \\ & \quad + \|\nabla v(\cdot, T_0)\|_{L^2}^2 + \|\nabla \omega(\cdot, T_0)\|_{L^2}^2. \end{aligned} \quad (33)$$

Finally we go to the estimate for H^3 -norm of v , ω , and b . In the following calculations, we will use the following commutator estimate due to Kato and Ponce [12]:

$$\begin{aligned} & \|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \\ & \leq (\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \end{aligned} \quad (34)$$

with $s > 0$, $\Lambda^s = (-\Delta)^{s/2}$ and $(1/p) = (1/p_1) + (1/q_1) = (1/p_2) + (1/q_2)$. Taking the operation Λ^3 on both sides of (1), then multiplying them by $\Lambda^3 v$, $\Lambda^3 \omega$, and $\Lambda^3 b$, and integrating by parts over $\mathbb{R}^3$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\Lambda^3 v|^2 + |\Lambda^3 \omega|^2 + |\Lambda^3 b|^2 dx \\ & + \int_{\mathbb{R}^3} |\Lambda^4 v|^2 dx + \int_{\mathbb{R}^3} |\Lambda^4 \omega|^2 dx + \int_{\mathbb{R}^3} |\Lambda^4 b|^2 dx \\ & + \int_{\mathbb{R}^3} |\Lambda^3 \operatorname{div} v|^2 dx + 2 \int_{\mathbb{R}^3} |\Lambda^3 \omega|^2 dx \\ & = - \int_{\mathbb{R}^3} [\Lambda^3 (v \cdot \nabla v) - v \cdot \nabla \Lambda^3 v] \Lambda^3 v dx \\ & + \int_{\mathbb{R}^3} \Lambda^3 \operatorname{curl} \omega \cdot \Lambda^3 v dx \\ & - \int_{\mathbb{R}^3} [\Lambda^3 (v \cdot \nabla \omega) - v \cdot \nabla \Lambda^3 \omega] \Lambda^3 \omega dx \\ & + \int_{\mathbb{R}^3} \Lambda^3 \operatorname{curl} v \cdot \Lambda^3 \omega dx \\ & + \int_{\mathbb{R}^3} [\Lambda^3 (b \cdot \nabla v) - b \cdot \nabla \Lambda^3 v] \Lambda^3 v dx \\ & - \int_{\mathbb{R}^3} [\Lambda^3 (v \cdot \nabla b) - v \cdot \nabla \Lambda^3 b] \Lambda^3 b dx \\ & + \int_{\mathbb{R}^3} [\Lambda^3 (b \cdot \nabla v) - b \cdot \nabla \Lambda^3 v] \Lambda^3 b dx \\ & \equiv A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7. \end{aligned} \quad (35)$$

Hence A_1 can be estimated as

$$\begin{aligned} A_1 & \leq C \|\nabla v\|_{L^3} \|\Lambda^3 v\|_{L^3}^2 \leq C \|\nabla v\|_{L^2}^{13/2} \|\Lambda^3 v\|_{L^2}^{1/4} \|\Lambda^4 v\|_{L^2}^{5/3} \\ & \leq \frac{1}{6} \|\Lambda^4 v\|_{L^2}^2 + C \|\nabla v\|_{L^2}^{13/2} \|\Lambda^3 v\|_{L^2}^{3/2}, \end{aligned} \quad (36)$$

where we used (33) with $s = 3$, $p = 3/2$, $p_1 = q_1 = p_2 = q_2 = 3$ and the following inequalities:

$$\begin{aligned} \|\nabla v\|_{L^3} & \leq C \|\nabla v\|_{L^2}^{3/4} \|\Lambda^3 v\|_{L^2}^{1/4}, \\ \|\Lambda^3 v\|_{L^3} & \leq C \|\nabla v\|_{L^2}^{1/6} \|\Lambda^4 v\|_{L^2}^{5/6}. \end{aligned} \quad (37)$$

If we use the existing estimate (31) for $T_0 < t < T$, (36) reduces to

$$A_1 \leq \frac{1}{6} \|\Lambda^4 v\|_{L^2}^2 + CC_0(e + y(t))^{(3/4)+(39/2)C\varepsilon}. \quad (38)$$

Using (37) again, we have

$$\begin{aligned} A_3 + A_5 + A_6 + A_7 & \leq \frac{1}{6} (\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4 \omega\|_{L^2}^2 + \|\Lambda^4 b\|_{L^2}^2) \\ & \quad + CC_0(e + y(t))^{(3/4)+(39/2)C\varepsilon}. \end{aligned} \quad (39)$$

For A_2 and A_4 , we have

$$\begin{aligned} A_2 + A_4 & \leq \frac{1}{6} (\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4 \omega\|_{L^2}^2) + C (\|\Lambda^3 v\|_{L^2}^2 + \|\Lambda^3 \omega\|_{L^2}^2) \\ & \leq \frac{1}{6} (\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4 \omega\|_{L^2}^2) + CC_0(e + y(t)). \end{aligned} \quad (40)$$

Inserting the above estimates (38)–(40) into (35), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |\Lambda^3 v|^2 + |\Lambda^3 \omega|^2 dx \\ & \leq CC_0(e + y(t))^{(3/4)+(39/2)C\varepsilon} + CC_0(e + y(t)). \end{aligned} \quad (41)$$

Gronwall's inequality implies the boundness of H^3 -norm of v , ω , and b provided that $39C\varepsilon < (1/2)$, which can be achieved by the absolute continuous property of integral (2). This completes the proof of Theorem 1.

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References

- [1] G. P. Galdi and S. Rionero, "A note on the existence and uniqueness of solutions of the micropolar fluid equations," *International Journal of Engineering Science*, vol. 15, no. 2, pp. 105–108, 1977.
- [2] M. A. Rojas-Medar, "Magneto-micropolar fluid motion: existence and uniqueness of strong solution," *Mathematische Nachrichten*, vol. 188, pp. 301–319, 1997.
- [3] E. E. Ortega-Torres and M. A. Rojas-Medar, "Magneto-micropolar fluid motion: global existence of strong solutions," *Abstract and Applied Analysis*, vol. 4, no. 2, pp. 109–125, 1999.
- [4] B. Yuan, "On regularity criteria for weak solutions to the micropolar fluid equations in Lorentz space," *Proceedings of the American Mathematical Society*, vol. 138, no. 6, pp. 2025–2036, 2010.

- [5] B. Yuan, "Regularity of weak solutions to magneto-micropolar fluid equations," *Acta Mathematica Scientia. Series B*, vol. 30, no. 5, pp. 1469–1480, 2010.
- [6] F. Xu, "Regularity criterion of weak solution for the 3D magneto-micropolar fluid equations in Besov spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2426–2433, 2012.
- [7] J. Geng, X. Chen, and S. Gala, "On regularity criteria for the 3D magneto-micropolar fluid equations in the critical Morrey-Campanato space," *Communications on Pure and Applied Analysis*, vol. 10, no. 2, pp. 583–592, 2011.
- [8] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, vol. 18, North-Holland Publishing, Amsterdam, The Netherlands, 1978.
- [9] Z. M. Chen and Z. Xin, "Homogeneity criterion for the Navier-Stokes equations in the whole spaces," *Journal of Mathematical Fluid Mechanics*, vol. 3, no. 2, pp. 152–182, 2001.
- [10] Y. Meyer, P. Gerard, and F. Oru, "Inégalités de Sobolev précisées," in *Séminaire sur les Équations aux Dérivées Partielles*, p. 8, Polytechnique, Palaiseau, France, 1997.
- [11] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, USA, 1968.
- [12] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.