Research Article

Global Dynamic Behavior of a Multigroup Cholera Model with Indirect Transmission

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For a multigroup cholera model with indirect transmission, the infection for a susceptible person is almost invariably transmitted by drinking contaminated water in which pathogens, *V. cholerae*, are present. The basic reproduction number $\mathcal{R}_0$ is identified and global dynamics are completely determined by $\mathcal{R}_0$. It shows that $\mathcal{R}_0$ is a globally threshold parameter in the sense that if it is less than one, the disease-free equilibrium is globally asymptotically stable; whereas if it is larger than one, there is a unique endemic equilibrium which is global asymptotically stable. For the proof of global stability with the disease-free equilibrium, we use the comparison principle; and for the endemic equilibrium we use the classical method of Lyapunov function and the graph-theoretic approach.

1. Introduction

Cholera, a waterborne gastroenteric infection, remains a significant threat to public health in the developing world. Outbreaks of cholera occur cyclically, usually twice per year in endemic areas, and the intensity of these outbreaks varies over longer periods [1]. Hence, in the last few decades, enormous attention is being paid to the cholera disease and several mathematical dynamic models have been developed to study the transmission of cholera [1–7]. In these papers, they consider the population is uniformly mixed, but many factors can lead to heterogeneity in a host population. So in this paper we divide different population into different groups, which can be divided geographically into communities, cities, and countries, to incorporate differential infectivity of multiple strains of the disease agent.

In the case of cholera, the transmission usually occurs through ingestion of contaminated water or feces rather than through casual human-human contact [1]. Therefore, direct contact of healthy person with an infected person is not a risk for contracting infection, whereas a healthy person may contract infection by drinking contaminated water in which

pathogens, *V. cholerae*, are present [2]. The members of this bacterial genus (*V. cholerae*) naturally colonize in lakes, rivers, and estuaries. Therefore we consider that cholera transmits to other individuals via bacteria in the aquatic environment and formulates a multi-group epidemic model for cholera. Let N be the total population which is divided into four epidemiological compartments, susceptible compartment S , infectious compartment I , recovered compartment R , and vaccinated compartment V . Let W be the density of *V. cholerae* in the aquatic environment. As a consequence of the increase in the density of virulent *V. cholerae* in the aquatic environment, humans become infected and begin to shed increasing numbers of bacteria into the aquatic environment, further elevating bacterial density and exacerbating the outbreak [1]. The growth rate of density of bacteria in the aquatic environment is assumed to be proportional to the number of infectious individuals. We assume that the immunity induced by vaccination is perfect; therefore individuals in vaccinated individuals V cannot be infected. The model is called a multi-group cholera SIRVW epidemic model.

In recent years, multi-group epidemic models have been used to describe the transmission dynamics of many

infectious disease in heterogeneous individuals, such as HIV/AIDS [8], dengue [9], West-Nile virus [10], sexually transmitted diseases [11]; and so on. It is well known that global dynamics of multi-group models with higher dimensions, especially the global stability of the endemic equilibrium, is a very challenging problem. Lajmanovich and York [12] proved global stability of the unique endemic equilibrium by using a quadratic global Lyapunov function on a class of n -group SIS models for gonorrhea; Hethcote [13] proved global stability of the endemic equilibrium for multi-group SIR model without vital dynamics; Thieme [14] proved global stability of the endemic equilibrium of multi-group SEIRS model under certain restrictions. However, they only proved global stability of the endemic equilibrium for multi-group model under some special conditions. In 2006, Guo et al. [15] have first succeeded to establish the complete global dynamics for a multi-group SIR model, by making use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov functions guided by graph theory. By using the results or ideas of [15], the papers [16, 17] proved the global stability of the endemic equilibrium for multi-group model with nonlinear incidence rates and the papers [18, 19] proved the global stability of the endemic equilibrium for multi-group model with distributed delays.

Distinguishing from these multi-group models with direct transmission from person to person, a multi-group cholera model with indirect transmission from the bacteria of the aquatic environment to person is proposed in this paper. We prove that the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$, while an endemic equilibrium exists uniquely and is globally asymptotically stable if $\mathcal{R}_0 > 1$.

The organization of this paper is as follows. In Section 2, we construct a multi-group cholera epidemiological and give some dynamic analysis on the disease-free equilibrium and the endemic equilibrium. An example is given in Section 3 and some conclusions are included in Section 4.

2. Mathematical Modeling and Analysis

For a multi-group epidemic model with cholera, the population of human is divided into n discrete groups, where $n \in \mathbb{N}$. Let $S_i(t)$, $I_i(t)$, $R_i(t)$, and $V_i(t)$ be the numbers of susceptible, infectious, recovered, and vaccinated individuals in group $i = 1, 2, \dots, n$ at time t , respectively. Let $W_i(t)$ be the density of bacteria in the aquatic environment in group $i = 1, 2, \dots, n$ at time t . Based on the assumptions in Section 1, the disease transmission rate of cholera between compartments S_i and W_j is denoted by β_{ij} , which means the susceptible individuals in the i th group can contact the bacteria of the aquatic environment in the j th ($j = 1, 2, \dots, n$) group. So the new infection that occurred in the i th group is given by $\sum_{j=1}^n \beta_{ij} S_i W_j$. The recruitment rate of individuals into $S_i(t)$ compartment with the i th group is given by a constant A_i . Within the i th group, it is assumed that natural death of human is d_i . A simple immunization policy is considered where the vaccination rate in $S_i(t)$ compartment is given by a constant γ_i and the losing

immunity rate from vaccination individuals is λ_i . We assume that individuals in $I_i(t)$ compartment recover with a rate constant r_i . In $W_i(t)$ compartment, the *brucella* shedding rate from $I_i(t)$ compartment is k_i , and the decaying rate of *brucella* is δ_i . So a general multi-group SIRVW epidemic model is described by the following system of differential equations:

$$\begin{aligned} \frac{dS_i}{dt} &= A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \sum_{j=1}^n \beta_{ij} S_i W_j, \\ \frac{dI_i}{dt} &= \sum_{j=1}^n \beta_{ij} S_i W_j - (d_i + r_i) I_i, \\ \frac{dR_i}{dt} &= r_i I_i - d_i R_i, \\ \frac{dV_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) V_i, \\ \frac{dW_i}{dt} &= k_i I_i - \delta_i W_i, \end{aligned} \quad (1)$$

$$i = 1, 2, \dots, n.$$

The parameters A_i , d_i , λ_i , γ_i , k_i , and δ_i are positive for all $i = 1, 2, \dots, n$, which is made for the biological justification. And we assume that β_{ij} is nonnegative for all $i, j = 1, 2, \dots, n$ and n -square matrix $B = (\beta_{ij})_{1 \leq i, j \leq n}$ is irreducible, which implies that every pair of groups is joined by an infectious path so that the presence of an infectious individual in the first group can cause infection in the second group.

Observe that the variable R_i does not appear in the first and last two equations of system (1); this allows us to consider the following reduced system:

$$\begin{aligned} \frac{dS_i}{dt} &= A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \sum_{j=1}^n \beta_{ij} S_i W_j, \\ \frac{dI_i}{dt} &= \sum_{j=1}^n \beta_{ij} S_i W_j - (d_i + r_i) I_i, \\ \frac{dV_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) V_i, \\ \frac{dW_i}{dt} &= k_i I_i - \delta_i W_i, \end{aligned} \quad (2)$$

$$i = 1, 2, \dots, n.$$

For each group i , adding the four equations in system (2) gives

$$\begin{aligned} \frac{d(S_i + I_i + V_i)}{dt} &= A_i - d_i (S_i + I_i + V_i) - r_i I_i \\ &\leq (A_i - d_i (S_i + I_i + V_i)), \end{aligned} \quad (3)$$

then it follows that

$$\lim_{t \rightarrow \infty} \sup (S_i + I_i + V_i) \leq \frac{A_i}{d_i}, \quad \lim_{t \rightarrow \infty} \sup W_i \leq \frac{k_i A_i}{d_i \delta_i}. \quad (4)$$

Therefore, omega limit sets of system (2) are contained in the following bounded region in the nonnegative cone of $\mathbb{R}^{4n}$:

$$X = \left\{ (S_i, I_i, V_i, W_i) \mid S_i, I_i, V_i, W_i \geq 0, 0 \leq (S_i + I_i + V_i) \leq \frac{A_i}{d_i}, W_i \leq \frac{k_i A_i}{d_i \delta_i}, i = 1, 2, \dots, n \right\}. \quad (5)$$

It can be verified that region X is positively invariant with respect to system (2). System (2) always has a disease-free equilibrium

$$P_0 = (S_1^0, 0, V_1^0, 0, \dots, S_i^0, 0, V_i^0, 0, \dots, S_n^0, 0, V_n^0, 0) \quad (6)$$

on the boundary of X , where

$$S_i^0 = \frac{A_i (\lambda_i + d_i)}{d_i (\lambda_i + d_i + \gamma_i)}, \quad V_i^0 = \frac{A_i \gamma_i}{d_i (\lambda_i + d_i + \gamma_i)}. \quad (7)$$

2.1. The Basic Reproduction Number. According to the next generation matrix formulated in papers [20–22], we define the basic reproduction number $\mathcal{R}_0$ of system (2). In order to

formulate $\mathcal{R}_0$, we order the infected variables first by disease state and then by group, that is,

$$I_1, W_1, I_2, W_2, \dots, I_n, W_n. \quad (8)$$

Consider the following auxiliary system:

$$\begin{aligned} \frac{dI_i}{dt} &= \sum_{j=1}^n \beta_{ij} S_j W_j - (d_i + r_i) I_i, \\ \frac{dW_i}{dt} &= k_i I_i - \delta_i W_i, \end{aligned} \quad (9)$$

$i = 1, 2, \dots, n.$

Follow the recipe from van den Driessche and Watmough [21] to obtain

$$F = \begin{pmatrix} 0 & \beta_{11} S_1^0 & 0 & \beta_{12} S_1^0 & \cdots & 0 & \beta_{1n} S_1^0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta_{21} S_2^0 & 0 & \beta_{22} S_2^0 & \cdots & 0 & \beta_{2n} S_2^0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \beta_{n1} S_n^0 & 0 & \beta_{n2} S_n^0 & \cdots & 0 & \beta_{nn} S_n^0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2n \times 2n},$$

$$V = \begin{pmatrix} d_1 + r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -k_1 & \delta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & d_2 + r_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -k_2 & \delta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_n + r_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_n & \delta_n \end{pmatrix}_{2n \times 2n}. \quad (10)$$

We can get the inverse of V , which equals

$$V^{-1} = \begin{pmatrix} \frac{1}{d_1 + r_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{k_1}{\delta_1 (d_1 + r_1)} & \frac{1}{\delta_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{d_2 + r_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{k_2}{\delta_2 (d_2 + r_2)} & \frac{1}{\delta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{d_n + r_n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{k_n}{\delta_n (d_n + r_n)} & \frac{1}{\delta_n} \end{pmatrix}_{2n \times 2n}. \quad (11)$$

Thus, the next generation matrix is FV^{-1} ,

$$FV^{-1} = \begin{pmatrix} A_{11} & \cdots & A_{1n} & B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} & B_{n1} & \cdots & B_{nm} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n \times 2n}, \quad (12)$$

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix}_{n \times n}.$$

So we can calculate the basic reproduction number of system (2),

$$\mathcal{R}_0 = \rho(FV^{-1}) = \rho(A), \quad (13)$$

where

$$A_{ij} = \frac{\beta_{ij} k_j S_i^0}{\delta_j (d_j + r_j)}, \quad S_i^0 = \frac{A_i (\lambda_i + d_i)}{d_i (\lambda_i + d_i + \gamma_i)}, \quad (14)$$

$$i = 1, 2, \dots, n,$$

and ρ denotes the spectral radius. As we will show, $\mathcal{R}_0$ is the key threshold parameters whose values completely characterize the global dynamics of system (2).

2.2. Global Stability of the Disease-Free Equilibrium of System (2). For the disease-free equilibrium P_0 of system (2), we have the following property.

Theorem 1. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium P_0 of system (2) is globally asymptotically stable in the region X .*

Proof. Let $M = F - V$, and define $s(M) = \max\{\text{Re } \lambda : \lambda \text{ is an eigenvalue of } M\}$, so $s(M)$ is a simple eigenvalue of M with a positive eigenvector [23]. By Theorem 2 in [21], there hold two equivalences:

$$\mathcal{R}_0 > 1 \iff s(M) > 0, \quad \mathcal{R}_0 < 1 \iff s(M) < 0. \quad (15)$$

To prove the locally stability of disease-free equilibrium, we check the hypotheses (A1)–(A5) in [21]. Hypotheses (A1)–(A4) are easily verified, while (A5) is satisfied if all eigenvalues of the $4n \times 4n$ matrix

$$J|_{P_0} = \begin{pmatrix} M & 0 \\ J_3 & J_4 \end{pmatrix}_{4n \times 4n} \quad (16)$$

have negative real parts, where $J_3 = -F$,

$$J_4 = \begin{pmatrix} -(d_1 + \gamma_1) & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \gamma_1 & -(d_1 + \lambda_1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(d_2 + \gamma_2) & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \gamma_2 & -(d_2 + \lambda_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(d_n + \gamma_n) & \lambda_n \\ 0 & 0 & 0 & 0 & \cdots & \gamma_n & -(d_n + \lambda_n) \end{pmatrix}_{2n \times 2n}. \quad (17)$$

Calculate the eigenvalues of J_4 :

$$s(J_4) = \max\{-d_1, \dots, -d_n, -(d_1 + \lambda_1 + \gamma_1), \dots, -(d_n + \lambda_n + \gamma_n)\} < 0. \quad (18)$$

If $\mathcal{R}_0 < 1$, then $s(M) < 0$ and $s(J|_{P_0}) < 0$, and the disease-free equilibrium P_0 of (2) is locally asymptotically stable.

Now we will prove that the disease-free equilibrium P_0 of system (2) is globally attractive when $\mathcal{R}_0 < 1$. From the third equation of system (2), we have

$$\begin{aligned} \frac{dV_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) V_i \\ &= \gamma_i (N_i - (I_i + V_i)) - (\lambda_i + d_i) V_i \\ &\leq \gamma_i \frac{A_i}{d_i} - (\lambda_i + \gamma_i + d_i) V_i. \end{aligned} \quad (19)$$

So we can have that, for a small enough positive number ϵ_1 , there exists $t_i > 0$, $i = 1, 2, \dots, n$ such that for all $t > t_i$,

$$V_i \leq \frac{A_i \gamma_i}{d_i (\lambda_i + \gamma_i + d_i)} + \epsilon_1 = V_i^0 + \epsilon_1. \quad (20)$$

Also from the equations of system (2), we have

$$\begin{aligned} \frac{dS_i}{dt} &= A_i + \lambda_i V_i - (\gamma_i + d_i) S_i - \sum_{j=1}^n \beta_{ij} S_i W_j \\ &\leq A_i + \lambda_i (V_i^0 + \epsilon_1) - (d_i + \gamma_i) S_i. \end{aligned} \quad (21)$$

Then

$$\lim_{t \rightarrow \infty} \sup S_i = \frac{A_i + \lambda_i (V_i^0 + \epsilon_1)}{d_i + \gamma_i} = S_i^0 + \epsilon_2, \quad \left(\epsilon_2 = \frac{\lambda_i \epsilon_1}{d_i + \gamma_i} \right). \quad (22)$$

From system (9) and $S_i \leq S_i^0 + \epsilon_2$ with all $t > t_i$. Thus, when $t > t_i$, we derive

$$\begin{aligned}\frac{dI_i}{dt} &= (S_i^0 + \epsilon_2) \sum_{j=1}^n \beta_{ij} W_j - (d_i + r_i) I_i, \\ \frac{dW_i}{dt} &= k_i I_i - \delta_i W_i, \\ i &= 1, 2, \dots, n.\end{aligned}\quad (23)$$

Consider the following auxiliary system

$$\begin{aligned}\frac{dI'_i}{dt} &= (S_i^0 + \epsilon_2) \sum_{j=1}^n \beta_{ij} W'_j - (d_i + r_i) I'_i, \\ \frac{dW'_i}{dt} &= k_i I'_i - \delta_i W'_i, \\ i &= 1, 2, \dots, n.\end{aligned}\quad (24)$$

Let M_0 be the matrix defined by

$$M_0 = \begin{pmatrix} 0 & \beta_{11} & 0 & \beta_{12} & \cdots & 0 & \beta_{1n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta_{21} & 0 & \beta_{22} & \cdots & 0 & \beta_{2n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \beta_{n1} & 0 & \beta_{n2} & \cdots & 0 & \beta_{nn} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2n \times 2n}, \quad (25)$$

and set $M_1 = M + \epsilon_2 M_0$. It follows from Theorem 2 in [21] that $\mathcal{R}_0 < 1$ if and only if $s(M) < 0$. Thus, there exists an $\epsilon_2 > 0$ small enough such that $s(M_1) < 0$. Using the Perron-Frobenius theorem, all eigenvalues of the matrix M_1 have negative real parts when $s(M_1) < 0$. Therefore it has

$$\begin{aligned}(I'_1(t), W'_1(t), I'_2(t), W'_2(t), \dots, I'_n(t), W'_n(t)) \\ \longrightarrow (0, 0, 0, 0, \dots, 0, 0), \quad t \longrightarrow \infty,\end{aligned}\quad (26)$$

which implies that the zero solution of system (24) is globally asymptotically stable. Using the comparison principle of Smith and Waltman [23], we know that

$$\begin{aligned}(I_1(t), W_1(t), I_2(t), W_2(t), \dots, I_n(t), W_n(t)) \\ \longrightarrow (0, 0, 0, 0, \dots, 0, 0), \quad t \longrightarrow \infty.\end{aligned}\quad (27)$$

By the theory of asymptotic autonomous system of Thieme [24], it is also known that

$$\begin{aligned}(S_1(t), V_1(t), \dots, S_n(t), V_n(t)) \\ \longrightarrow (S_1(0), V_1(0), \dots, S_n(0), V_n(0)), \quad t \longrightarrow \infty.\end{aligned}\quad (28)$$

So P_0 is globally attractive when $\mathcal{R}_0 < 1$. It follows that the disease-free equilibrium P_0 of (2) is globally asymptotically stable when $\mathcal{R}_0 < 1$. This completes the proof. $\square$

2.3. The Uniform Persistence and Unique Positive Solution of System (2). In this section, we give the proof of the uniform persistence and the unique positive solution of system (2). Define

$$\begin{aligned}X_0 &= \{(S_i, I_i, V_i, W_i) \in X \mid I_i, W_i > 0, i = 1, 2, \dots, n\}, \\ \partial X_0 &= X \mid X_0.\end{aligned}\quad (29)$$

Theorem 2. When $\mathcal{R}_0 > 1$, there exists a positive constant ϵ_1 such that when $\|I_i(0)\| < \epsilon_1, \|W_i(0)\| < \epsilon_1$ for $(S_i(0), I_i(0), V_i(0), W_i(0)) \in X_0$

$$\limsup_{t \rightarrow \infty} \max \{I_i(t), W_i(t)\} > \epsilon_1, \quad i = 1, 2, \dots, n. \quad (30)$$

Proof. Consider the following system:

$$\begin{aligned}\frac{dS_i}{dt} &= A_i + \lambda_i V_i - (\gamma_i + d_i) S_i, \\ \frac{dV_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) V_i, \\ i &= 1, 2, \dots, n.\end{aligned}\quad (31)$$

Using Corollary 3.2 in Zhao and Jing [25], it then follows that system (31) has a unique positive equilibrium $(S_1^0, V_1^0, \dots, S_n^0, V_n^0)$ which is globally asymptotically stable.

As to $\mathcal{R}_0 > 1 \Leftrightarrow s(M) > 0$, choose $\epsilon > 0$ small enough such that $s(M_2) > 0$, where $M_2 = M - \epsilon M_0$. Let us consider a perturbed system

$$\begin{aligned}\frac{dS_i}{dt} &= A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \epsilon_1 S_i \sum_{j=1}^n \beta_{ij}, \\ \frac{dV_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) V_i, \\ i &= 1, 2, \dots, n.\end{aligned}\quad (32)$$

From our previous analysis of system (32), we can restrict $\epsilon_1 > 0$ small enough such that (32) admits a unique positive equilibrium $(S_i^0(\epsilon_1), V_i^0(\epsilon_1), i = 1, 2, \dots, n)$ which is globally asymptotically stable. $S_i^0(\epsilon_1)$ is continuous in ϵ_1 , so we can further restrict ϵ_1 small enough such that $S_i^0(\epsilon_1) > S_i^0 - \epsilon, i = 1, 2, \dots, n$.

For the sake of contradiction of Theorem 2, there is a $T > 0$ such that $I_i(t) < \epsilon_1, W_i(t) < \epsilon_1, i = 1, 2, \dots, n$, for all $t \geq T$. Then for $t \geq T$, we have

$$\begin{aligned}\frac{dS_i}{dt} &\geq A_i - (d_i + \gamma_i) S_i + \lambda_i R_i - \epsilon_1 S_i \sum_{j=1}^n \beta_{ij}, \\ \frac{dR_i}{dt} &= \gamma_i S_i - (\lambda_i + d_i) R_i, \\ i &= 1, 2, \dots, n.\end{aligned}\quad (33)$$

Since the equilibrium $(S_i^0(\epsilon_1), V_i^0(\epsilon_1), i = 1, 2, \dots, n)$ of (32) is globally asymptotically stable and $S_i^0(\epsilon_1) > S_i^0 - \epsilon$,

$i = 1, 2, \dots, n$. There exists a $T_1 > T > 0$ such that $S_i(t) > S_i^0 - \varepsilon$, $i = 1, 2, \dots, n$ for $t > T_1$. As a consequence, for $t > T_1$, there holds

$$\begin{aligned} \frac{dI_i}{dt} &\geq (S_i^0 - \varepsilon) \sum_{j=1}^n \beta_{ij} W_j - (d_i + r_i) I_i, \\ \frac{dW_i}{dt} &= k_i I_i - \delta_i W_i, \end{aligned} \quad (34)$$

$i = 1, 2, \dots, n$.

Consider the following system

$$\begin{aligned} \frac{dI'_i}{dt} &= (S_i^0 - \varepsilon) \sum_{j=1}^n \beta_{ij} W'_j - (d_i + r_i) I'_i, \\ \frac{dW'_i}{dt} &= k_i I'_i - \delta_i W'_i, \end{aligned} \quad (35)$$

$i = 1, 2, \dots, n$.

Since the matrix M_2 has positive eigenvalue $s(M_2)$ with a positive eigenvector. It is easy to see that

$$\begin{aligned} (I'_1(t), W'_1(t), I'_2(t), W'_2(t), \dots, I'_n(t), W'_n(t)) \\ \longrightarrow (\infty, \infty, \infty, \infty, \dots, \infty, \infty), \quad t \longrightarrow \infty. \end{aligned} \quad (36)$$

Using the comparison principle of Smith and Waltman [23], we also know that

$$\begin{aligned} (I_1(t), W_1(t), I_2(t), W_2(t), \dots, I_n(t), W_n(t)) \\ \longrightarrow (\infty, \infty, \infty, \infty, \dots, \infty, \infty), \quad t \longrightarrow \infty, \end{aligned} \quad (37)$$

which leads to a contradiction, therefore we claim that

$$\limsup_{t \rightarrow \infty} \max \{I_i(t), W_i(t)\} > \varepsilon_i, \quad i = 1, 2, \dots, n. \quad (38)$$

This completes the proof. $\square$

We also have the following result of system (2).

Theorem 3. *If $\mathcal{R}_0 > 1$, then system (2) admits at least one positive equilibrium, and there is a positive constant ε such that every solution $(S_i(t), I_i(t), V_i(t), W_i(t))$ of the system (2) with $(S_i(0), I_i(0), V_i(0), W_i(0)) \in X_0$ satisfies*

$$\min \left\{ \liminf_{t \rightarrow \infty} I_i(t), \liminf_{t \rightarrow \infty} W_i(t) \right\} \geq \varepsilon, \quad i = 1, 2, \dots, n, \quad (39)$$

which implies that system (2) is uniformly persistent.

Proof. Now we prove that system (2) is uniformly persistent with respect to $(X_0, \partial X_0)$. By the form of (2), it is easy to see that both X and X_0 are positively invariant and ∂X_0 is relatively closed in X . Furthermore system (2) is point dissipative. Let

$$\begin{aligned} M_{\partial} &= \{(S_i(0), I_i(0), V_i(0), W_i(0)) \mid (S_i(t), I_i(t), V_i(t), W_i(t)) \\ &\in \partial X_0, \forall t \geq 0, i = 1, 2, \dots, n\}. \end{aligned} \quad (40)$$

It is easy to show that

$$M_{\partial} = \{(S_i(t), 0, V_i(t), 0) \mid S_i(t), V_i(t) \geq 0, i = 1, 2, \dots, n\}. \quad (41)$$

Noting that $\{(S_i(t), 0, V_i(t), 0) \mid S_i(t), V_i(t) \geq 0, i = 1, 2, \dots, n\} \subseteq M_{\partial}$. We only need to prove $M_{\partial} \subseteq \{(S_i(t), 0, V_i(t), 0) \mid S_i(t), V_i(t) \geq 0, i = 1, 2, \dots, n\}$. Assume $(S_i(0), I_i(0), V_i(0), W_i(0))$, $i = 1, 2, \dots, n \in M_{\partial}$. It suffices to show that $I_i(t) = 0$, $W_i(t) = 0$, for all $t \geq 0$, $i = 1, 2, \dots, n$. Suppose not. Then there exist an i_0 , $1 \leq i_0 \leq n$, and $t_0 \geq 0$ such that $I_{i_0}(t_0) > 0$, $W_{i_0}(t_0) > 0$ and we partition $\{1, 2, \dots, n\}$ into two sets Q_1 and Q_2 such that

$$\begin{aligned} (I_i(t_0), W_i(t_0))^T &= 0, \quad \forall i \in Q_1, \\ (I_i(t_0), W_i(t_0))^T &> 0, \quad \forall i \in Q_2. \end{aligned} \quad (42)$$

Q_1 is nonempty due to the definition of M_{∂} . Q_2 is nonempty since $I_{i_0}(t_0) > 0$, $W_{i_0}(t_0) > 0$. For any $i \in Q_2$ and we have that

$$\frac{dW_i(t_0)}{dt_0} = k_i I_i(t_0) - \delta_i W_i(t_0) > k_i I_i(t_0), \quad i \in Q_2. \quad (43)$$

It follows that there is an $\eta > 0$ such that $I_i(t) > 0$, for $t_0 < t < t_0 + \eta$, $i \in Q_2$. This means that $(S_i(t), I_i(t), V_i(t), W_i(t))$, $i = 1, 2, \dots, n$ does not belong to ∂X_0 for $t_0 < t < t_0 + \eta$, which contradicts the assumption that $(S_i(0), I_i(0), V_i(0), W_i(0))$, $i = 1, 2, \dots, n \in M_{\partial}$. This proves the system (41).

P_0 is globally asymptotically stable for system (2). It is clear that there is only an equilibrium P_0 in M_{∂} and by aforementioned claim, it then follows that P_0 is isolated invariant set in X , $W^s(P_0) \cap X_0 = \emptyset$. Clearly, every orbit in M_{∂} converges to P_0 , P_0 is acyclic in M_{∂} . Using Theorem 4.6 in Thieme [26], we conclude that the system (2) is uniformly persistent with respect to $(X_0, \partial X_0)$. By the result of [27, 28], system (2) has an equilibrium $(S_1^*, I_1^*, V_1^*, W_1^*, \dots, S_n^*, I_n^*, V_n^*, W_n^*) \in X_0$. We further claim that $S_i^*, V_i^* > 0$, $i = 1, 2, \dots, n$. Suppose that $S_i^* = V_i^* = 0$, $i = 1, 2, \dots, n$; from of (2), we can get $I_i^* = W_i^* = 0$, $i = 1, 2, \dots, n$. It is a contradiction. Then $(S_1^*, I_1^*, V_1^*, W_1^*, \dots, S_n^*, I_n^*, V_n^*, W_n^*) \in X_0$ is a componentwise positive equilibrium of system (2). This completes the proof. $\square$

The following theorem shows that there exists a unique positive solution for system (2) when $\mathcal{R}_0 > 1$.

Theorem 4. *If $\mathcal{R}_0 > 1$, then there only exists a unique positive equilibrium P^* for system (2).*

Proof. Consider the following system:

$$\begin{aligned} A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \sum_{j=1}^n \beta_{ij} S_i W_j &= 0, \\ \sum_{j=1}^n \beta_{ij} S_i W_j - (d_i + r_i) I_i &= 0, \end{aligned}$$

$$\begin{aligned}
\gamma_i S_i - (\lambda_i + d_i) V_i &= 0, \\
k_i I_i - \delta_i W_i &= 0, \\
i &= 1, 2, \dots, n.
\end{aligned} \tag{44}$$

We have that

$$\begin{aligned}
S_i &= \frac{d_i + \lambda_i}{d_i (d_i + \lambda_i + \gamma_i)} (A_i - (d_i + r_i) I_i), \\
W_i &= \frac{k_i I_i}{\delta_i}, \quad V_i = \frac{\gamma_i S_i}{d_i + \lambda_i}, \\
i &= 1, 2, \dots, n.
\end{aligned} \tag{45}$$

Hence, the equilibrium of system (2) is equal to the following system:

$$B_i (A_i - n_i I_i) \sum_{j=1}^n \beta_{ij} I_j - n_i I_i = 0, \quad i = 1, 2, \dots, n, \tag{46}$$

where

$$B_i = \frac{k_i (d_i + \lambda_i)}{d_i \delta_i (d_i + \lambda_i + \gamma_i)}, \quad n_i = d_i + r_i, \quad i = 1, 2, \dots, n. \tag{47}$$

Therefore, we only need to prove that (46) has a unique positive equilibrium when $\mathcal{R}_0 > 1$. Use the method in [12] to demonstrate the unique positive equilibrium of (46). First we prove that $I_i^* = h$, $i = 1, 2, \dots, n$, is the only positive solution of (46). Assume that $I_i^* = h$ and $I_i^* = k$ are two positive solutions of (46), both nonzero. If $h \neq k$, then $h_i \neq k_i$ for some i ($i = 1, 2, \dots, n$). Assume without loss of generality that $h_1 > k_1$ and moreover that $h_1/k_1 \geq h_i/k_i$ for all i ($i = 1, 2, \dots, n$). Since h and k are positive solutions of (46), we substitute them into (46). We obtain

$$\begin{aligned}
0 &= B_1 (A_1 - n_1 h_1) \sum_{j=1}^n \beta_{1j} h_j - n_1 h_1 \\
&= B_1 (A_1 - n_1 k_1) \sum_{j=1}^n \beta_{1j} k_j - n_1 k_1,
\end{aligned} \tag{48}$$

so

$$\begin{aligned}
0 &= B_1 (A_1 - n_1 h_1) \sum_{j=1}^n \beta_{1j} h_j \frac{k_1}{h_1} - n_1 k_1 \\
&= B_1 (A_1 - n_1 k_1) \sum_{j=1}^n \beta_{1j} k_j - n_1 k_1,
\end{aligned} \tag{49}$$

$$B_1 (A_1 - n_1 h_1) \sum_{j=1}^n \beta_{1j} h_j \frac{k_1}{h_1} = B_1 (A_1 - n_1 k_1) \sum_{j=1}^n \beta_{1j} k_j.$$

But $(h_i/h_1)k_1 \leq k_i$ and $B_1(A_1 - n_1 h_1) < B_1(A_1 - n_1 k_1)$; thus from the above equalities we get

$$\begin{aligned}
&B_1 (A_1 - n_1 h_1) \sum_{j=1}^n \beta_{1j} h_j \frac{k_1}{h_1} \\
&\leq B_1 (A_1 - n_1 h_1) \sum_{j=1}^n \beta_{1j} k_j \\
&< B_1 (A_1 - n_1 k_1) \sum_{j=1}^n \beta_{1j} k_j.
\end{aligned} \tag{50}$$

This is a contradiction, so there is only one positive solution $I_i^* = h$, $i = 1, 2, \dots, n$, of (46). So when $\mathcal{R}_0 > 1$, there only exists a unique positive equilibrium for system (2). $\square$

2.4. Global Stability of the Unique Endemic Solution of System (2). In this section, we prove that the unique endemic equilibrium of system (2) is globally asymptotically stable in X_0 . In order to prove global stability of the endemic equilibrium, the Lyapunov function will be used. In the following, we also use a Lyapunov function to prove global stability of the endemic equilibrium.

Theorem 5. *If $\mathcal{R}_0 > 1$, the unique positive equilibrium P^* of system (2) is globally asymptotically stable in X_0 .*

Proof. Following [15] we define

$$\xi_{ij} = \beta_{ij} S_i^* W_j^*, \quad 1 \leq i, j \leq n, \quad n \geq 2, \tag{51}$$

$$\mathbf{B} = \begin{pmatrix} \sum_{j \neq 1}^n \xi_{1j} & -\xi_{21} & \cdots & -\xi_{n1} \\ -\xi_{12} & \sum_{j \neq 2}^n \xi_{2j} & \cdots & -\xi_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\xi_{1n} & -\xi_{2n} & \cdots & \sum_{j \neq n}^n \xi_{nj} \end{pmatrix}_{n \times n}, \tag{52}$$

which is a Laplacian matrix whose column sums are zero and which is irreducible. Therefore, it follows from Lemma 2.1 of [15] that the solution space of linear system

$$\mathbf{B}\zeta = 0, \tag{53}$$

has dimension 1 with a basis

$$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^T = (c_1, c_2, \dots, c_n)^T, \tag{54}$$

where c_i denotes the cofactor of the i th diagonal entry of $\mathbf{B}$. Note that from (53) we have that

$$\sum_{j=1}^n \zeta_i \xi_{ij} = \sum_{j=1}^n \zeta_j \xi_{ji}, \quad i = 1, 2, \dots, n. \tag{55}$$

For such $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ we define a Lyapunov function

$$\begin{aligned} L(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) &= \sum_{i=1}^n \zeta_i \left(S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*} \right. \\ &\quad \left. + V_i - V_i^* - V_i^* \ln \frac{V_i}{V_i^*} \right. \\ &\quad \left. + \frac{d_i + r_i}{k_i} \left(W_i - W_i^* - W_i^* \ln \frac{W_i}{W_i^*} \right) \right), \end{aligned} \quad (56)$$

where $\mathbf{S} = (S_1, S_2, \dots, S_n)$, $\mathbf{I} = (I_1, I_2, \dots, I_n)$, $\mathbf{V} = (V_1, V_2, \dots, V_n)$, and $\mathbf{W} = (W_1, W_2, \dots, W_n)$. It is easy to see that $L(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) \geq 0$ for all $(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) \geq 0$ and the equality $L(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) = 0$ holds if and only if $(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) = (\mathbf{S}^*, \mathbf{I}^*, \mathbf{V}^*, \mathbf{W}^*)$. The derivative along the trajectories of system (2) is

$$\begin{aligned} L'(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) &= \sum_{i=1}^n \zeta_i \left(A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \sum_{j=1}^n \beta_{ij} S_i W_j \right. \\ &\quad \left. - \frac{S_i^*}{S_i} \left(A_i - (d_i + \gamma_i) S_i + \lambda_i V_i - \sum_{j=1}^n \beta_{ij} S_i W_j \right) \right. \\ &\quad \left. + \sum_{j=1}^n \beta_{ij} S_i W_j - (d_i + r_i) I_i \right. \\ &\quad \left. - \frac{I_i^*}{I_i} \left(\sum_{j=1}^n \beta_{ij} S_i W_j - (d_i + r_i) I_i \right) + \gamma_i S_i \right. \\ &\quad \left. - (\lambda_i + d_i) V_i - \frac{V_i^*}{V_i} (\gamma_i S_i - (\lambda_i + d_i) V_i) \right. \\ &\quad \left. + \frac{d_i + r_i}{k_i} \left(k_i I_i - \delta_i W_i - \frac{W_i^*}{W_i} (k_i I_i - \delta_i W_i) \right) \right) \\ &= L_1 + L_2 + L_3. \end{aligned} \quad (57)$$

From system (44), we have

$$A_i = (d_i + \gamma_i) S_i^* - \lambda_i V_i^* + \sum_{j=1}^n \beta_{ij} S_i^* W_j^*, \quad (58)$$

$$\sum_{j=1}^n \beta_{ij} S_i^* W_j^* = (d_i + r_i) I_i^* = \frac{\delta_i (d_i + r_i) W_i^*}{k_i}. \quad (59)$$

So

$$\begin{aligned} L_1 &= \sum_{i=1}^n \zeta_i \left(\sum_{j=1}^n \beta_{ij} S_i^* W_j - \frac{\delta_i (d_i + r_i) W_i}{k_i} \right), \\ L_2 &= \sum_{i=1}^n \zeta_i \left((d_i + \gamma_i) S_i^* - \lambda_i V_i^* - (d_i + \gamma_i) S_i + \lambda_i V_i \right. \\ &\quad \left. + \frac{S_i^*}{S_i} ((d_i + \gamma_i) S_i^* - \lambda_i V_i^* - (d_i + \gamma_i) S_i + \lambda_i V_i) \right. \\ &\quad \left. + \gamma_i S_i - (\lambda_i + d_i) V_i \right. \\ &\quad \left. + \frac{V_i^*}{V_i} (\gamma_i S_i - (\lambda_i + d_i) V_i) \right) \\ &= \sum_{i=1}^n \zeta_i \left(d_i S_i^* \left(2 - \frac{S_i}{S_i^*} - \frac{S_i^*}{S_i} \right) \right. \\ &\quad \left. + \lambda_i V_i^* \left(2 - \frac{S_i V_i^*}{S_i^* V_i} - \frac{S_i^* V_i}{S_i V_i^*} \right) \right. \\ &\quad \left. + d_i V_i^* \left(3 - \frac{V_i}{V_i^*} - \frac{S_i^*}{S_i} - \frac{S_i V_i^*}{S_i^* V_i} \right) \right) \leq 0, \\ L_3 &= \sum_{i=1}^n \zeta_i \left(3 \sum_{j=1}^n \beta_{ij} S_i^* W_j^* - \sum_{j=1}^n \beta_{ij} S_i^* W_j^* \frac{S_i^*}{S_i} \right. \\ &\quad \left. - \sum_{j=1}^n \beta_{ij} S_i W_j \frac{I_i^*}{I_i} - (d_i + r_i) I_i \frac{W_i^*}{W_i} \right). \end{aligned} \quad (60)$$

Now we claim that

$$\sum_{i=1}^n \zeta_i \sum_{j=1}^n \beta_{ij} S_i^* W_j = \sum_{i=1}^n \zeta_i \frac{\delta_i (d_i + r_i) W_i}{k_i}. \quad (61)$$

Appealing to (51), (55) and (59),

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n \zeta_i \beta_{ij} S_i^* W_j \\ &= \sum_{i=1}^n \sum_{j=1}^n \zeta_j \beta_{ji} S_j^* W_i = \sum_{i=1}^n \sum_{j=1}^n \frac{W_i}{W_i^*} \zeta_j \beta_{ji} S_j^* W_i^* \\ &= \sum_{i=1}^n \frac{W_i}{W_i^*} \sum_{j=1}^n \zeta_j \xi_{ji} = \sum_{i=1}^n \frac{W_i}{W_i^*} \sum_{j=1}^n \zeta_i \xi_{ij} \\ &= \sum_{i=1}^n \zeta_i \frac{\delta_i (d_i + r_i) W_i}{k_i}. \end{aligned} \quad (62)$$

From (61) we have

$$\begin{aligned}
L'(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) &\leq \sum_{i=1}^n \zeta_i \left(3 \sum_{j=1}^n \beta_{ij} S_i^* W_j^* - \sum_{j=1}^n \beta_{ij} S_i^* W_j^* \frac{S_i^*}{S_i} \right. \\
&\quad \left. - \sum_{j=1}^n \beta_{ij} S_i W_j \frac{I_i^*}{I_i} - (d_i + r_i) I_i \frac{W_i^*}{W_i} \right) \quad (63) \\
&= \sum_{i,j=1}^n \zeta_i \xi_{ij} \left(3 - \frac{S_i^*}{S_i} - \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} - \frac{W_i^* I_i}{W_i I_i^*} \right) \\
&=: H_n(S_1, I_1, W_1, \dots, S_n, I_n, W_n).
\end{aligned}$$

Next we show that $H_n \leq 0$ for all $(S_1, I_1, W_1, \dots, S_n, I_n, W_n) \in X_0$ by applying the graph-theoretic approach developed in [29–31]. As in [29], $L = G(B)$ denotes the directed graph associated with matrix $\mathbf{B}$, Q presents a subgraph of L , CQ denotes the unique elementary cycle of Q , $E(CQ)$ presents the set of directed arcs in CQ , and $l = l(Q)$ denotes the number of arcs in CQ . Then H_n can be rewritten as

$$H_n = \sum_Q H_{n,Q}, \quad (64)$$

where

$$\begin{aligned}
H_{n,Q} &= \prod_{(r,m) \in E(CQ)} \xi_{rm} \\
&\quad \times \left(3l - \sum_{(i,j) \in E(CQ)} \left(\frac{S_i^*}{S_i} + \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} + \frac{W_i^* I_i}{W_i I_i^*} \right) \right). \quad (65)
\end{aligned}$$

For instance,

$$\begin{aligned}
H_1 &= H_1(S_1, I_1, W_1) \\
&= \sum_{i,j=1} \zeta_i \xi_{ij} \left(3 - \frac{S_i^*}{S_i} - \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} - \frac{W_i^* I_i}{W_i I_i^*} \right) \leq 0, \\
H_2 &= H_2(S_1, I_1, W_1, S_2, I_2, W_2) \\
&= \sum_{i,j=1}^2 \zeta_i \xi_{ij} \left(3 - \frac{S_i^*}{S_i} - \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} - \frac{W_i^* I_i}{W_i I_i^*} \right) \\
&= \xi_{11} \xi_{21} \left(3 - \frac{S_1^*}{S_1} - \frac{S_1 W_1 I_1^*}{S_1^* W_1^* I_1} - \frac{W_1^* I_1}{W_1 I_1^*} \right) \quad (66) \\
&\quad + \xi_{22} \xi_{12} \left(3 - \frac{S_2^*}{S_2} - \frac{S_2 W_2 I_2^*}{S_2^* W_2^* I_2} - \frac{W_2^* I_2}{W_2 I_2^*} \right) \\
&\quad + \xi_{12} \xi_{21} \left(6 - \frac{S_1^*}{S_1} - \frac{S_1 W_1 I_1^*}{S_1^* W_1^* I_1} - \frac{W_1^* I_1}{W_1 I_1^*} - \frac{S_2^*}{S_2} \right. \\
&\quad \left. - \frac{S_2 W_2 I_2^*}{S_2^* W_2^* I_2} - \frac{W_2^* I_2}{W_2 I_2^*} \right) \leq 0.
\end{aligned}$$

Note that for each unicycle graph Q , it is easy to see that

$$\prod_{(i,j) \in E(CQ)} \frac{S_i^*}{S_i} \cdot \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} \cdot \frac{W_i^* I_i}{W_i I_i^*} = \prod_{(i,j) \in E(CQ)} \frac{W_i^* W_j}{W_i W_j^*} = 1. \quad (67)$$

Therefore,

$$\sum_{(i,j) \in E(CQ)} \left(\frac{S_i^*}{S_i} + \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} + \frac{W_i^* I_i}{W_i I_i^*} \right) \geq 3l, \quad (68)$$

and hence $H_{n,Q} \leq 0$ for each Q , and $H_{n,Q} = 0$ if and only if

$$\frac{S_i^*}{S_i} = \frac{S_i W_j I_i^*}{S_i^* W_j^* I_i} = \frac{W_i^* I_i}{W_i I_i^*}, \quad (i, j) \in E(CQ). \quad (69)$$

Thus

$$L'(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) \leq H_n \leq 0. \quad (70)$$

The equality $L'(\mathbf{S}, \mathbf{I}, \mathbf{V}, \mathbf{W}) = 0$ holds if and only if $S_i = S_i^*$, $I_i = I_i^*$, $V_i = V_i^*$, and $W_i = W_i^*$ for all $i = 1, 2, \dots, n$. Therefore, following from LaSalle's Invariance Principle [32], the unique endemic equilibrium P^* of system (2) is globally asymptotically stable. This completes the proof. $\square$

3. A Numerical Example

Consider the system (1), when $i = 2$, one has the two-group model as follows:

$$\frac{dS_1}{dt} = A_1 - (d_1 + \gamma_1) S_1 + \lambda_1 V_1 - (\beta_{11} S_1 W_1 + \beta_{12} S_1 W_2),$$

$$\frac{dI_1}{dt} = \beta_{11} S_1 W_1 + \beta_{12} S_1 W_2 - (d_1 + r_1) I_1,$$

$$\frac{dR_1}{dt} = r_1 I_1 - d_1 R_1,$$

$$\frac{dV_1}{dt} = \gamma_1 S_1 - (\lambda_1 + d_1) V_1,$$

$$\frac{dW_1}{dt} = k_1 I_1 - \delta_1 W_1,$$

$$\frac{dS_2}{dt} = A_2 - (d_2 + \gamma_2) S_2 + \lambda_2 V_2 - (\beta_{21} S_2 W_1 + \beta_{22} S_2 W_2),$$

$$\frac{dI_2}{dt} = \beta_{21} S_2 W_1 + \beta_{22} S_2 W_2 - (d_2 + r_2) I_2,$$

$$\frac{dR_2}{dt} = r_2 I_2 - d_2 R_2,$$

$$\frac{dV_2}{dt} = \gamma_2 S_2 - (\lambda_2 + d_2) V_2,$$

$$\frac{dW_2}{dt} = k_2 I_2 - \delta_2 W_2.$$

(71)

We can give the basic reproduction number of system (71), which is

$$\mathcal{R}'_0 = \frac{A_{11} + A_{22} + \sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}}{2}, \quad (72)$$

where

$$A_{ij} = \frac{\beta_{ij}k_j S_i^0}{\delta_j(d_j + r_j)}, \quad S_i^0 = \frac{A_i(\lambda_i + d_i)}{d_i(\lambda_i + d_i + \gamma_i)}, \quad i = 1, 2. \quad (73)$$

Taking $A_1 = 150$, $A_2 = 220$, $d_1 = 0.1$, $d_2 = 0.1$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $r_1 = 1$, $r_2 = 1$, $k_1 = 10$, $k_2 = 10$, $\delta_1 = 8$, $\delta_2 = 8$ and using Matlab ODE solver, we run numerical simulations for two cases.

If $\beta_{11} = 0.00048$, $\beta_{12} = 0.0004$, $\beta_{21} = 0.0004$, and $\beta_{22} = 0.00045$, we have $\mathcal{R}'_0 \approx 0.9804 < 1$. Hence the disease-free equilibrium of system (71) is globally asymptotically stable (see Figure 1(a)). If $\beta_{11} = 0.0025$, $\beta_{12} = 0.001$, $\beta_{21} = 0.001$, and $\beta_{22} = 0.0020$, we have $\mathcal{R}'_0 \approx 3.6594 > 1$. Hence the endemic equilibrium of system (71) is globally asymptotically stable (see Figure 1(b)).

4. Conclusion

Cholera epidemic has become a major health problem for many developing countries. From good understanding of the transmission dynamics of cholera in many emergent epidemic regions, the heterogeneous host population can be divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions. Hence, in this paper, we proposed a multi-group cholera SIRVW epidemiological model. In order to distinguish many multi-group models with direct transmission from person to person, we only considered this multi-group cholera model with indirect transmission from the bacteria of the aquatic environment to person. Firstly, the basic reproduction number $\mathcal{R}_0$ of this model is given. Then, it is found that the model has two non-negative equilibria, the disease-free equilibrium and the endemic equilibrium. The disease-free equilibrium exists without any condition whereas the endemic equilibrium exists provided $\mathcal{R}_0 > 1$. Finally, through the analysis of the model it has been found that the global asymptotic behavior of multi-group SIRVW model is completely determined by the size of $\mathcal{R}_0$. That is, the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$ while an endemic equilibrium exists uniquely and is globally asymptotically stable if $\mathcal{R}_0 > 1$. By running numerical simulations for the cases of two-groups model, we can see that the disease-free equilibrium of system (71) is globally stable when $\mathcal{R}'_0 < 1$ and the unique endemic equilibrium of system (71) is globally stable when $\mathcal{R}'_0 > 1$.

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FIGURE 1: (a) The disease dies out in both groups. (b) The disease persists in both groups. Initial conditions are: $S_1(0) = 280$, $I_1(0) = 40$, $R_1(0) = 10$, $V_1(0) = 130$, $W_1(0) = 250$, $S_2(0) = 260$, $I_2(0) = 20$, $R_2(0) = 10$, $V_2(0) = 130$, $W_2(0) = 300$.

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