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Research Article

Synchronization of an Uncertain Fractional-Order Chaotic System via Backstepping Sliding Mode Control

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Backstepping control approach combined with sliding mode control (SMC) technique is utilized to realize synchronization of uncertain fractional-order strict-feedback chaotic system. A backstepping SMC method is presented to compensate the uncertainty which occurs in the slave system. Moreover, the newly proposed control scheme is applied to implement synchronization of fractional-order Duffing-Holmes system. The simulation results demonstrate that the backstepping SMC method is robust against the modeling uncertainties and external disturbances.

1. Introduction

Fractional calculus is a generalization of the ordinary integration and differentiation to noninteger-order case. In recent years, it has found widespread application in the fields of physics, applied mathematics, and engineering [1–3]. Moreover, it has been generally realized that fractional integrals and derivatives are more suitable to describe memory and hereditary properties of various materials and processes than its integer-order counterpart [4].

Chaotic behavior has been observed in the laboratory in a variety of systems, including electrical circuits, lasers, chemical reactions, fluid dynamics, and analog computers [5]. Over the past few years, more and more researchers have turned their attention to chaotic dynamics of fractional-order system. It has been shown that chaotic behavior of an integer-order nonlinear system is preserved when the order becomes fractional [6]. As a consequence, a number of fractional-order chaotic systems have been proposed, including fractional-order variants of Chua's system [7], fractional-order Chen system [8], fractional-order Lorenz, Rössler, and Liu systems [9–11].

Synchronization control is one of the important research areas in chaos theory. During the past three decades, chaos synchronization has been attracting interest from researchers in various fields [12–14]. It should be pointed out that synchronization problem of fractional-order chaotic systems was first reported by Deng and Li [15]. Afterwards, different synchronization control methods have been successfully applied to chaos synchronization of factional order systems, such as sliding mode control [16, 17], adaptive fuzzy control [18, 19], nonlinear feedback control [20, 21], the open-plus-closed-loop control [22], observer-based control [23], and some other methods [24–26].

Backstepping approach was first proposed in [27] for designing stabilizing controls for a special type of nonlinear dynamical systems, which is referred to as the strict-feedback system. The technique consists in a recursive procedure that skillfully constructs the Lyapunov function and designs the virtual control input [28–30]. When this recursive procedure terminates, a feedback design for the real control input results [31]. On the other hand, sliding mode control (SMC) is one of the popular strategies to deal with uncertain systems. The main feature of SMC is the robustness against parameter uncertainties and external disturbances. In this paper, combining the merits of backstepping control and SMC, a backstepping sliding mode controller is devised to implement chaos synchronization of uncertain fractional-order strict feedback chaotic system.

The rest of this paper is organized as follows. In Section 2, definition and lemma in fractional calculus are presented.

In Section 3, the backstepping sliding mode controller is systematically designed. In Section 4, synchronization problem of uncertain fractional-order Duffing-Holmes system is handled by the proposed method. Some concluding remarks are made in Section 5.

2. Preliminaries

There are three most frequently used definitions for fractional derivatives [1], that is, Riemann-Liouville, Grünwald-Letnikov, and Caputo definitions. Caputo definition is widely used in engineering applications since it takes on the same form as for integer-order differential equations in the initial conditions. Therefore, the following sections are based on Caputo derivative.

Definition 1 (see [1]). The Caputo fractional derivative of order α of a continuous function $f : R^+ \to R$ is defined as

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m, \end{cases} \tag{1}$$

where Γ is Γ-function, and

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \Gamma(z+1) = z\Gamma(z). \tag{2}$$

It should be noticed that the fractional integral of order $\alpha > 0$ is denoted by $D_t^{-\alpha}$.

In what follows, we list several basic facts of fractional derivatives and integrals which will be used in the stability and stabilization analysis. The readers can refer to [1] for more details.

Fact 1. For $\alpha = n$, where n is an integer, the operation $D_t^n f(t)$ gives the same result as classical calculus of integer-order n. In particular, when $\alpha = 1$, the operation $D_t^1 f(t)$ coincides with the ordinary derivative $df(t)/dt$.

Fact 2. For $\alpha = 0$, the operation $D_t^\alpha f(t)$ is the identity operation

$$D_t^0 f(t) = f(t). \tag{3}$$

Fact 3. Similary to integer-order calculus, fractional differentiation and fractional integration are both linear operations

$$D_t^\alpha [af(t) + bg(t)] = aD_t^\alpha f(t) + bD_t^\alpha g(t). \tag{4}$$

where a and b are constants.

Fact 4. The additive law of exponents (semigroup property) holds

$$D_t^\alpha D_t^\beta f(t) = D_t^\beta D_t^\alpha f(t) = D_t^{\alpha+\beta} f(t). \tag{5}$$

Fact 5. For $\alpha > 0$, the following equation holds:

$$D_t^\alpha D_t^{-\alpha} f(t) = D_t^0 f(t) = f(t) \tag{6}$$

which means that the fractional differentiation operator is a left inverse to the fractional integration operator of the same order α.

Lemma 2 (see [32]). *Let $x = 0$ be an equilibrium point for either Caputo or RL fractional nonautonomous system*

$$D_t^q x(t) = f(x,t), \tag{7}$$

where $q \in (0,1)$ and $f(x,t)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$. Assume that there exits a Lyapunov function $V(t,x(t))$ satisfying

$$\begin{gathered} \alpha_1 \|x\|^a \le V(t,x) \le \alpha_2 \|x\|, \\ \dot{V}(t,x) \le -\alpha_3 \|x\|, \end{gathered} \tag{8}$$

where $\alpha_1, \alpha_2, \alpha_3$, and a are positive constants and $\|\cdot\|$ denotes an arbitrary norm. Then the equilibrium point of system (7) is Mittag-Leffler (asymptotically) stable.

3. Main Results

In synchronization task, there are two dynamical systems, which are, respectively, called master system and slave system. From the view point of control, the task is to design a controller which obtains signals from the master system to tune the behavior of the slave system. In this paper, we consider synchronization of fractional-order strict feedback system described by

$$\begin{gathered} D_t^q x_1 = x_2, \\ D_t^q x_2 = f(t, x_1, x_2), \end{gathered} \tag{9}$$

which is regarded as master system and the slave system is defined as

$$\begin{gathered} D_t^q y_1 = y_2, \\ D_t^q y_2 = f(t, y_1, y_2) + \Delta f(t, y_1, y_2) + d(t) + u(t), \end{gathered} \tag{10}$$

where $0 < q \le 1$, $X = (x_1, x_2)^T$ and $Y = (y_1, y_2)^T$ are the states of systems (9) and (10), $\Delta f(t, y_1, y_2)$ denotes uncertainty, $d(t)$ is the external disturbance, and $u(t)$ is the control input.

Defining the synchronization error as $e_i = y_i - x_i$ $(i = 1, 2)$, then the error system is given by

$$\begin{gathered} D_t^q e_1 = e_2, \\ D_t^q e_2 = f(t, y_1, y_2) - f(t, x_1, x_2) + F(t, Y) + u(t), \end{gathered} \tag{11}$$

where $F(t, Y) = \Delta f(t, y_1, y_2) + d(t)$. Suppose γ is the upper bound of function F; that is, $\|F(t, Y)\| \le \gamma$.

Step 1. We start with the scalar system

$$D_t^q z_1 = e_2, \tag{12}$$

where $z_1 = e_1$, and e_2 is a virtual controller selected as $e_2 = \alpha(z_1)$.

Choose the candidate Lyapunov function as

$$V_1 = \frac{1}{2} z_1^2 > 0; \tag{13}$$

then the time derivative of V_1 along the trajectories of system (12) is

$$\dot{V} = z_1 \dot{z}_1 = z_1 D_t^{1-q} \left(D_t^q z_1\right) = z_1 D_t^{1-q} e_2. \tag{14}$$

Choose $D_t^{1-q} e_2 = -k_1 z_1$; that is, $\alpha(z_1) = -k_1 D_t^{q-1} z_1$, where k_1 is a positive constant. Thus, we have $\dot{V} = -k_1 z_1^2 \leq 0$.

Step 2. Let $z_2 = e_2 - \alpha(z_1)$, so

$$\begin{aligned} D_t^q z_2 &= D_t^q e_2 - D_t^q \alpha\left(z_1\right) \\ &= f\left(t, y_1, y_2\right) - f\left(t, x_1, x_2\right) + F\left(t, Y\right) \\ &\quad + u\left(t\right) + k_1 D_t^{2q-1} z_1. \end{aligned} \tag{15}$$

Design the switching surface as

$$S = k_2 D_t^{-1} z_1 + D_t^{q-1} z_2, \tag{16}$$

where k_2 is the sliding surface parameter to be designed later.

According to SMC method, the condition which guarantees the trajectory of the system arrives at the sliding surface is $S\dot{S} < 0$, and when in the sliding mode, the switching surface and its derivative must satisfy the following conditions:

$$S\left(t\right) = 0, \qquad \dot{S}\left(t\right) = 0. \tag{17}$$

In view of (15), it means that

$$\begin{aligned} \dot{S} &= k_2 z_1 + D_t^q z_2 \\ &= k_2 z_1 + f\left(t, Y\right) - f\left(t, X\right) + F\left(t, Y\right) + k_1 D_t^{2q-1} z_1 + u\left(t\right) \\ &= 0. \end{aligned} \tag{18}$$

Thus, the equivalent control $u_{\text{eq}}(t)$ in the case of uncertainty-free is calculated by

$$u_{\text{eq}}\left(t\right) = f\left(t, X\right) - f\left(t, Y\right) - k_2 z_1 - k_1 D_t^{2q-1} z_1, \tag{19}$$

To satisfy the sliding mode condition, the reaching law is chosen as

$$u_r = -k_3 S - \gamma \operatorname{sgn}\left(S\right), \tag{20}$$

where k_3 is a positive constant. Hence, the total control law is

$$\begin{aligned} u &= u_{\text{eq}} + u_r \\ &= f\left(t, X\right) - f\left(t, Y\right) - k_2 z_1 - k_1 D_t^{2q-1} z_1 - k_3 S - \gamma \operatorname{sgn}\left(S\right). \end{aligned} \tag{21}$$

Selecting

$$V_2 = |S| \tag{22}$$

as a composite Lyapunov function, we have

$$\begin{aligned} \dot{V}_2 &= \operatorname{sgn}\left(S\right) \dot{S} \\ &= \operatorname{sgn}\left(S\right) \left(k_2 z_1 + D_t^q z_2\right) \\ &= \operatorname{sgn}\left(S\right) \Big(k_2 z_1 + f\left(t, Y\right) - f\left(t, X\right) \\ &\qquad + F\left(t, Y\right) + k_1 D_t^{2q-1} z_1 + u\left(t\right)\Big). \end{aligned} \tag{23}$$

Substituting the control law (21) into the right hand side of (23), we have

$$\begin{aligned} \dot{V}_2 &= \operatorname{sgn}\left(S\right) \Big[k_2 z_1 + f\left(t, Y\right) - f\left(t, X\right) \\ &\qquad + F\left(t, Y\right) + k_1 D_t^{2q-1} z_1 \\ &\qquad + \Big(f\left(t, X\right) - f\left(t, Y\right) - k_2 z_1 \\ &\qquad\quad - k_1 D_t^{2q-1} z_1 - k_3 S - \gamma \operatorname{sgn}\left(S\right)\Big)\Big] \\ &= \operatorname{sgn}\left(S\right) \left[F\left(t, Y\right) - k_3 S - \gamma \operatorname{sgn}\left(S\right)\right] \\ &\leq \gamma - k_3 \operatorname{sgn}\left(S\right) S - \gamma \operatorname{sgn}\left(S\right) \operatorname{sgn}\left(S\right) \\ &= -k_3 \left|S\right|. \end{aligned} \tag{24}$$

Hence, according to Lemma 2, system (15) is asymptotically stable.

From the above two steps, we obtain $\lim_{t\to\infty} z_i = 0$ $(i = 1, 2)$. The properties $z_1 = e_1$, $z_2 = e_2 - \alpha(z_1) = e_2 + k_1 D_t^{q-1} z_1$ imply that $\lim_{t\to\infty} e_i = 0$ $(i = 1, 2)$. Thus, the global synchronization of systems (9) and (10) is achieved.

4. Applications

In order to demonstrate the effectiveness of the proposed control scheme, numerical simulations are made for synchronization of fractional-order Duffing-Holmes system. In the simulation, Adams-Bashforth-Moulton predictor-corrector algorithm is used, and the detailed descriptions of this algorithm are available in [33].

4.1. System Description. In [20], an integer-order chaotic Duffing-Holmes system has been studied. Let us consider the fractional-order case described by

$$\begin{aligned} D_t^q x_1 &= x_2, \\ D_t^q x_2 &= x_1 - a x_2 - x_1^3 + b \cos t. \end{aligned} \tag{25}$$

Initial conditions of master system and slave system are set as $x_1(0) = 0.2$, $x_2(0) = 0.2$ and $y_1(0) = 0.1$, $y_2(0) = -0.2$, respectively.

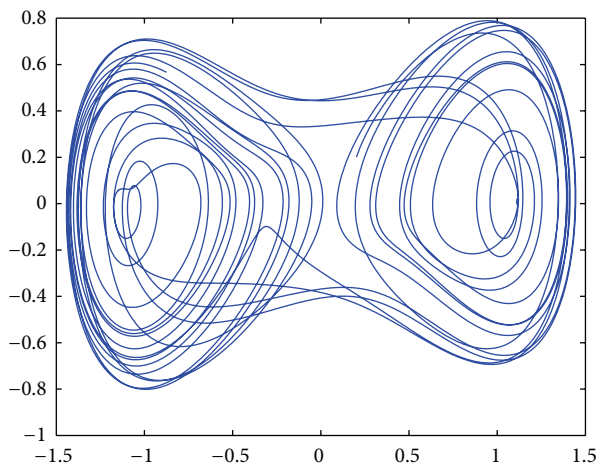

FIGURE 1: Phase portrait of Duffing-Holmes system (25) with fractional-order $q = 0.98$.

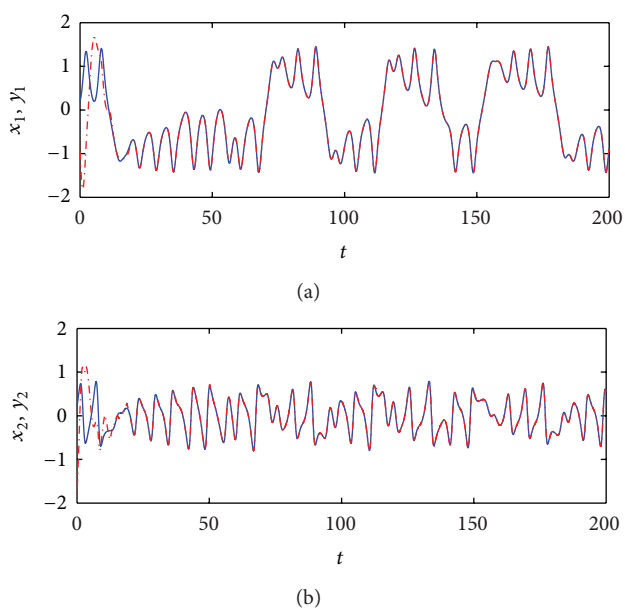

FIGURE 2: Synchronization performance of fractional-order Duffing-Holmes systems (25) and (26).

Phase portrait of the chaotic dynamics in system (25) when $q = 0.98$ are shown in Figure 1. Parameters a and b are set as 0.25 and 0.3, respectively.

The slave system is perturbed by the uncertainty and disturbance, which is described by

$$\begin{aligned} D_t^q y_1 &= y_2, \\ D_t^q y_2 &= y_1 - a y_2 - y_1^3 + b \cos t + \Delta f(t, Y) + d(t) + u, \end{aligned} \tag{26}$$

where $\Delta f(t, Y)$ and $d(t)$ are chosen as $0.1 \sin(t)\sqrt{y_1^2 + y_2^2}$ and $0.1 \sin(t)$, respectively.

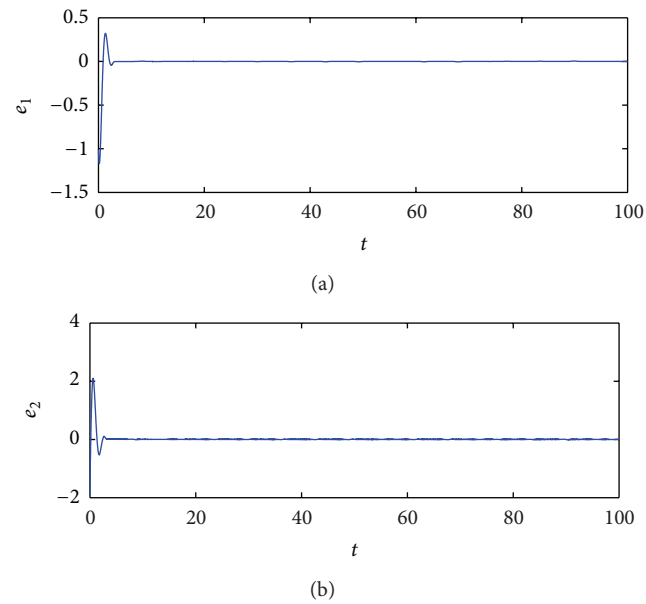

FIGURE 3: Synchronization error between fractional-order Duffing-Holmes systems (25) and (26).

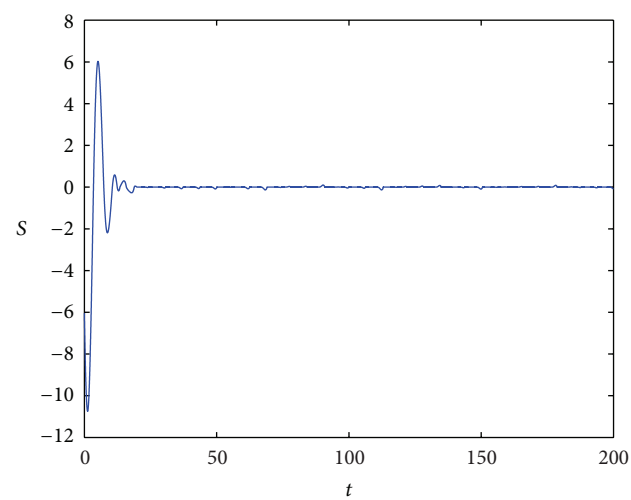

FIGURE 4: Switching surface S of the sliding mode controller (27).

According to the scheme proposed in Section 3, the control law is designed as follows:

$$\begin{aligned} u = {} & a e_2 - (1 + k_2) e_1 - k_1 D_t^{2q-1} e_1 + y_1^3 \\ & - x_1^3 - \gamma \operatorname{sgn}(S) - k_3 S, \end{aligned} \tag{27}$$

in which $S = k_2 D_t^{-1} z_1 + D_t^{q-1} z_2 = k_2 D_t^{-1} e_1 + D_t^{q-1} e_2 + k_1 D_t^{2q-2} e_1$. When the parameters are chosen as $q = 0.98$, $k_1 = k_2 = k_3 = 2$, $\gamma = 0.2$, the simulation results are shown in Figures 2–5. Figures 2 and 3 show the synchronization performance between drive-response system (25)-(26) and the time response of the error system, respectively. The sliding surface and the control input u are displayed in Figures 4 and 5.

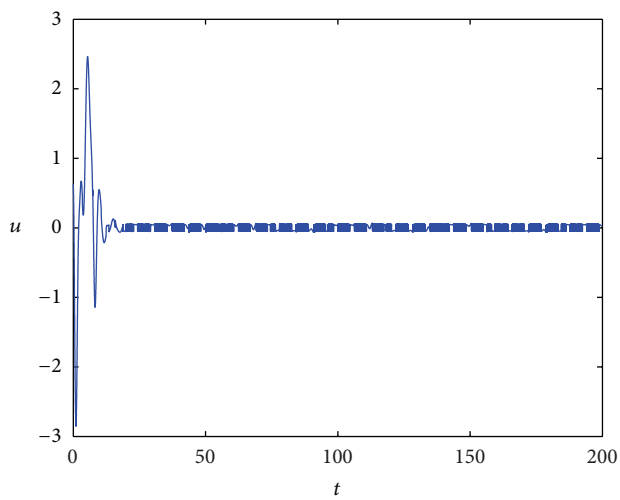

Figure 5: Control input u of the slave system (26).

5. Conclusions

In this paper, the backstepping SMC method is developed for fractional-order strict-feedback chaotic system. The Lyapunov function, the virtual control, the switching surface, and the actual control are systematically designed, respectively. The proposed approach, which combines the merits of backstepping control and SMC, is robust against the modeling uncertainties and external disturbances. Numerical results demonstrate that the convergence speed of the synchronized error system is satisfactory. In addition, the strategy can be generalized to the investigation of synchronization of other fractional-order strict feedback chaotic or hyperchaotic systems.

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