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Research Article

Upper and Lower Solutions for m-Point Impulsive BVP with One-Dimensional p-Laplacian

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Upper and lower solutions theories are established for a kind of m-point impulsive boundary value problems with p-Laplacian. By using such techniques and the Schauder fixed point theorem, the existence of solutions and positive solutions is obtained. Nagumo conditions play an important role in the nonlinear term involved with the first-order derivatives.

1. Introduction

In this paper, we study the following m-point impulsive boundary value problem with one-dimensional p-Laplacian:

$$\begin{gathered}\left(\phi_p\left(u'(t)\right)\right)' + q(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in J', \\ \Delta u|_{t=t_k} = I_{1k}\left(u(t_k)\right), \quad k = 1, 2, \ldots, n, \\ \Delta\phi_p\left(u'\right)\big|_{t=t_k} = I_{2k}\left(u(t_k), u'(t_k)\right), \quad k = 1, 2, \ldots, n, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \qquad u'(1) = 0,\end{gathered} \tag{1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. $J = [0,1]$, t_k ($k = 1, 2, \ldots, n$, where n is a fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_n < 1$. $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$. $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\sum_{i=1}^{m-2} a_i < 1$. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $k = 1, 2, \ldots, n$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. For an introduction of the basic theory of impulsive differential equations in R^n, see Lakshmikantham et al. [1], Baĭnov and Simeonov [2], Samoilenko and Perestyuk [3], and the references therein. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations (see for instance [4–9] and the references therein).

At the same time, it is well known that the method of lower and upper solutions is a powerful tool for proving the existence results for a large class of boundary value problems. For a few of such works, we refer the readers to [10–14].

In [15], Cabada and Pouso considered

$$\begin{gathered}\left(\phi_p\left(u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \quad \text{for a.e. } t \in I = [a, b], \\ g\left(u(a), u'(a), u'(b)\right) = 0, \qquad u(b) = h(u(a)),\end{gathered} \tag{2}$$

by using the upper and lower method, the authors get the existence of solution to the above BVP.

In [16], Lü et al. studied

$$\begin{gathered}\left(\phi_p\left(u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \quad \text{for a.e. } t \in I = [0, 1], \\ u(0) = 0, \qquad u(1) = 0,\end{gathered} \tag{3}$$

by giving conditions on f involving pairs of lower and upper solutions, they get the existence of at least three solutions to the above BVP.

In [12], Shen and Wang studied

$$\begin{aligned} x''(t) &= f\left(t, x(t), x'(t)\right), \quad \text{for } t \in J,\ t \neq t_k, \\ \delta x(t_k) &= I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, p, \\ \delta x'(t_k) &= J_k\left(x(t_k), x'(t_k)\right), \quad k = 1, 2, \ldots, p, \\ g\left(x(0), x'(0)\right) &= 0, \qquad h\left(x(1), x'(1)\right) = 0, \end{aligned} \tag{4}$$

they prove the existence of solutions to the problem under the assumption that there exist lower and upper solutions associated with the problem.

Motivated by the works mentioned above, in this paper, we considered BVP (1), the main tool is upper and lower method, and the Schäuder fixed point theorem. We not only get the existence of solutions, but also the existence of positive solutions.

The main structure of this paper is as follows. In Section 2, we give the preliminary and present some lemmas in order to prove our main results. Section 4 presents the main theorems of this paper, and at the end of Section 4, we give an example to illustrate our main results.

2. Preliminary

Define $PC[0,1] = \{u \mid u : [0,1] \to R$ is continuous at $t \neq t_k$, $u(t_k) = u(t_k^-)$, $u(t_k^+)$ exists, left continuous$\}$, and $PC^1[0,1] = \{u \mid u \in PC[0,1], u' : [0,1] \to R$ is continuous at $t \neq t_k$, $u'(t_k^+)$ and $u'(t_k^-)$ exists, $u'(t_k) = u'(t_k^-)$, $k = 1, 2, \ldots, n\}$. Then, $PC^1[0,1]$ is a real Banach space with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{0\le t\le 1}|u(t)|$.

By a solution of (1), we mean a function $u \in E$ which satisfies (1), where $E = PC[0,1] \cap C^2(J')$.

Definition 1. A function $\alpha \in E$, with $\phi_p(\alpha') \in PC^1(0,1)$, will be called a lower solution of (1) if

$$\begin{aligned} &\left(\phi_p\left(\alpha'(t)\right)\right)' + q(t) f\left(t, \alpha(t), \alpha'(t)\right) \ge 0, \quad t \in J', \\ &\Delta\alpha|_{t=t_k} \le I_{1k}\left(\alpha(t_k)\right), \quad k = 1, 2, \ldots, n, \\ &\Delta\phi_p(\alpha')\big|_{t=t_k} \le I_{2k}\left(\alpha(t_k), \alpha'(t_k)\right), \quad k = 1, 2, \ldots, n, \\ &\alpha(0) \le \sum_{i=1}^{m-2} a_i \alpha(\eta_i), \qquad \alpha'(1) < 0. \end{aligned} \tag{5}$$

A function $\beta \in E$, with $\phi_p(\beta') \in PC^1(0,1)$, satisfying the reversed inequalities is an upper solution of problem (1).

Next, we define the Nagumo condition we are going to use. Note that the condition does not depend on the boundary data of the problem.

Definition 2. We say that f satisfies a Nagumo condition relative to the pair α and β, with $\alpha, \beta \in E$, $\alpha \le \beta$ in $[0,1]$, if there exists a function $\psi : C([0,+\infty), [0,+\infty))$ such that

$$\left|f(t,x,y)\right| \le \psi\left(|y|\right), \quad \forall (t,x,y) \in P, \tag{6}$$

where $P = \{(t,x,y) \in [0,1]\times[0,+\infty)\times R; \alpha(t) \le x(t) \le \beta(t)\}$, and also that

$$\int_{\phi_p(\nu)}^{\infty} \frac{ds}{\psi\left(\phi_p^{-1}(s)\right)} > \int_0^1 q(s)\, ds, \tag{7}$$

here, $\nu = \max_{k=1,2,\ldots,n}\{|(\beta(t_{k+1})-\alpha(t_k))/(t_{k+1}-t_k)|, |(\alpha(t_{k+1})-\beta(t_k))/(t_{k+1}-t_k)|\}$.

From Definition 2, we can find a real number $L > \nu > 0$ such that

$$\int_{\phi_p(\nu)}^{\phi_p(L)} \frac{ds}{\psi\left(\phi_p^{-1}(s)\right)} > \int_0^1 q(t)\, dt. \tag{8}$$

In addition, we assume that the following three conditions hold:

(H_1) $q \in C(0,1)$ with $q > 0$ on $(0,1)$ and $\int_0^1 q(s)ds < \infty$;

(H_2) $f \in C([0,1] \times R^2, R)$;

(H_3) $I_{1k}, I_{2k} \in C(R,R)$. $I_{2k}(y,z)$ is nondecreasing in $z \in [-L, L]$ for all $1 \le k \le n$.

Firstly, we define

$$P_{\alpha,\beta}(t,x) = \begin{cases} \beta(t), & x(t) \ge \beta(t), \\ x(t), & \alpha(t) \le x(t) \le \beta(t) \quad \forall x \in [0,+\infty), \\ \beta(t), & x(t) \le \alpha(t). \end{cases} \tag{9}$$

One can find the next result, with its proof, in [17].

Lemma 3. *For each $u \in E$, the next two properties hold:*

(i) *$dP_{\alpha,\beta}(t,u)/dt$ exists for a.e. $t \in J'$,*

(ii) *if $u, u_m \in E$ and $u_m \to u$ in E, then*

$$\frac{d}{dt}P_{\alpha,\beta}(t, u_m(t)) \longrightarrow \frac{d}{dt}P_{\alpha,\beta}(t, u(t)) \quad \text{for a.e. } t \in J'. \tag{10}$$

We consider the following modified problem:

$$\begin{aligned} &\left(\phi_p\left(u'(t)\right)\right)'(t) + q(t) f^*\left(t, u(t), \frac{d}{dt}P_{\alpha,\beta}(t,u)\right) = 0, \\ &\qquad t \in J', \\ &\Delta u|_{t=t_k} = I_{1k}\left(P_{\alpha,\beta}(t,u)\right), \quad k = 1, 2, \ldots, n, \\ &\Delta\phi_p(u')\big|_{t=t_k} = I_{2k}\left(P_{\alpha,\beta}(t,u), h\left(t, u'\right)\right), \quad k = 1, 2, \ldots, n, \\ &u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \qquad u'(1) = 0, \end{aligned} \tag{11}$$

with

$$f^*(t,x,y) = f\left(t, P_{\alpha,\beta}(t,x), h(t,y)\right) + \tanh\left(x - P_{\alpha,\beta}(t,x)\right), \tag{12}$$

where h is defined by

$$h(t,y) = \begin{cases} L, & y(t) > L, \\ y(t), & |y(t)| \le L, \\ L, & y(t) < -L. \end{cases} \quad \forall y \in R, \tag{13}$$

Thus f^* is a continuous function on $[0,1] \times [0,\infty) \times R$ and satisfies

$$\begin{gathered} |f^*(t,x,y)| \le \psi(|y|) + 1, \quad \forall |y| < L, \\ |f^*(t,x,y)| \le M, \quad \forall (t,x,y) \in P, \end{gathered} \tag{14}$$

for some constant M. Moreover, we may choose M so that $\|\alpha\|_\infty, \|\beta\|_\infty < M$.

Lemma 4. *Suppose that $\sum_{i=1}^{m-2} a_i \ne 1$, denote $a = \sum_{i=1}^{m-2} a_i$, then the following boundary value problem:*

$$\begin{gathered} \left(\phi_p\left(u'(t)\right)\right)' + v(t) = 0, \quad t \in J', \\ \Delta u|_{t=t_k} = I_{1k}\left(u(t_k)\right), \quad k = 1,2,\ldots,n, \\ \Delta\phi_p(u')\big|_{t=t_k} = I_{2k}\left(u(t_k), u'(t_k)\right), \quad k = 1,2,\ldots,n, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \qquad u'(1) = 0, \end{gathered} \tag{15}$$

has a unique solution as follows:

$$\begin{aligned} u(t) = {} & \frac{1}{1-a}\sum_{i=1}^{m-2} a_i \Bigg(\sum_{t_k<\eta_i} I_{1k}\left(u(t_k)\right) \\ & + \int_0^{\eta_i} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds \Bigg) \\ & + \int_0^t \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds \\ & + \sum_{t_k<t} I_{1k}\left(u(t_k)\right). \end{aligned} \tag{16}$$

Proof.

$$\left(\phi_p\left(u'(t)\right)\right)' = -v(t). \tag{17}$$

For $t \in (t_n, 1)$, integrating (17) from t to 1, we get

$$u'(t) = \phi_p^{-1}\left(\int_t^1 v(s)\,ds\right). \tag{18}$$

For $t \in (t_{n-1}, t_n]$, integrating (17) from t to t_n, we get

$$u'(t) = \phi_p^{-1}\left[\left(\int_t^{t_n} v(s)\,ds\right) + \phi_p\left(u'(t_n^-)\right)\right]. \tag{19}$$

Combining (18), (19), and the boundary condition, for $t \in (t_{n-1}, t_n]$, we have

$$\begin{aligned} u'(t) &= \phi_p^{-1}\left[\left(\int_t^{t_n} v(s)\,ds\right) + \phi_p\left(u'(t_n^+)\right) - I_{2n}\left(u(t_n), u'(t_n)\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^{t_n} v(s)\,ds\right) + \int_{t_n}^1 v(s)\,ds - I_{2n}\left(u(t_n), u'(t_n)\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^1 v(s)\,ds\right) - I_{2n}\left(u(t_n), u'(t_n)\right)\right]. \end{aligned} \tag{20}$$

For $t \in (t_{n-2}, t_{n-1}]$, integrating (17) from t to t_{n-1}, and combining (20), we have

$$\begin{aligned} u'(t) &= \phi_p^{-1}\left[\left(\int_t^{t_{n-1}} v(s)\,ds\right) + \phi_p\left(u'(t_{n-1}^-)\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^{t_{n-1}} v(s)\,ds\right) + \phi_p\left(u'(t_{n-1}^+)\right) - I_{2(n-1)}\left(u(t_{n-1}), u'(t_{n-1})\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^{t_{n-1}} v(s)\,ds\right) + \int_{t_{n-1}}^1 v(s)\,ds - I_{2n}\left(u(t_n), u'(t_n)\right) - I_{2(n-1)}\left(u(t_{n-1}), u'(t_{n-1})\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^1 v(s)\,ds\right) - I_{2n}\left(u(t_n), u'(t_n)\right) - I_{2(n-1)}\left(u(t_{n-1}), u'(t_{n-1})\right)\right] \\ &= \phi_p^{-1}\left[\left(\int_t^1 v(s)\,ds\right) - \sum_{t<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right]. \end{aligned} \tag{21}$$

By induction, for $t \in (0,1)$, there holds

$$u'(t) = \phi_p^{-1}\left[\left(\int_t^1 v(s)\,ds\right) - \sum_{t<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right]. \tag{22}$$

Integrating (22) from 0 to t_1, we have

$$u(t_1) = u(0) + \int_0^{t_1} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds. \tag{23}$$

For $t \in (t_1, t_2]$, integrating (22) from t_1 to t, we have

$$u(t) = u(t_1^+) + \int_{t_1}^{t} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds. \tag{24}$$

By the boundary condition $\Delta u|_{t=t_k} = I_{1k}(u(t_k))$, (23), and (24), for $t \in (t_1, t_2]$, we have

$$u(t) = u(0) + I_{11}(u(t_1)) + \int_0^{t} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds. \tag{25}$$

Repeating in the above manner, for $t \in [0, 1]$, we have

$$u(t) = u(0) + \sum_{t_k<t} I_{1k}(u(t_k)) + \int_0^{t} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds. \tag{26}$$

By (26), we have

$$u(\eta_i) = u(0) + \sum_{t_k<\eta_i} I_{1k}(u(t_k)) + \int_0^{\eta_i} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds, \quad i = 0, 1, \ldots, m-2. \tag{27}$$

By the boundary condition $u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i)$, we obtain

$$u(0) = \frac{1}{1-a} \sum_{i=1}^{m-2} a_i \left(\sum_{t_k<\eta_i} I_{1k}(u(t_k)) + \int_0^{\eta_i} \phi_p^{-1}\left(\int_s^1 v(\tau)\,d\tau - \sum_{s<t_k} I_{2k}\left(u(t_k), u'(t_k)\right)\right) ds\right). \tag{28}$$

Substituting (26) into (28), we have (16) which holds. The proof is complete. □

Lemma 5. *If u is a solution of BVP* (11), *$\alpha(t)$ and $\beta(t)$ are lower and upper solutions of* (1), *respectively, $\alpha \le \beta$, and*

$$I_{1k}(\alpha(t_k)) \le I_{1k}(u) \le I_{1k}(\beta(t_k)), \quad k = 1, \ldots, p, \quad \text{for } \alpha(t_k) \le u \le \beta(t_k), \tag{29}$$

then,

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0, 1]. \tag{30}$$

Proof. Denote $y(t) = u(t) - \beta(t)$, we will only see that $u(t) \le \beta(t)$ for every $t \in J$. An analogous reasoning shows that $u(t) \ge \alpha(t)$ for all $t \in J$.

Otherwise, if $u(t) \le \beta(t)$, $t \in [0, 1]$ does not hold, then $\sup_{0\le t\le 1}(u(t) - \beta(t)) > 0$. Since $u'(1) - \beta'(1) < 0$, there are three cases.

Case 1. Denote $t_0 = 0$, $t_{n+1} = 1$. There exists $p \in \{0, 1, 2, \ldots, n\}$, $\bar{t}_0 \in (t_p, t_{p+1})$ such that $y(\bar{t}_0) = \sup_{0\le t\le 1} y(t) > 0$. Then, $\sup_{0\le t\le 1} y(t) = \max_{0\le t\le 1} y(t)$. Let $\bar{t}_1 = \max\{t \mid y(t) \le 0,\ t \in [t_p, \bar{t}_0)\}$, $\bar{t}_2 = \min\{t \mid y(t) \le 0,\ t \in (\bar{t}_0, t_{p+1}]\}$. If $y(t) > 0$ for all $t \in [t_p, \bar{t}_0)$, let $\bar{t}_1 = t_p$, and if $y(t) > 0$ for all $t \in (\bar{t}_0, t_{p+1}]$, let $\bar{t}_2 = t_{p+1}$. Thus,

$$\begin{aligned} &\left[\phi_p(u'(t))\right]' - \left[\phi_p(\beta'(t))\right]' \\ &\quad \ge f(t, \beta(t), \beta'(t)) \\ &\qquad + \tanh(u(t) - \beta(t)) - f(t, \beta(t), \beta'(t)) \\ &\quad = \tanh(u(t) - \beta(t)) > 0, \quad \forall t \in (\bar{t}_1, \bar{t}_2). \end{aligned} \tag{31}$$

As a consequence, $\phi_p(u'(t)) - \phi_p(\beta'(t)) > \phi_p(u'(\bar{t}_0)) - \phi_p(\beta'(\bar{t}_0)) = 0$ for all $t \in (\bar{t}_0, \bar{t}_2)$ and since ϕ_p is increasing and, in particular, one to one, we have

$$u'(t) > \beta'(t), \quad \forall t \in (\bar{t}_0, \bar{t}_2). \tag{32}$$

Thus, $u(\bar{t}_2) - \beta(\bar{t}_2) > u(\bar{t}_0) - \beta(\bar{t}_0) > 0$, which is a contradiction.

Case 2. According to Case 1, if $\sup_{0\le t\le 1} y(t) > 0$, then $\sup_{0\le t\le 1} y(t) = y(t_k^-) = y(t_k)$ or $\sup_{0\le t\le 1} y(t) = y(t_k^+)$, $k = 0, 1, 2, \ldots, n$. Since $y'(1) < 0$, $y(1)$ cannot be $\sup_{0\le t\le 1} y(t)$.

Next, we claim that for $k = 0, 1, 2, \ldots, n-1$, if $\sup_{t\in[t_k,t_{k+1}]} y(t) > 0$, then $\sup_{t\in[t_k,t_{k+1}]} y(t) = y(t_k^+)$. Otherwise, $\sup_{t\in[t_k,t_{k+1}]} y(t) = y(t_{k+1}^-) = y(t_{k+1})$. Suppose that $\sup_{0\le t\le t_1} y(t) = y(t_1^-) = y(t_1) > 0$ and $y'(t_1) = y'(t_1^-) \ge 0$.

From the boundary condition and (29), we have

$$\begin{aligned} & u(t_1^+) - u(t_1) \\ & \quad = u(t_1^+) - u(t_1^-) \\ & \quad = I_{1k}\left(P_{\alpha,\beta}(t,u)\right) = I_{1k}(\beta(t_1)) \\ & \quad = \beta(t_1^+) - \beta(t_1^-) = \beta(t_1^+) - \beta(t_1), \\ & \phi_p\left(u'(t_1^+)\right) - \phi_p\left(u'(t_1)\right) \\ & \quad = I_{2k}\left(P_{\alpha,\beta}(t_1,u), h\left(t_1,u'\right)\right) \\ & \quad = I_{2k}\left(\beta(t_1), \beta'(t_1)\right) \\ & \quad \geq \phi_p\left(\beta'(t_1^+)\right) - \phi_p\left(\beta'(t_1)\right). \end{aligned} \tag{33}$$

Hence,

$$\begin{aligned} y(t_1^+) &= u(t_1^+) - \beta(t_1^+) = u(t_1) - \beta(t_1) > 0, \\ y'(t_1^+) &= u'(t_1^+) - \beta'(t_1^+) \geq u'(t_1) - \beta'(t_1) \geq 0. \end{aligned} \tag{34}$$

Suppose that $y'(t_1^+) = 0$ and y is nonincreasing on some interval $(t_1, t_1+\gamma) \subset (t_1, t_2)$, where $\gamma > 0$ is sufficiently small such that $y(t) > 0$ on $t \in (t_1, t_1+\gamma)$. For $t \in (t_1, t_1+\gamma)$,

$$\begin{aligned} & \left[\phi_p\left(u'(t)\right)\right]' - \left[\phi_p\left(\beta'(t)\right)\right]' \\ & \quad \geq f\left(t, \beta(t), \frac{d}{dt}P_{\alpha,\beta}(t,u)\right) + \tanh\left(u(t) - \beta(t)\right) \\ & \quad - f\left(t, \beta(t), \beta'(t)\right) \\ & \quad = \tanh\left(u(t) - \beta(t)\right) > 0. \end{aligned} \tag{35}$$

Note that $f(t, \beta(t), (d/dt)P_{\alpha,\beta}(t,u)) - f(t, \beta(t), \beta'(t)) \to 0$ as $\gamma \to 0^+$. Hence, when $\gamma > 0$ is sufficiently small, $[\phi_p(u'(t))]' - [\phi_p(\beta'(t))]' > 0$, $t \in (t_1, t_1+\gamma)$, which contradicts the assumption of monotonicity of y. Thus, we obtain

$$\begin{aligned} 0 < y(t_1^+) < y(t_2) &= y(t_2^-), \\ y'(t_2) &= y'(t_2^-) \geq 0. \end{aligned} \tag{36}$$

We use the preceding procedure and deduce by induction that

$$y(t_k) > 0, \quad y'(t_k) \geq 0, \quad k = 1, 2, \ldots, n+1, \tag{37}$$

which contradicts that $y'(1) < 0$. Essentially, by the same analysis, we can get that $y(t_k) > 0$, $k = 2, 3, \ldots, n$ cannot hold.

Case 3. $u(0) - \beta(0) = \sup_{0\leq t\leq 1}(u(t) - \beta(t)) = \max_{0\leq t\leq 1}(u(t) - \beta(t)) > 0$. Easily, it holds that $u'(0^+) - \beta'(0^+) \leq 0$. While by the boundary condition and $\sum_{i=1}^{m-2} a_i < 1$, we have

$$\begin{aligned} u(0) - \beta(0) &\leq \sum_{i=1}^{m-2} a_i \left(u(\eta_i) - \beta(\eta_i)\right) \\ &< \max_{i=1,2,\ldots,m-2} \left\{u(\eta_i) - \beta(\eta_i)\right\}, \end{aligned} \tag{38}$$

which is also a contradiction.

Consequently, $u(t) \leq \beta(t)$ holds for all $t \in [0,1]$. The proof is complete. □

Lemma 6. *If u is a solution of* (11), *then $-L < u'(t) < L$ for all $t \in [0,1]$.*

Proof. Let $u \in E$ be a solution of (11). From Lemma 5, we have $\alpha(t) \leq u(t) \leq \beta(t)$, and so

$$\left(\phi_p\left(u'(t)\right)\right)' + q(t) f\left(t, u(t), u'(t)\right) = 0 \quad \text{for } t \in J'. \tag{39}$$

By the mean value theorem, there exists $\xi_0 \in (t_k, t_{k+1})$ with $u'(\xi_0) = (u(t_{k+1}) - u(t_k))/(t_{k+1} - t_k)$, $k = 1, 2, \ldots, n$, and as a result,

$$\begin{aligned} -L < -\nu \leq \frac{\alpha(t_{k+1}) - \beta(t_k)}{t_{k+1} - t_k} &\leq u'(\xi_0) \\ &\leq \frac{\beta(t_{k+1}) - \alpha(t_k)}{t_{k+1} - t_k} \leq \nu < L. \end{aligned} \tag{40}$$

Let $\nu_0 = |u'(\xi_0)|$. Suppose that there exists a point in the interval $[0,1]$ for which $u' > L$ or $u' < -L$. From the continuity of u', we can choose $\xi_1 \in [0,1]$, such that one of the following situations holds:

(i) $u'(\xi_0) = \nu_0$, $u'(\xi_1) = L$ and $\nu_0 \leq u'(t) \leq L$ for all $t \in (\xi_0, \xi_1)$;

(ii) $u'(\xi_1) = L$, $u'(\xi_0) = \nu_0$ and $\nu_0 \leq u'(t) \leq L$ for all $t \in (\xi_1, \xi_0)$;

(iii) $u'(\xi_0) = -\nu_0$, $u'(\xi_1) = -L$ and $-L \leq u'(t) \leq -\nu_0$ for all $t \in (\xi_0, \xi_1)$;

(iv) $u'(\xi_1) = -L$, $u'(\xi_0) = -\nu_0$ and $-L \leq u'(t) \leq -\nu_0$ for all $t \in (\xi_1, \xi_0)$.

Without loss of generality, suppose that $-L \leq \nu_0 \leq u'(t) \leq L$ for all $t \in (\xi_0, \xi_1)$. Then,

$$\left(\phi_p\left(u'(t)\right)\right)' n + q(t) f\left(t, u(t), u'(t)\right) = 0 \quad \text{for } t \in (\xi_0, \xi_1), \tag{41}$$

and so

$$\begin{aligned} \left|\left(\phi_p\left(u'(t)\right)\right)'\right| &= \left|q(t) f\left(t, u(t), u'(t)\right)\right| \\ &\leq q(t) \left|\psi\left(\left|u'(t)\right|\right)\right|, \quad \text{for } t \in (\xi_0, \xi_1). \end{aligned} \tag{42}$$

As a result,

$$\begin{aligned} \int_{\phi_p(\nu_0)}^{\phi_p(L)} \frac{ds}{\psi\left(\phi_p^{-1}(s)\right)} &= \int_{\xi_0}^{\xi_1} \frac{\left|\left(\phi_p\left(u'(t)\right)\right)'\right|}{\psi\left(\left|u'(t)\right|\right)} dt \\ &\leq \int_{\xi_0}^{\xi_1} q(t)\, dt < \int_0^1 q(t)\, dt. \end{aligned} \tag{43}$$

Note also that $\phi_p^{-1}(s) \geq 0$ for $s \in [\phi_p(\nu_0), \phi_p(L)]$, so we have $\nu_0 \leq \nu$ and thus $\phi_p(\nu_0) \leq \phi_p(\nu)$, which leads to the following:

$$\int_{\phi_p(\nu_0)}^{\phi_p(L)} \frac{du}{\psi\left(\phi_p^{-1}(u)\right)} \geq \int_{\phi_p(\nu)}^{\phi_p(L)} \frac{1}{\psi\left(\phi_p^{-1}(s)\right)} ds > \int_0^1 q(t)\, dt, \tag{44}$$

a contradiction. The proof is complete. □

Theorem 7 (Schäuder fixed point theorem). *Let P be a convex subset of a normed Linear space E. Each continuous, compact map $T : P \to P$ has a fixed point.*

3. The Main Results

Theorem 8. *Suppose that conditions (H_1)–(H_3) hold. Then, BVP* (1) *has at least one solution $u \in E \cap C^2(J')$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad -L < u'(t) < L, \quad t \in [0,1]. \tag{45}$$

Proof. Solving (11) is equivalent to finding a $u \in E$ which satisfies

$$\begin{aligned} u(t) &= \frac{1}{1-a}\sum_{i=1}^{m-2} a_i \Bigg(\sum_{t_k<\eta_i} I_{1k}^*(u(t_k)) + \int_0^{\eta_i} \phi_p^{-1}\Bigg(\int_s^1 F_u^*(\tau)\, d\tau - \sum_{s<t_k} I_{2k}^*\left(u(t_k), u'(t_k)\right)\Bigg) ds \Bigg) \\ &\quad + \left(\int_0^t \phi_p^{-1}\left(\int_s^1 F_u^* d\tau \right) ds - \sum_{s<t_k} I_{2k}^*\left(u(t_k), u'(t_k)\right) \right) + \sum_{t_k<t} I_{1k}^*(u(t_k)), \end{aligned} \tag{46}$$

where $F_u^*(t) = q(t) f^*(t, u(t), u'(t))$, $I_{1k}^*(u(t_k)) = I_{1k}(P_{\alpha,\beta}(t, u))$, $I_{2k}^*(u(t_k), u'(t_k)) = I_{2k}(P_{\alpha,\beta}(t, u), h(t, u'))$.

Now, define the following operator $T : E \to E$ by

$$\begin{aligned} (Tu)(t) &= \frac{1}{1-a}\sum_{i=1}^{m-2} a_i \Bigg(\sum_{t_k<\eta_i} I_{1k}^*(u(t_k)) + \int_0^{\eta_i} \phi_p^{-1}\Bigg(\int_s^1 F_u^*(\tau)\, d\tau - \sum_{s<t_k} I_{2k}^*\left(u(t_k), u'(t_k)\right)\Bigg) ds \Bigg) \\ &\quad + \int_0^t \phi_p^{-1}\left(\int_s^1 F_u^*(\tau)\, d\tau - \sum_{s<t_k} I_{2k}^*\left(u(t_k), u'(t_k)\right) \right) ds + \sum_{t_k<t} I_{1k}^*(u(t_k)). \end{aligned} \tag{47}$$

It is obvious that $T : E \to E$ is completely continuous.

By the Schäuder fixed point Theorem 7, we can easily obtain that T has a fixed point $u \in E$, which is a solution of BVP (11). And by Lemmas 5 and 6, we know that $\alpha(t) \leq u(t) \leq \beta(t)$, $-L < u'(t) < L$, then BVP (11) becomes BVP (1), therefore $u(t)$ is a solution of BVP (1). The proof is complete. □

Theorem 9. *Suppose that conditions (H_1)–(H_3) hold. Assume that there exist two lower solutions α_1 and α_2 and two upper solutions β_1 and β_2 for problem* (1)*, satisfying the following:*

(i) $\alpha_1 \leq \alpha_2 \leq \beta_2$;

(ii) $\alpha_1 \leq \beta_1 \leq \beta_2$;

(iii) $\alpha_2 \nleq \beta_1$, *which means that there exists $t \in [0,1]$ such that $\alpha_2(t) > \beta_1(t)$;*

(iv) *if u is a solution of* (1) *with $u \geq \alpha_2$, then $u > \alpha_2$ on* $(0,1)$;

(v) *if u is a solution of* (1) *with $u \leq \beta_1$, then $u < \beta_1$ on* $(0,1)$.

If f satisfies the Nagumo condition with respect to α_1, β_2, then problem (1) *has at least three solutions u_1, u_2, and u_3 satisfying*

$$\alpha_1 \leq u_1 \leq \beta_1, \quad \alpha_2 \leq u_2 \leq \beta_2, \quad u_3 \nleq \beta_1, \quad u_3 \ngeq \alpha_2. \tag{48}$$

Proof. We consider the following modified problem:

$$\begin{aligned} &\left(\phi_p\left(u'(t)\right)\right)' + q(t) f^*\left(t, u(t), \frac{d}{dt} P_{\alpha_1,\beta_2}(t,u)\right) = 0, \quad t \in J', \\ &\Delta u|_{t=t_k} = \overline{I}_{1k}(u(t_k)), \quad k = 1, 2, \ldots, n, \\ &\Delta\phi_p(u')\big|_{t=t_k} = \overline{I}_{2k}\left(u(t_k), u'(t_k)\right), \quad k = 1, 2, \ldots, n, \\ &u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \qquad u'(1) = 0. \end{aligned} \tag{49}$$

Now, define the following operator $\overline{T} : E \to E$ by

$$\begin{aligned} \left(\overline{T}u\right)(t) &= \frac{1}{1-a}\sum_{i=1}^{m-2} a_i \Bigg(\sum_{t_k<\eta_i} \overline{I}_{1k}(u(t_k)) + \int_0^{\eta_i} \phi_p^{-1}\Bigg(\int_s^1 \overline{F}_u(\tau)\, d\tau - \sum_{s<t_k} \overline{I}_{2k}\left(u(t_k), u'(t_k)\right)\Bigg) ds \Bigg) \end{aligned}$$

$$+\int_0^t \phi_p^{-1}\left(\int_s^1 \overline{F}_u(\tau)\,d\tau - \sum_{s<t_k} \overline{I}_{2k}\left(u(t_k), u'(t_k)\right)\right) ds + \sum_{t_k<t} \overline{I}_{1k}(u(t_k)), \tag{50}$$

where $\overline{F}_u = q(t) f^*(t, u(t), (d/dt)P_{\alpha_1,\beta_2}(t,u))$, $\overline{I}_{1k}(u(t_k)) = I_{1k}(P_{\alpha_1,\beta_2}(t_k,u))$, and $\overline{I}_{2k} = I_{2k}(P_{\alpha_1,\beta_2}(t,u), h(t,u'))$.

It is standard that $\overline{T} : E \to E$ is completely continuous.

Let $N > (4/(1-a))((MQ)^{1/(p-1)} + nI)$, where $M = \max\{f(t,x,y) \mid (t,x,y) \in [0,1] \times [\alpha_1,\beta_2] \times [-L,L]\}$, $Q = \int_0^1 q(t)dt$, $I = \max\{\max_{t_k<1}(I_{1k}(P_{\alpha_1,\beta_2}(t_k,u)), I_{2k}(P_{\alpha,\beta}(t,u), h(t,u'))\}$.

Let $\Omega = \{u \in E : \|u\| < N\}$.

It is immediate from the argument above that $\overline{T}(\overline{\Omega}) \subset \Omega$. Thus,

$$\deg\left(I - \overline{T}, \Omega, 0\right) = 1. \tag{51}$$

Let

$$\Omega_{\alpha_2} = \{u \in \Omega : u > \alpha_2 \text{ on } (0,1)\}, \qquad \Omega^{\beta_1} = \{u \in \Omega : u < \beta_1 \text{ on } (0,1)\}. \tag{52}$$

Since $\alpha_2 \not\leq \beta_1$, $\alpha_2 > -\overline{M}$, $\beta_1 < \overline{M}$ (i.e., choose $\overline{M}$ such that $\|\alpha_2\|_\infty, \|\beta_1\|_\infty < M$). It follows that $\Omega^{\beta_1} \neq \emptyset \neq \Omega_{\alpha_2}$, $\Omega^{\beta_1} \cap \Omega_{\alpha_2} = \emptyset$, and $\Omega \setminus (\overline{\Omega^{\beta_1} \cup \Omega_{\alpha_2}}) \neq \emptyset$.

By assumptions (iv) and (v), there are no solutions in $\partial\Omega^{\beta_1} \cup \partial\Omega_{\alpha_2}$. Thus,

$$\begin{aligned}\deg\left(I - \overline{T}, \Omega, 0\right) &= \deg\left(I - \overline{T}, \Omega \setminus \left(\overline{\Omega^{\beta_1} \cup \Omega_{\alpha_2}}\right), 0\right) \\ &\quad + \deg\left(I - \overline{T}, \Omega^{\beta_1}, 0\right) \\ &\quad + \deg\left(I - \overline{T}, \Omega_{\alpha_2}, 0\right).\end{aligned} \tag{53}$$

We show that $\deg(I - \overline{T}, \Omega^{\beta_1}, 0) = \deg(I - \overline{T}, \Omega_{\alpha_2}, 0) = 1$, then

$$\deg\left(I - \overline{T}, \Omega \setminus \left(\overline{\Omega^{\beta_1} \cup \Omega_{\alpha_2}}\right), 0\right) = -1, \tag{54}$$

and there are solutions in $\overline{\Omega^{\beta_1} \cup \Omega_{\alpha_2}}$, Ω^{β_1}, Ω_{α_2} as required.

We now show $\deg(I - \overline{T}, \Omega_{\alpha_2}, 0) = 1$. The proof that $\deg(I - \overline{T}, \Omega^{\beta_1}, 0) = 1$ is similar and hence omitted. We define $I - W$, the extension to $\overline{\Omega}$ of the restriction of $I - \overline{T}$ to $\overline{\Omega}_{\alpha_2}$ as follows. Let

$$\begin{aligned} w(t,x,y) &= f\left(t, P_{\alpha_2,\beta_2}(t,x), h(t,y)\right) \\ &\quad + \tanh\left(x - P_{\alpha_2,\beta_2}(t,x)\right),\end{aligned} \tag{55}$$

where $P_{\alpha_2,\beta_2}(t,x)$ (in $P_{\alpha_2,\beta_2}(t,x)$ replace α_1 by α_2) and h are previously defined. Thus, w is a continuous function on $[0,1] \times R^2$ and satisfies

$$\begin{aligned} |w(t,x,y)| &\leq \psi(|y|) + \frac{\pi}{2}, \quad \text{for } |y| < L, \\ |w(t,x,y)| &\leq M_1, \quad \text{for } (t,x,y) \in [0,1] \times R^2,\end{aligned} \tag{56}$$

for some constants M_1. Moreover, we may choose M_1 so that $\|\alpha_2\|_\infty, \|\beta_2\|_\infty < M_1$.

Consider the following problem:

$$\begin{aligned} &\left(\phi_p\left(u'(t)\right)\right)' + q(t)\, w\left(t, u(t), \frac{d}{dt}P_{\alpha_2,\beta_2}(t,u)\right) = 0, \quad t \in J', \\ &\Delta u|_{t=t_k} = \widetilde{I}_{1k}(u(t_k)), \quad k = 1,2,\ldots,n. \\ &\Delta\phi_p(u')\big|_{t=t_k} = \widetilde{I}_{2k}\left(u(t_k), u'(t_k)\right), \quad k = 1,2,\ldots,n. \\ &u(0) = \sum_{i=1}^{m-2} a_i u(\eta_i), \qquad u'(1) = 0.\end{aligned} \tag{57}$$

Now, define the following operator:

$$\begin{aligned}(Wu)(t) &= \frac{1}{1-a}\sum_{i=1}^{m-2} a_i \left[\int_0^{\eta_i} \phi_p^{-1}\left(\int_s^1 q(\tau) W_u^*(\tau)\,d\tau\right) ds + \sum_{t_k<\eta_i} \overline{I}_{1k}(u(t_k))\right] \\ &\quad + \int_0^t \phi_p^{-1}\left(\int_s^1 q(\tau) W_u^*(\tau)\,d\tau - \sum_{s<t_k} \widetilde{I}_{2k}\left(u(t_k), u'(t_k)\right)\right) ds + \sum_{t_k<t} \widetilde{I}_{1k}(u(t_k)),\end{aligned} \tag{58}$$

where $W_u^*(t) = q(t)w(t, u(t), (d/dt)P_{\alpha_2,\beta_2}(t,u))$, $\widetilde{I}_{1k}(u(t_k)) = I_{1k}(P_{\alpha_2,\beta_2}(t_k,u))$, and $\widetilde{I}_{2k} = I_{2k}(P_{\alpha_2,\beta_2}(t,u), h(t,u'))$. Again, it is easy to check (from a previous argument and (v)) that u is a solution of (57) if $u \in \Omega_{\alpha_2}$ and $Wu = u$ (note that $W : E \to E$ is compact).

Thus, $\deg(I - \overline{T}, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) = 0$. Moreover, it is easy to see that $W(\overline{\Omega}) \subset \Omega$. By assumptions (iv) and (v), there are no solutions in $\partial\Omega_{\alpha_2} \cap \partial\Omega^{\beta_1}$. Thus,

$$\begin{aligned}\deg\left(I - \overline{T}, \Omega_{\alpha_2}, 0\right) &= \deg\left(I - W, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0\right) \\ &\quad + \deg\left(I - W, \Omega_{\alpha_2}, 0\right) \\ &= \deg(I - W, \Omega, 0) = 1.\end{aligned} \tag{59}$$

Thus there are three solutions, as required. The proof is complete. □

A slight modification of the argument in Theorem 9 yields the next result.

Theorem 10. *Suppose that conditions (H_1)–(H_3) hold. Assume that there exist two lower solutions α_1 and α_2 and two upper solutions β_1 and β_2 for problem* (1), *satisfying*

(i) $\alpha_1 \le \alpha_2 \le \beta_2$;

(ii) $\alpha_1 \le \beta_1 \le \beta_2$;

(iii) $\alpha_2 \nleq \beta_1$;

(iv) *there exists* $0 < \bar{\varepsilon} < \min_{t\in[0,1]}\{\alpha_2(t) - \alpha_1(t), \beta_2(t) - \beta_1(t)\}$ *such that all* $\varepsilon \in (0, \bar{\varepsilon}]$, *the function* $\alpha_2 - \varepsilon$, *and* $\beta_1 + \varepsilon$ *are, respectively, lower and upper solution of* (1);

(v) $\alpha_2 - \bar{\varepsilon} \nleq \beta_1 + \bar{\varepsilon}$.

If f satisfies the Nagumo condition with respect to α_1, β_2, then problem (1) *has at least three solutions u_1, u_2, and u_3 satisfying*

$$\begin{aligned} \alpha_1 \le u_1 \le \beta_1, \qquad \alpha_2 \le u_2 \le \beta_2, \\ u_3 \nleq \beta_1, \qquad u_3 \ngeq \alpha_2. \end{aligned} \tag{60}$$

Proof. In the proof of Theorem 9, define

$$\begin{aligned} \Omega_{\alpha_2} &= \{u \in \Omega : u > \alpha_2 - \bar{\varepsilon} \text{ on } (0,1)\}, \\ \Omega_{\beta_1} &= \{u \in \Omega : u < \beta_1 + \bar{\varepsilon} \text{ on } (0,1)\}, \end{aligned} \tag{61}$$

where Ω is defined in Theorem 9. □

Theorem 11. *Let $f \in C([0,1] \times (0,+\infty) \times (0,+\infty), [0,+\infty))$ and q satisfies (H_1). Furthermore, the following conditions hold.*

(i) *BVP* (1) *has a pair of positive upper and lower solutions β, $\alpha \in X$ satisfying*

$$\alpha(t) \le \beta(t), \quad t \in [0,1]. \tag{62}$$

(ii) *f satisfies a Nagumo condition relative to the pair α and β.*

Then, BVP (1) *has at least one positive solution such that*

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0,1]. \tag{63}$$

Proof. It can be proved easily, we omit it here. □

4. Examples

Example 1. Consider the following BVP:

$$\begin{aligned} &\left(\phi_p\left(u'(t)\right)\right)' + q(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in J', \\ &\Delta u|_{t=t_k} = \frac{u(t_k)}{14}, \quad k = 1,2,\ldots,n, \\ &\Delta\phi_p(u')\big|_{t=t_k} = -24u'(t_k) + u(t_k), \quad k = 1,2,\ldots,n, \\ &u(0) = \frac{1}{5}u\left(\frac{1}{3}\right) + \frac{1}{6}u\left(\frac{2}{3}\right), \qquad u'(1) = 0, \end{aligned} \tag{64}$$

where $q(t) = 1$, $f(t, u(t), u'(t)) = 4t^2 + \sin^2 u(t) + 20u'(t)^3$. Let $p = 3$; then, $q = 3/2$, which means that $\phi_p^{-1} = \phi_q$.

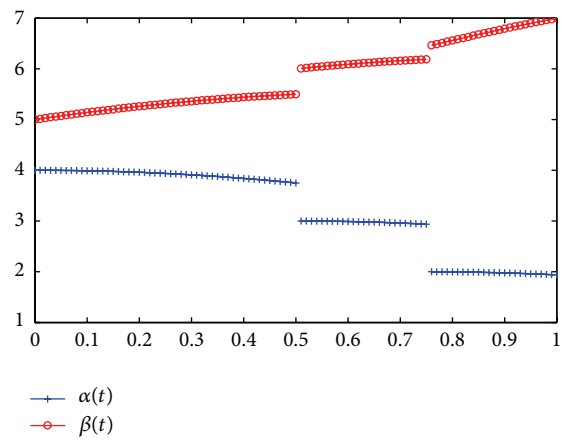

Figure 1

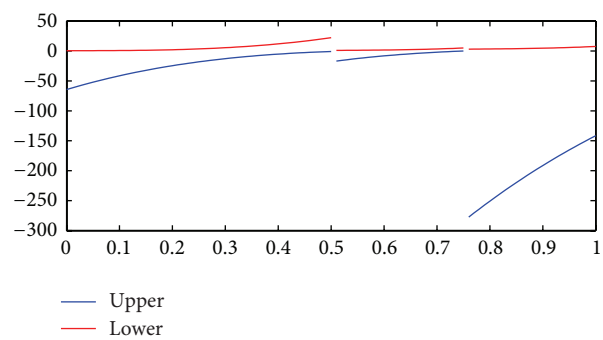

Figure 2

It is clear that

$$\begin{aligned} \alpha(t) &= \begin{cases} -t^2 + 4, & t \in \left[0, \frac{1}{2}\right], \\ -\left(t - \frac{1}{2}\right)^2 + 3, & t \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ -\left(t - \frac{3}{4}\right)^2 + 2, & t \in \left(\frac{3}{4}, 1\right], \end{cases} \\ \beta(t) &= \begin{cases} -\left(t - \frac{3}{4}\right)^2 + \frac{89}{16}, & t \in \left[0, \frac{1}{2}\right], \\ -(t-1)^2 + \frac{25}{4}, & t \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ -(t-2)^2 + 8, & t \in \left(\frac{3}{4}, 1\right] \end{cases} \end{aligned} \tag{65}$$

are lower and upper solutions of BVP (64), respectively. The figures of $\alpha(t)$ and $\beta(t)$ are as shown in Figure 1.

Clearly, $\alpha(t) \le \beta(t)$. After substituting $\alpha(t)$, $\beta(t)$ into $(\phi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0$, $t \in J'$, denote upper $= (\phi_p(\beta'(t)))' + q(t)f(t, \beta(t), \beta'(t))$ and lower $= (\phi_p(\alpha'(t)))' + q(t)f(t, \alpha(t), \alpha'(t))$, we can get Figure 2. From Figure 2, we can see that when $t \in J'$, $(\phi_p(\beta'(t)))' + q(t)f(t, \beta(t), \beta'(t)) \le 0$ and $(\phi_p(\beta'(t)))' + q(t)f(t, \beta(t), \beta'(t)) \ge 0$. Other inequalities can be verified easily. So, according to Definition 1, we can get that $\beta(t)$ and $\alpha(t)$ are upper and lower solutions of BVP (64), respectively.

Let $\psi(y) = 20y^3 + 5$, and it is clear that f satisfies Nagumo condition relative to α and β. Obviously, all the conditions of Theorem 11 hold. Hence, (64) has at least one positive solution $\alpha(t) \leq u(t) \leq \beta(t)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1989.

[2] D. D. Baĭnov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 1989.

[3] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science: Series A: Monographs and Treatises*, World Scientific, Singapore, 1995.

[4] Z. L. Wei, "Periodic boundary value problems for second order impulsive integrodifferential equations of mixed type in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 195, no. 1, pp. 214–229, 1995.

[5] S. G. Hristova and D. D. Bainov, "Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations," *Journal of Mathematical Analysis and Applications*, vol. 197, no. 1, pp. 1–13, 1996.

[6] X. Liu and D. Guo, "Periodic boundary value problems for a class of second-order impulsive integro-differential equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 284–302, 1997.

[7] R. P. Agarwal and D. O'Regan, "Multiple nonnegative solutions for second order impulsive differential equations," *Applied Mathematics and Computation*, vol. 114, no. 1, pp. 51–59, 2000.

[8] W. Ding and M. Han, "Periodic boundary value problem for the second order impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 155, no. 3, pp. 709–726, 2004.

[9] E. K. Lee and Y. H. Lee, "Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equations," *Applied Mathematics and Computation*, vol. 158, no. 3, pp. 745–759, 2004.

[10] C. D. Coster and P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Elsevier, New York, NY, USA, 2006.

[11] D. Jiang, "Upper and lower solutions method and a singular superlinear boundary value problem for the one-dimensional p-Laplacian," *Computers & Mathematics with Applications*, vol. 42, no. 6-7, pp. 927–940, 2001.

[12] J. Shen and W. Wang, "Impulsive boundary value problems with nonlinear boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 4055–4062, 2008.

[13] I. Rachůnková and J. Tomeček, "Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 525–539, 2004.

[14] D. Jiang, "Upper and lower solutions method and a superlinear singular boundary value problem," *Computers & Mathematics with Applications*, vol. 44, no. 3-4, pp. 323–337, 2002.

[15] A. Cabada and R. L. Pouso, "Existence results for the problem $(\phi(u'))' = f(t, u, u')$ with nonlinear boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 2, pp. 221–231, 1999.

[16] H. Lü, D. O'Regan, and R. P. Agarwal, "Triple solutions for the one-dimensional p-Laplacian," *Glasnik Matematički*, vol. 38, no. 2, pp. 273–284, 2003.

[17] M. X. Wang, A. Cabada, and J. J. Nieto, "Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions," *Annales Polonici Mathematici*, vol. 58, no. 3, pp. 221–235, 1993.