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Research Article

Almost Periodic Solutions for Second Order Dynamic Equations on Time Scales

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We firstly introduce the concept and the properties of C^m almost periodic functions on time scales, which generalizes the concept of almost periodic functions on time scales and the concept of $C^{(n)}$-almost periodic functions. Secondly, we consider the existence and uniqueness of almost periodic solutions for second order dynamic equations on time scales by Schauder's fixed point theorem and contracting mapping principle. At last, we obtain alternative theorems for second order dynamic equations on time scales.

1. Introduction

The theory of dynamic equations on time scales was first introduced by Hilger [1]. The study of dynamic equations on time scales helps to avoid studying results twice, once for differential equations and once for difference equations. In recent years, the theory of first order and second order dynamic equations on time scales has been studied, and some important results have been presented in [2–6]. However, to the best of our knowledge, there are no results on the existence of almost periodic solutions for the second order dynamic equations on time scales. The aim of this paper is to consider the existence of almost periodic solutions for second order dynamic equations on time scales.

The concept of almost periodicity was first introduced by Bohr [7] and later generalized by Bochner, Fink, N'Guérékata, and Shen and Yi and others (see [8–11]). Recently, Guan and Wang [12] and Li and Wang [13, 14] developed the theory of almost periodic functions on time scales, which do not only unify the almost periodic functions on $\mathbb{R}$ and the almost periodic sequences on $\mathbb{Z}$ but also extend to nontrivial time scales, for example, q-difference equations.

The existence and uniqueness solutions for second order dynamic equations have become important in recent years in mathematical models and they rise in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. The existence of oscillatory and nonoscillatory solutions for second order equations has been studied in [15–18] (and the references therein).

This paper is concerned with the second order dynamic equation as follows:

$$x^{\Delta\Delta} = g\left(t, x, x^{\Delta}\right), \tag{1}$$

where $g : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is almost periodic in $t \in \mathbb{T}$ uniformly. Such a type of equation appears in many problems of applications, such as Brillouin focusing systems [19, 20], nonlinear elasticity [21], and Ermakov-Pinney equations [22, 23].

In this paper, we consider the existence of almost periodic solutions (1) and present alternative theorems for second order dynamic equations. In order to do this, we introduce a new concept called C^m almost periodicity, which generalizes the concept of almost periodic functions on time scales and the concept of $C^{(n)}$-almost periodic functions on $\mathbb{R}$ introduced by Adamczak [24], Bugajewski and N'Guérékata [25]. The study of almost periodic solutions on time scales has tremendous potential for applications in mathematical models of real processes and phenomena.

This paper is organized as follows. In Section 2, we recall some properties on time scales. In Section 3, we give the concept and properties of C^m almost periodicity,

and C^m uniformly almost periodicity on time scales. The most important part of this paper are Sections 4 and 5. In Section 4, by exponential dichotomy on time scales and fixed point theorems, the existence and uniqueness theorems of almost periodic solutions for second order dynamic equations are obtained. In Section 5, we show the alternative theorems for second order dynamic equations on time scales by topological degree method.

2. Preliminaries on Time Scales

Suppose that $\mathbb{T}$ is an arbitrary time scale, that is, a nonempty closed subset of $\mathbb{R}$. Set $\mathbb{T}^+ = \mathbb{T} \cap \mathbb{R}^+$, $[a,b]_{\mathbb{T}}((a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}, (a,b)_{\mathbb{T}}$, resp.) $= \mathbb{T} \cap [a,b]((a,b], [a,b), (a,b)$, resp.). First, we recall some results of time scales in [5, 6].

Definition 1 (see [5, 6]). For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the graininess function $\mu : \mathbb{T} \to [0,\infty)$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the backwards graininess function $\nu : \mathbb{T} \to [0,\infty)$ are given by

$$\begin{aligned} \sigma(t) &:= \inf\{s \in \mathbb{T} : s > t\}, \qquad \mu(t) := \sigma(t) - t, \\ \rho(t) &:= \sup\{s \in \mathbb{T} : s < t\}, \qquad \nu(t) = t - \rho(t), \end{aligned} \tag{2}$$

respectively. Set $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$, where ϕ denotes the empty set.

If $\sigma(t) > t$, then the point t is called right-scattered, while if $\rho(t) < t$, then the point t is termed left-scattered. If $t < \sup T$ and $\sigma(t) = t$, then the point t is called right-dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Set

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \rho(\sup \mathbb{T}, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases} \tag{3}$$

Definition 2 (see [5, 6]). (i) Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then f is called Δ-differentiable at $t \in \mathbb{T}^\kappa$ if the limit

$$\lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \tag{4}$$

provided that this limit exists as a finite number θ, and θ is called the Δ-derivative of f at $t \in \mathbb{T}^\kappa$ and we denote it by $\theta = f^\Delta(t)$.

(ii) If $F^\Delta(t) = f(t)$, then the Δ-integral is defined by

$$\int_a^t f(s)\,\Delta s = F(t) - F(a). \tag{5}$$

Definition 3 (see [5, 6]). A function $f : \mathbb{T} \to \mathbb{R}^n$ is called rd-continuous if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$.

Obviously, $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n) \subset C(\mathbb{T}, \mathbb{R}^n)$. If f is Δ-differential, then f is continuous. If $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$, then f is Δ-integral.

Definition 4 (see [5, 6]). Suppose that $p, q \in \mathscr{R} = \{p : \mathbb{T} \to \mathbb{R} : 1 + \mu(t)p(t) \neq 0 \text{ for } t \in \mathbb{T}^\kappa\}$. Then

(i) $(p \oplus q)(t) := p(t) + \mu(t)p(t)q(t)$ for all $t \in \mathbb{T}^\kappa$;

(ii) $\ominus p(t) := -p(t)/(1 + \mu(t)p(t))$ for all $t \in \mathbb{T}^\kappa$.

Definition 5 (see [5, 6]). Suppose that $p \in \mathscr{R}$; then

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau\right) \quad \text{for } s,t \in \mathbb{T}, \tag{6}$$

where

$$\xi_{\mu(\tau)}(p(\tau)) = \begin{cases} p(\tau) & \text{if } \mu(\tau) = 0 \\ \dfrac{1}{\mu(\tau)} \log(1 + \mu(\tau)p(\tau)) & \text{if } \mu(\tau) \neq 0. \end{cases} \tag{7}$$

The following lemmas are concerned with the properties of $e_p(t,s)$.

Lemma 6 (see [6, 26]). *Suppose that $p, q \in \mathscr{R}$, $s, t \in \mathbb{T}$. Then*

(i) $e_0(t,s) = 1$, $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $1/e_p(t,s) = e_{\ominus p}(t,s) = e_p(s,t)$;

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vi) $e_p^\Delta(\cdot,s) = pe_p(\cdot,s)$;

(vii) $(1/e_p(\cdot,s))^\Delta = -p(\cdot)/e_p^\sigma(\cdot,s)$;

(viii) $e_p(t,\cdot)^\Delta = -pe_p^\sigma(t,\cdot)$;

(ix) let $p > 0$ be a constant, and let the graininess function μ be uniformly bounded with $\sup_{t\in\mathbb{T}}\mu(t) = M > 0$. Then for $t \geq s$,

$$\begin{aligned} e^{p(t-s)} &\geq e_p(t,s) \geq (1 + pM)^{(t-s)/M}, \\ e^{-p(t-s)} &\leq e_{\ominus p}(t,s) \leq \left(\frac{1}{1 + pM}\right)^{(t-s)/M}. \end{aligned} \tag{8}$$

Lemma 7 (see [5]). *Let $f : \mathbb{T} \to \mathbb{R}$ be function. Then*

(i) $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$;

(ii) $\int_{\rho(t)}^t f(s)\Delta s = \nu(t)f(\rho(t))$.

3. C^m Almost Periodic Functions

In this section, we will introduce a new concept called C^m almost periodic and C^m uniformly almost periodic on time scales. First, we recall the conceptions and the properties of almost periodic functions and uniformly almost periodic functions in the works [12–14].

Definition 8 (see [12]). Suppose that $\mathbb{T}$ is a time scale.

(i) A real number τ is called a translation invariant for $\mathbb{T}$, if

$$\tau + t \in \mathbb{T} \quad \text{for } t \in \mathbb{T}. \tag{9}$$

The notation $\gamma(\mathbb{T})$ denotes the set of all translation invariants of $\mathbb{T}$.

(ii) If there exist $\alpha, \beta \in \overline{\gamma}(\mathbb{T}) = \gamma(\mathbb{T}) - \{0\}$ with $\alpha\beta < 0$, then $\mathbb{T}$ is called two-way translation invariant time scale or almost periodic time scale.

There exist lots of two-way translation invariant time scales. For example, $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$, $\mathbb{P}_{a,b} = \bigcup_{k\in\mathbb{Z}}[k(a+b), k(a+b)+a]$ $(a, b > 0)$ and so on.

Lemma 9 (see [12]). *If $\mathbb{T}$ is a two-way translation invariant time scale, then* $\sup \mathbb{T} = \infty$, $\inf \mathbb{T} = -\infty$.

Thus, $\mathbb{T}^\kappa = \mathbb{T}$ follows from $\mathbb{T}$ being a two-way translation invariant time scale.

Lemma 10. *Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. Then there exists a $M > 0$ such that $\mu(t), \nu(t) \le M$ for all $t \in \mathbb{T}$.*

Proof. Since $\mathbb{T}$ is a two-way translation invariant time scale, there is a $M > 0$ with $M \in \overline{\gamma}(\mathbb{T})$ such that $t \pm M \in \mathbb{T}$ for all $t \in \mathbb{T}$. Thus $\sigma(t) \le M + t$ and $\rho(t) \ge t - M$ for all $t \in \mathbb{T}$. Furthermore, $\mu(t), \nu(t) \le M$ for all $t \in \mathbb{T}$. □

If $\mathbb{T}$ is a two-way translation invariant time scale, set

$$\begin{gathered} \vartheta^+ := \min [0, \infty)_{\mathbb{T}}, \\ M := \sup_{t\in\mathbb{T}} \mu(t), \\ \|x\| := \max_{1\le i\le n} \sup_{t\in\mathbb{T}} \left|x^i(t)\right|, \end{gathered} \tag{10}$$

where $x(t) = (x^1(t), \ldots, x^n(t))^\tau \in BC(\mathbb{T}, \mathbb{R}^n)$ with $BC(\mathbb{T}, \mathbb{R}^n)$ denoting the set of all bounded functions in $C(\mathbb{T}, \mathbb{R}^n)$.

Definition 11 (see [12]). Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called (Bohr) almost periodic if for every $\varepsilon > 0$ there exists a $l_\varepsilon > 0$ such that every interval of length l_ε contains at least one $\tau \in \gamma(\mathbb{T})$ satisfying

$$\left\|f(t+\tau) - f(t)\right\| < \varepsilon \quad \text{for } t \in \mathbb{T}. \tag{11}$$

Definition 12 (see [13]). Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called (Bochner) almost periodic if any sequence $\{s_n'\} \subset \gamma(\mathbb{T})$; there exists a subsequence $\{s_n\} \subset \{s_n'\}$ such that $f(t + s_n)$ converges uniformly for $t \in \mathbb{T}$.

In [14], Li and Wang show that Definition 11 is equivalent to Definition 12. In the following, the notation $\mathrm{AP}(\mathbb{T}, \mathbb{R}^n)$ denotes the set of all almost periodic functions in $C(\mathbb{T}, \mathbb{R}^n)$.

Lemma 13 (see [13]). *Suppose that $f, g \in AP(\mathbb{T}, \mathbb{R}^n)$ and $\alpha \in \mathbb{R}$ is a constant. Then*

(i) $f + g, \alpha f \in AP(\mathbb{T}, \mathbb{R}^n)$;

(ii) f is bounded on $\mathbb{T}$;

(iii) f is uniformly continuous on $\mathbb{T}$;

(iv) $F = \int_0^t f(s)\Delta s$ is almost periodic if and only if f is bounded on $\mathbb{T}$.

Definition 14. A function $f \in C(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$ is called almost periodic in $t \in \mathbb{T}$ uniformly for all $x \in \mathbb{R}^m$ if $f(\cdot, x)$ is almost periodic for all x in each compact subset of $\mathbb{R}^m$.

The notation $\mathrm{UAP}(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$ denotes the set of all uniformly almost periodic functions in $C(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$.

Similar to the proofs in [14], Lemmas 15 and 16 hold.

Lemma 15. *Suppose that $f, g \in UAP(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$ and $\alpha \in \mathbb{R}$ is a constant. Then*

(i) $f + g, \alpha f \in UAP(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$;

(ii) for each compact subset $S \subset \mathbb{R}^m$, f is bounded and uniformly continuous on $\mathbb{T} \times S$;

(iii) $F(t, x) = \int_0^t f(s, x)\Delta s \in UAP(\mathbb{T}\times\mathbb{R}^m, \mathbb{R}^n)$ if and only if $F(t, x)$ is bounded on $\mathbb{T}\times S$ with $S \subset \mathbb{R}^m$ being compact.

Lemma 16. *Let $f \in UAP(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^n)$ and $\phi \in AP(\mathbb{T}, \mathbb{R}^m)$. Then $F(t) = f(t, \phi(t)) \in AP(\mathbb{T}, \mathbb{R}^n)$.*

Lemma 17. *Let $f(t) = (f^1(t), \ldots, f^n(t))^\tau \in C(\mathbb{T}, \mathbb{R}^{mn})$ with $f^i \in C(\mathbb{T}, \mathbb{R}^m)$ for $i = 1, \ldots, n$. Then $f \in AP(\mathbb{T}, \mathbb{R}^{mn})$ if and only if $f^i \in AP(\mathbb{T}, \mathbb{R}^m)$ for $i = 1, \ldots, n$.*

Proof. If $f \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^{mn})$, then for each $\varepsilon > 0$, there exists a $l_\varepsilon > 0$ such that every interval of length l_ε contains a τ implying that

$$\max_{1\le i\le n} \sup_{t\in\mathbb{T}} \left\{\left\|f^i(t+\tau) - f^i(t)\right\|\right\} = \left\|f(t+\tau) - f(t)\right\| < \varepsilon. \tag{12}$$

Thus for $i = 1, \ldots, n$, $\|f^i(t+\tau) - f^i(t)\| < \varepsilon$; that is, $f^i \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^m)$ for $i = 1, \ldots, n$.

If $f^i \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^m)$ for $i = 1, \ldots, n$, then for each sequence $\{t_k''\} \subset \gamma(\mathbb{T})$, there exists a subsequence $\{t_k'\}$ such that $f^1(t+t_k')$ converges uniformly as $k \to \infty$. Furthermore, there exists a subsequence $\{t_k\} \subset \{t_k'\}$ such that $f^2(t + t_k)$ converges uniformly as $k \to \infty$. Inductively, there exists a subsequence $\{s_k\} \subset \{t_k''\}$ such that $f^i(t + s_k)$ converges uniformly as $k \to \infty$ for $i = 1, \ldots, n$. Moreover, $(f^1(t + s_k), \ldots, f^m(t + s_k))^\tau$ converges uniformly as $k \to \infty$ which implies that $f \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^{mn})$. □

Set

$$\|y\|_m = \sum_{j=0}^{m} \left\|y^{\Delta^j}(t)\right\|, \tag{13}$$

where $y^{\Delta^0} = y$. Then $(BC^m(\mathbb{T},\mathbb{R}^n), \|\cdot\|_m)$ is a Banach space, where $BC^m(\mathbb{T},\mathbb{R}^n)$ denotes the set of all bounded functions in $C^m(\mathbb{T},\mathbb{R}^n)$, and $\|\cdot\|_0 = \|\cdot\|$.

Now we introduce a new concept called C^m almost periodic.

Definition 18. Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. A function $f \in C^m(\mathbb{T},\mathbb{R}^n)$ is called C^m (Bohr) almost periodic if for every $\varepsilon > 0$ there exists a $l_\varepsilon > 0$ such that every interval of length l_ε contains at least one $\tau \in \gamma(\mathbb{T})$ satisfying

$$\|f(t+\tau) - f(t)\|_m < \varepsilon \quad \text{for } t \in \mathbb{T}. \tag{14}$$

Definition 19. Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. A function $f \in C^m(\mathbb{T},\mathbb{R}^n)$ is called C^m (Bochner) almost periodic if any sequence $\{s_n'\} \subset \overline{\gamma}(\mathbb{T})$; there exists a subsequence $\{s_n\} \subset \{s_n'\}$ such that $f(t+s_n)$ converges uniformly for $t \in \mathbb{T}$.

Theorem 20. *Let $f : \mathbb{T} \to \mathbb{R}^n$. Then f is C^m (Bohr) almost periodic if and only if $f, f^\Delta, \ldots, f^{\Delta^m}$ are (Bohr) almost periodic.*

Proof. If $f \in \mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$, then for each $\varepsilon > 0$, there is a $l_\varepsilon > 0$ such that for every interval of length $l_\varepsilon > 0$ contains a $\tau \in \gamma(\mathbb{T})$ satisfying

$$\begin{aligned} &\left\|f^{\Delta^i}(t+\tau) - f^{\Delta^i}(t)\right\| \\ &\quad < \sum_{j=0}^{m} \left\|f^{\Delta^j}(t+\tau) - f^{\Delta^j}(t)\right\| \\ &\quad = \|f(t+\tau) - f(t)\|_m < \varepsilon \quad \text{for } i = 0,1,\ldots,m, \end{aligned} \tag{15}$$

which implies $f, f^\Delta, \ldots, f^{\Delta^m}$ are (Bohr) almost periodic.

If $f, f^\Delta, \ldots, f^{\Delta^m}$ are almost periodic, by Lemma 17, $(f, f^\Delta, \ldots, f^{\Delta^m})^\tau \in \mathrm{AP}(\mathbb{T},\mathbb{R}^{(m+1)n})$. That is, for each $\varepsilon > 0$, there is a $l_\varepsilon > 0$ such that for every interval of length $l_\varepsilon > 0$ contains a τ satisfying

$$\left\|f^{\Delta^j}(t+\tau) - f^{\Delta^j}(t)\right\| < \frac{\varepsilon}{m+1} \quad \text{for } j = 0,\ldots,m. \tag{16}$$

Thus,

$$\|f(t+\tau) - f(t)\|_m = \sum_{j=0}^{m} \left\|f^{\Delta^j}(t+\tau) - f^{\Delta^j}(t)\right\| < \varepsilon. \tag{17}$$

Therefore, $f \in \mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$. □

If $f : \mathbb{T} \to \mathbb{R}^n$ is C^m (Bochner) almost periodic, the following theorem also holds.

Theorem 21. *Let $f : \mathbb{T} \to \mathbb{R}^n$. Then f is C^m (Bochner) almost periodic if and only if $f, f^\Delta, \ldots, f^{\Delta^m}$ are (Bochner) almost periodic.*

Since $f : \mathbb{T} \to \mathbb{R}^n$ is (Bohr) almost periodic if and only if $f : \mathbb{T} \to \mathbb{R}^n$ is (Bochner) almost periodic, according to Theorems 20 and 21, Definition 18 is equivalent to Definition 19. In the following, the notations $\mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$, denotes the set of all almost periodic functions in $C^m(\mathbb{T},\mathbb{R}^n)$. Obviously, $\mathrm{AP}^0(\mathbb{T},\mathbb{R}^n) = \mathrm{AP}(\mathbb{T},\mathbb{R}^n)$, and

$$\mathrm{AP}^{m+1}(\mathbb{T},\mathbb{R}^n) \subset \mathrm{AP}^m(\mathbb{T},\mathbb{R}^n) \subset BC^m(\mathbb{T},\mathbb{R}^n). \tag{18}$$

According to Definition 19, we obtain the following theorem.

Theorem 22. *Suppose that $f, g \in AP^m(\mathbb{T},\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$ is a constant. Then $f+g, \alpha f, f^\vee \in AP^m(\mathbb{T},\mathbb{R}^n)$, where $f^\vee(t) = f(-t)$.*

Theorem 23. *Let $\{f_k\} \subset C^m(\mathbb{T},\mathbb{R}^n)$ be a sequence of C^m almost periodic functions. If $\lim_{k\to\infty} f_k(t) = f(t)$ holds uniformly on $\mathbb{T}$, then $f \in AP^m(\mathbb{T},\mathbb{R}^n)$.*

Proof. Noting that $\{f_k\}$ converge uniformly, thus for any $\varepsilon > 0$, there exists a $N = N(\varepsilon) > 0$ such that

$$\begin{aligned} \|f_N(t) - f(t)\|_m &< \frac{\varepsilon}{3}, \\ \|f_N(t+\tau) - f(t+\tau)\|_m &< \frac{\varepsilon}{3} \quad \forall \tau \in \gamma(\mathbb{T}). \end{aligned} \tag{19}$$

Since $f_N \in \mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$, there exists a $l_{\varepsilon/3}$ such that every interval of length $l_{\varepsilon/3}$ contains a $\tau_0 \in \overline{\gamma}(\mathbb{T})$ such that

$$\|f_N(t+\tau_0) - f_N(t)\|_m < \frac{\varepsilon}{3}. \tag{20}$$

Then

$$\begin{aligned} &\|f(t+\tau_0) - f(t)\|_m \\ &\quad \le \|f(t+\tau_0) - f_N(t+\tau_0)\|_m \\ &\qquad + \|f_N(t+\tau_0) - f_N(t)\|_m \\ &\qquad + \|f_N(t) - f(t)\|_m \\ &\quad < \varepsilon, \end{aligned} \tag{21}$$

which yields that $f \in \mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$. □

Therefore, $\mathrm{AP}^m(\mathbb{T},\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_m$.

Similar to the proof of Lemma 13, Theorem 24 holds.

Theorem 24. *Let $f \in AP^m(\mathbb{T},\mathbb{R}^n)$. If $F(t) = \int_a^t f(s)\Delta s$ is bounded, then $F \in AP^{m+1}(\mathbb{T},\mathbb{R}^n)$.*

Theorem 25. *Suppose that $f \in UAP(\mathbb{T}\times\mathbb{R}^{mn},\mathbb{R}^n)$, $\phi \in AP^{m-1}(\mathbb{T},\mathbb{R}^n)$. Then $F(t) = f(t,\phi(t),\ldots,\phi^{\Delta^{m-1}}(t)) \in AP(\mathbb{T},\mathbb{R}^n)$.*

Proof. Since $\phi \in \mathrm{AP}^{m-1}(\mathbb{T},\mathbb{R}^n)$, then for each sequence $\{t_n''\} \subset \gamma(\mathbb{T})$ there exists a subsequence $\{t_n'\} \subset \{t_n''\}$ and function $\psi = (\psi_0,\ldots,\psi_{m-1})^\tau \in C(\mathbb{T},\mathbb{R}^{mn})$ such that

$$\lim_{n\to\infty} \phi^{\Delta^i}(t+t_n') = \psi_i(t) \quad \text{for } t \in \mathbb{T},\ i = 0,\ldots,m-1, \tag{22}$$

holds uniformly.

By Theorem 25 and Theorem 3.20 in [14], $\psi \in H((\phi,\dots,\phi^{\Delta^{m-1}})^\tau)$ with $H((\phi,\dots,\phi^{\Delta^{m-1}})^\tau)$ being hull of $(\phi,\dots,\phi^{\Delta^{m-1}})^\tau$ and $H((\phi,\dots,\phi^{\Delta^{m-1}})^\tau)$ is compact. Thus there exists $\{t_n\} \subset \{t_n'\}$ such that

$$\begin{gathered}\lim_{n\to\infty} f(t+t_n, x_1,\dots,x_m) = g(t, x_1,\dots,x_m) \\ \text{for } t\in\mathbb{T},\ (x_1,\dots,x_n)^\tau \in H\left(\left(\phi,\dots,\phi^{\Delta^{m-1}}\right)^\tau\right)\end{gathered} \tag{23}$$

holds uniformly.

Set $G(t) = g(t,\psi_1(t),\dots,\psi_{m-1}(t))$. Then,

$$\begin{aligned}&\|F(t+t_n) - G(t)\| \\ &\quad\le \left\|f\left(t+t_n, \phi(t+t_n),\dots,\phi^{\Delta^{m-1}}(t+t_n)\right)\right. \\ &\qquad \left. - f(t+t_n,\psi_0(t),\dots,\psi_{m-1}(t))\right\| \\ &\qquad + \|f(t+t_n,\psi_0(t),\dots,\psi_{m-1}(t)) \\ &\qquad\quad - g(t,\psi_0(t),\dots,\psi_{m-1}(t))\| \\ &\quad < \varepsilon,\end{aligned} \tag{24}$$

by the uniform continuity of f, (22), and (23). Therefore $F \in \mathrm{AP}(\mathbb{T},\mathbb{R}^n)$. □

4. Almost Periodic Solutions

In this section, we will consider the existence and uniqueness of almost periodic solutions of (1). To investigate (1), we will consider the following auxiliary equation:

$$x^{\Delta\Delta} = x + f\left(t, x, x^\Delta\right), \tag{25}$$

where $f(t,x,y) = g(t,x,y) - x$. Obviously, $f \in \mathrm{UAP}(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$ follows from $g \in \mathrm{UAP}(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$.

Lemma 26 (see [13, 14]). *Let A be a $n\times n$-matrix in $\mathscr{R}(\mathbb{T},\mathbb{R}^{n\times n}) = \{A \in Gl(n,n) : \det(I_n + \mu(t)A) \neq 0\}$, where I_n is $n\times n$ identity matrix. If the linear system*

$$x^\Delta = A(t)x \tag{26}$$

admits an exponential dichotomy on $\mathbb{T}$, that is, there exist a projection P on $\mathbb{R}^n$ and positive constants K_i and α_i for $i = 1,2$ such that

$$\begin{gathered}\left\|\Phi(t)P\Phi^{-1}(s)\right\| \le K_1 e_{\ominus\alpha_1}(t,s) \quad \text{for } t\ge s \\ \left\|\Phi(t)(I_n - P)\Phi^{-1}(s)\right\| \le K_2 e_{\ominus\alpha_2}(s,t) \quad \text{for } t\le s,\end{gathered} \tag{27}$$

then the system

$$x^\Delta = A(t)x + f(t) \tag{28}$$

has a bounded solution $x(t)$ as follows:

$$\begin{aligned}x(t) = &\int_{-\infty}^t \Phi(t)P\Phi^{-1}(\sigma(s))f(s)\Delta s \\ &- \int_t^\infty \Phi(t)(I-P)\Phi^{-1}(\sigma(s))f(s)\Delta s.\end{aligned} \tag{29}$$

Moreover, if A and f are almost periodic functions, then (28) admits a unique almost periodic solution as (29).

We will rewrite (25) by the following lemma.

Lemma 27. *Suppose that $k \in \mathbb{R}$ with $k \neq 0$ is chosen. If x is the solution of (25), then $(x,(x^\Delta - x)/k)^\tau$ is the solution of vector equation*

$$X^\Delta = AX + F(t,X), \tag{30}$$

where

$$\begin{gathered}X = \begin{pmatrix} x_1 \\ x_2\end{pmatrix}, \qquad A = \begin{pmatrix} 1 & k \\ 0 & -1\end{pmatrix}, \\ F(t,X) = \begin{pmatrix} 0 \\ \dfrac{f(t,x_1,x_1+kx_2)}{k}\end{pmatrix}.\end{gathered} \tag{31}$$

However, if $X = (x_1,x_2)^\tau$ is the solution of (30), then x_1 is the solution of (25).

Lemma 28. *Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. If $k^2\mu(t) + 2k \le 2/3$ for $t \in \mathbb{T}$, then the homogeneous equation*

$$X^\Delta = AX = \begin{pmatrix} 1 & k \\ 0 & -1\end{pmatrix} X \tag{32}$$

admits an exponential dichotomy on $\mathbb{T}$. Furthermore, the solution of (32) with the initial value $X(\vartheta^+) = I$ is

$$e_A(t,\vartheta^+) = e_1(t,\vartheta^+)\begin{pmatrix} 1 & k\displaystyle\int_{\vartheta^+}^t (1+\mu(\tau))^{-1}\Delta\tau \\ 0 & 1\end{pmatrix}. \tag{33}$$

Proof. For $i = 1,2$, the matrix A satisfies

$$\begin{aligned}&|a_{ii}(t)| - \sum_{j\neq i}|a_{ji}(t)| - \frac{1}{2}\mu(t)\left(\sum_{j\neq i}|a_{ji}(t)|\right)^2 \\ &\quad \ge 2k + k^2\mu(t).\end{aligned} \tag{34}$$

Thus by Theorem 5.1 in [27], (32) admits an exponential dichotomy on $\mathbb{T}$.

According to Theorem 5.35 in [5], we have

$$e_A(t,\vartheta^+) = r_1P_0 + r_2P_1 = r_1P_0 + r_2(A - I_2)P_0, \tag{35}$$

where $(r_1,r_2)^\tau$ is the solution of

$$\begin{pmatrix} r_1^\Delta \\ r_2^\Delta\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} r_1 \\ r_2\end{pmatrix}, \qquad \begin{pmatrix} r_1(\vartheta^+) \\ r_2(\vartheta^+)\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}. \tag{36}$$

Thus (33) holds. □

Lemma 29. *Suppose that $k \in \mathbb{T}$ with $k \neq 0$. Then $x \in AP^1(\mathbb{T},\mathbb{R})$ if and only if $(x,(x^\Delta - x)/k)^\tau \in AP(\mathbb{T},\mathbb{R}^2)$.*

Proof. If $x \in \mathrm{AP}^1(\mathbb{T},\mathbb{R})$ holds, from Theorem 20 (or Theorem 21), $x, x^\Delta \in \mathrm{AP}(\mathbb{T},\mathbb{R})$. According to Lemmas 13 and 17, we have $(x,(x^\Delta - x)/k)^\tau \in \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$.

If $(x,(x^\Delta - x)/k)^\tau \in \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ holds, then $x,(x^\Delta - x)/k \in \mathrm{AP}(\mathbb{T},\mathbb{R})$. Thus $x^\Delta \in \mathrm{AP}(\mathbb{T},\mathbb{R})$ and $x \in \mathrm{AP}^1(\mathbb{T},\mathbb{R})$. □

Lemma 30. *Suppose that $\mathbb{T}$ is a two-way translation invariant time scale. Assume that* (26) *admits an exponential dichotomy on $\mathbb{T}$, with a projection P on $\mathbb{R}^n$ and positive constants K_i and α_i for $i = 1,2$ such that* (27) *holds. If $\|A\| = \max_{1\le i,j\le n}\sup_{\mathbb{T}}|a_{ij}(t)| < \infty$ is bounded, where $A(t) = (a_{ij}(t))_{n\times n}$, then for any fixed $h \in \mathbb{R}$, the families of functions $\{\Phi(\cdot)P\Phi^{-1}(s) : s \in (-\infty, t-h]_{\mathbb{T}}\}$ and $\{\Phi(\cdot)(I-P)\Phi^{-1}(s) : s \in [t+h,\infty)_{\mathbb{T}}\}$ are equicontinuous on $[\vartheta^+,\infty)_{\mathbb{T}}$ and $(-\infty,\vartheta^+]_{\mathbb{T}}$, respectively.*

Proof. The proof of equicontinuity of $\{\Phi(\cdot)(I-P)\Phi^{-1}(s) : s \in [t+h,\infty)_{\mathbb{T}}\}$ on $(-\infty,\vartheta^+]_{\mathbb{T}}$ is similar to that of $\{\Phi(\cdot)P\Phi^{-1}(s) : s \in (-\infty,t-h]_{\mathbb{T}}\}$. Thus, we only prove the equi-continuity of $\{\Phi(\cdot)P\Phi^{-1}(s) : s \in (-\infty,t-h]_{\mathbb{T}}\}$.

Since Φ is the fundamental solutions of (26), the function $\Phi(\cdot)P\Phi^{-1}(s)$ is Δ-differential on $[\vartheta^+,\infty)$. Moreover, for fixed $h > 0$ and $\omega \in \overline{\gamma}(\mathbb{T})$ chosen, we have

$$\begin{aligned}
&\int_{\vartheta^+}^{\infty}\left\|\left(\Phi(t)P\Phi^{-1}(s)\right)^\Delta\right\|\Delta t \\
&\quad\le \int_{\vartheta^+}^{\infty} K_1 e_{\ominus\alpha_1}(t,s)\|A\|\Delta t \\
&\quad\le \|A\|\sum_{n=0}^{\infty}\int_{n\omega}^{(n+1)\omega} K_1\left(\frac{1}{1+M\alpha_1}\right)^{(t-s)/M}\Delta t \\
&\quad= \frac{\|A\|K_1\omega(1+M\alpha_1)^{-h/M}}{1-(1+M\alpha_1)^{-\omega/M}} < \infty,
\end{aligned} \tag{37}$$

which implies that $(\Phi(t)P\Phi^{-1}(s))^\Delta \in L^1([0,\infty)_{\mathbb{T}})$. By $\Phi(\cdot)P\Phi^{-1}(s)$ being Δ-differential Δ-a.e. for $s \in (-\infty,t-h]_{\mathbb{T}}$ and Theorem 4.1 in [28], $\Phi(\cdot)P\Phi(s)$ is absolutely continuous; thus the family $\{\Phi(\cdot)P\Phi^{-1}(s) : s \in (-\infty,t-h]_{\mathbb{T}}\}$ is equi-continuous on $(-\infty,\vartheta^+]_{\mathbb{T}}$. □

Theorem 31. *Suppose that there is a $k > 0$ such that $k^2\mu(t) + 2k \le 2/3$ for all $t \in \mathbb{T}$. Assume $f \in UAP(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$ takes bounded sets into bounded sets. Then* (25) *admits a almost periodic solution.*

Proof. Set the operator $\Gamma : \mathrm{AP}(\mathbb{T},\mathbb{R}^2) \to \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ by

$$\begin{aligned}
\Gamma X(t) = &\int_{-\infty}^{t}\Phi(t)P\Phi^{-1}(\sigma(s))F(s,X(s))\Delta s \\
&-\int_{t}^{\infty}\Phi(t)(I-P)\Phi^{-1}(\sigma(s))F(s,X(s))\Delta s,
\end{aligned} \tag{38}$$

where Φ is a fundamental solution matrix of (32). From Theorem 25 and Lemma 26, the operator $\Gamma : \mathrm{AP}(\mathbb{T},\mathbb{R}^2) \to \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ is well defined, $\|\Gamma X\|$ is bounded, and the solution of (30) is equivalent to the fixed point of Γ.

Since $f \in \mathrm{UAP}(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$, then for each compact subset $S \subset \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$, f is bounded and uniformly continuous on $\mathbb{T}\times S$. Thus, for each $\varepsilon > 0$ and compact subset $S \subset \mathbb{R}^2$, there is a compact subset $S_1 \subset S$ such that $X = (x_1,x_2)^\tau$, $Y = (y_1,y_2)^\tau \in S_1$ with $\|X-Y\| < \delta$ implies that $\|F(t,X) - F(t,Y)\| = \|f(t,x_1,x_1+kx_2) - g(t,y_1,y_1+ky_2)\|/k < \varepsilon$. Then

$$\begin{aligned}
\|\Gamma X - \Gamma Y\| \le &\int_{-\infty}^{t}\left\|\Phi(t)P\Phi^{-1}(\sigma(s))\right\| \\
&\times\|F(s,X(s)) - F(s,Y(s))\|\Delta s \\
&+\int_{t}^{\infty}\left\|\Phi(t)(I_2-P)\Phi^{-1}(\sigma(s))\right\| \\
&\times\|F(s,X(s)) - F(s,Y(s))\|\Delta s \\
\le &\left(\int_{-\infty}^{t}K_1 e_{\ominus\alpha_1}(t,\sigma(s))\Delta s\right. \\
&\left.+\int_{t}^{\infty}K_2 e_{\ominus\alpha_2}(\sigma(s),t)\Delta s\right)\varepsilon \\
\le &\left(\frac{K_1(1+M\alpha_1)}{\alpha_1}+\frac{K_2}{\alpha_2}\right)\varepsilon,
\end{aligned} \tag{39}$$

which implies that Γ is continuous.

In the following, we will show that $\Gamma : \mathrm{AP}(\mathbb{T},\mathbb{R}^2) \to \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ is equi-continuous. If $X = (x,x^\Delta) \in \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ and there exists a $C > 0$ such that $\|X\| \le C$, then there exists $B > 0$ such that

$$\sup_{t\in\mathbb{T},\|X\|\le C}\|F(t,X)\| = \sup_{t\in\mathbb{T},\|X\|\le C}\left\|\frac{f(t,x,x+kx^\Delta)}{k}\right\| \le B. \tag{40}$$

Without loss of generality, we suppose that $t_1 > t_2$, and $\Phi(\vartheta^+) = I_2$. Let $\varepsilon > 0$, and let $X \in \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$. We note that

$$\begin{aligned}
&\Gamma X(t_1) - \Gamma X(t_2) \\
&\quad= \int_{-\infty}^{t_2}(\Phi(t_1)-\Phi(t_2))P\Phi^{-1} \\
&\qquad\times(\sigma(s))F(s,X(s))\Delta s \\
&\qquad+\int_{t_2}^{t_1}\Phi(t_1)P\Phi^{-1}(\sigma(s))F(s,X(s))\Delta s \\
&\qquad+\int_{t_2}^{t_1}\Phi(t_2)(I-P)\Phi^{-1}(\sigma(s))F(s,X(s))\Delta s \\
&\qquad-\int_{t_1}^{\infty}(\Phi(t_1)-\Phi(t_2))(I-P)\Phi^{-1}(\sigma(s)) \\
&\qquad\times F(s,X(s))\Delta s \\
&\quad= J_1 + J_2 + J_3 - J_4.
\end{aligned} \tag{41}$$

From Lemma 6, we have

$$\begin{aligned}\|J_2\| &= \left\|\int_{t_2}^{t_1} \Phi(t_1) P\Phi^{-1}(\sigma(s)) F(s, X(s)) \Delta s\right\| \\ &\le B\int_{t_2}^{t_1} \left\|\Phi(t_1) P\Phi^{-1}(\sigma(s))\right\| \Delta s \\ &\le B\int_{t_2}^{t_1} K_1 e_{\ominus\alpha_1}(t_1, \sigma(s)) \Delta s \\ &= BK_1\int_{t_2}^{t_1} (1+\mu(s)\alpha_1) e_{\alpha_1}(s, t_1) \Delta s \\ &\le \frac{BK_1(1+M\alpha_1)}{\alpha_1}\left(1 - e_{\alpha_1}(t_2, t_1)\right), \\ \|J_3\| &\le B\int_{t_2}^{t_1} K_2 e_{\ominus\alpha_2}(\sigma(s), t_2) \Delta s \\ &= \frac{BK_2}{\alpha_2}\left(1 - e_{\ominus\alpha_2}(t_1, t_2)\right).\end{aligned} \tag{42}$$

Thus, there is a $\delta_1 > 0$ such that $0 < t_1 - t_2 < \delta_1$ implies that $\|J_2\|, \|J_3\| < \varepsilon/4$.

In the following, we will prove there exists a $\delta_2 > 0$ such that $0 < t_1 - t_2 < \delta_2$ implies that $\|J_1\| < \varepsilon/4$ and $\|J_4\| < \varepsilon/4$. We will prove that by the following cases:

(i) $t_1 > t_2 > \vartheta^+$;

(ii) $t_1 \ge \vartheta^+ \ge t_2$;

(iii) $\vartheta^+ > t_1 > t_2$.

Suppose that case (i) holds. By Theorem 4.1 in [28] $\Phi(t)$ is absolutely continuous; that is, there exists a $\delta_3 > 0$ such that $0 < t_1 - t_2 < \delta_3$ implies that

$$\|\Phi(t_1) - \Phi(t_2)\| < \frac{\alpha_2\varepsilon}{4BK_2}. \tag{43}$$

Then

$$\begin{aligned}\|J_4\| &= \left\|\int_{t_1}^{\infty} (\Phi(t_1) - \Phi(t_2)) \Phi(\vartheta^+) \right. \\ &\qquad \left. \times (I-P)\Phi^{-1}(\sigma(s)) F(s, X(s)) \Delta s\right\| \\ &\le B\|\Phi(t_1) - \Phi(t_2)\| \\ &\quad \times \int_{t_1}^{\infty} \left\|\Phi(\vartheta^+)(I-P)\Phi^{-1}(\sigma(s))\right\| \Delta s \\ &\le BK_2\|\Phi(t_1) - \Phi(t_2)\| \int_{t_1}^{\infty} e_{\ominus\alpha_2}(\sigma(s), \vartheta^+) \Delta s \\ &\le \frac{BK_2}{\alpha_2} e_{\ominus\alpha_2}(t_1, \vartheta^+) \|\Phi(t_1) - \Phi(t_2)\| \\ &\le \frac{BK_2}{\alpha_2}\|\Phi(t_1) - \Phi(t_2)\| < \frac{\varepsilon}{4}.\end{aligned} \tag{44}$$

If $t_2 = \rho(t_2) = \sup\{s \in \mathbb{T} : s < t_2\} = \sup\{t_2 - h \in \mathbb{T} : h \in \mathbb{R}^+\}$, then there exist h_1, h_2 such that $0 < h_2 < h_1 < \varepsilon/16BK_1(1+M\alpha_1)$ and $t_2 - h_1, t_2 - h_2 \in \mathbb{T}$. By the equi-continuity of $\Phi(\cdot)P\Phi(s)$, there exists a $\delta_4 > 0$ such that $0 < t_1 - t_2 < \delta_4$ implies that for given $\omega \in \gamma(\mathbb{T})$ we have

$$\begin{aligned}&\left\|\Phi(t_1) P\Phi^{-1}(t_2 - \tau) - \Phi(t_2) P\Phi^{-1}(t_2 - \tau)\right\| \\ &\quad < \frac{\varepsilon\left((1+M\alpha_1)^{\omega/M} - 1\right)}{8BK_1\omega(1+M\alpha_1)^{1+\omega/M+h_2/M}} \quad \text{for } \tau \ge h_1.\end{aligned} \tag{45}$$

Thus,

$$\begin{aligned}\|J_1\| &\le \left\|\int_{-\infty}^{t_2-h_1} (\Phi(t_1) - \Phi(t_2)) \right. \\ &\qquad \left. \times P\Phi^{-1}(\sigma(s)) F(s, X(s)) \Delta s\right\| \\ &\quad + \left\|\int_{t_2-h_1}^{t_2} (\Phi(t_1) - \Phi(t_2)) \right. \\ &\qquad \left. \times P\Phi^{-1}(\sigma(s)) F(s, X(s)) \Delta s\right\| \\ &\le \int_{[h_1,\infty)_{\mathbb{T}}} \left\|(\Phi(t_1) - \Phi(t_2)) P\Phi^{-1}(t_2 - h_2)\right\| \\ &\qquad \times \left\|\Phi(t_2 - h_2) P\Phi^{-1}(\sigma(t_2 - \tau))\right\| \\ &\qquad \times \left\|F(t_2 - \tau, X(t_2 - \tau))\right\| \Delta\tau \\ &\quad + \int_{t_2-h_1}^{t_2} \left(\left\|\Phi(t_1) P\Phi^{-1}(\sigma(s))\right\| \right. \\ &\qquad \left. + \left\|\Phi(t_2) P\Phi^{-1}(\sigma(s))\right\|\right) \\ &\qquad \times \|F(s, X(s))\| \Delta s \\ &< \frac{\varepsilon\left((1+M\alpha_1)^{\omega/M} - 1\right)}{8\omega(1+M\alpha_1)^{1+\omega/M+h_2/M}} \\ &\quad \times \int_{[h_1,\infty)_{\mathbb{T}}} e_{\ominus\alpha_1}(t_2 - h_2, \sigma(t_2 - \tau)) \Delta\tau \\ &\quad + BK_1(1+M\alpha_1) \\ &\quad \times \int_{t_2-h_1}^{t_2} \left(e_{\alpha_1}(s, t_1) + e_{\alpha_1}(s, t_2)\right) \Delta s \\ &\le \frac{\varepsilon\left((1+M\alpha_1)^{\omega/M} - 1\right)}{8\omega(1+M\alpha_1)^{1+\omega/M+h_2/M}} \\ &\quad \times \sum_{n=0}^{\infty} \int_{n\omega}^{(n+1)\omega} (1+\mu(t_2 - \tau)\alpha_1) e_{\ominus\alpha_1} \\ &\qquad \times (t_2 - h_2, t_2 - \tau) \Delta\tau + BK_1(1+M\alpha_1) h_1 \\ &\quad \times \left((1+M\alpha_1)^{(t_2-t_1)/M} + 1\right)\end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon\left(\left(1+M\alpha_1\right)^{\omega/M}-1\right)}{8\omega\left(1+M\alpha_1\right)^{1+\omega/M+h_2/M}} \\
&\quad\times\sum_{n=0}^{\infty}\int_{n\omega}^{(n+1)\omega}\left(1+M\alpha_1\right) \\
&\qquad\times\left(\frac{1}{1+M\alpha_1}\right)^{(\tau-h_2)/M}\Delta\tau+\frac{\varepsilon}{8} \\
&\le \frac{\varepsilon}{4}.
\end{aligned}
\tag{46}
$$

If $t_2 > \rho(t_2)$, then

$$
\begin{aligned}
J_1 = &\int_{-\infty}^{\rho(t_2)}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right) \\
&\times P\Phi^{-1}\left(\sigma\left(s\right)\right)F\left(s,X\left(s\right)\right)\Delta s \\
&+\int_{\rho(t_2)}^{t_2}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right) \\
&\times P\Phi^{-1}\left(\sigma\left(s\right)\right)F\left(s,X\left(s\right)\right)\Delta s.
\end{aligned}
\tag{47}
$$

Similar to the proof above, we can show that there is a $\delta_3' > 0$ such that $t_1 - t_2 < \delta_3'$ implies that

$$
\begin{aligned}
&\left\|\int_{-\infty}^{\rho(t_2)}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right)\right. \\
&\qquad\left.\times P\Phi^{-1}\left(\sigma\left(s\right)\right)F\left(s,X\left(s\right)\right)\Delta s\right\| \\
&\le \int_{\vartheta^+}^{\infty}\left\|\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right)P\Phi^{-1}\left(\rho\left(t_2\right)\right)\right\| \\
&\qquad\times\left\|\Phi\left(\rho\left(t_2\right)\right)P\Phi^{-1}\left(\sigma\left(\rho\left(t_2\right)-\tau\right)\right)\right\| < \frac{\varepsilon}{8}.
\end{aligned}
\tag{48}
$$

Note that

$$
\begin{aligned}
&\left\|\int_{\rho(t_2)}^{t_2}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right)\right. \\
&\qquad\left.\times P\Phi^{-1}\left(\sigma\left(s\right)\right)F\left(s,X\left(s\right)\right)\Delta s\right\| \\
&= \left\|\int_{\rho(t_2)}^{t_2}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right)P\Phi^{-1}\left(s\right)\right. \\
&\qquad\left.\times\left(I+\mu\left(s\right)A\right)^{-1}F\left(s,X\left(s\right)\right)\Delta s\right\| \\
&\le \left\|\nu\left(t_2\right)\right\|\left\|\Phi\left(t_1\right)-\Phi\left(t_2\right)\right\|\left\|P\right\| \\
&\qquad\times\left\|\Phi^{-1}\left(\rho\left(t_2\right)\right)\left(I+\mu\left(\rho\left(t_2\right)\right)A\right)^{-1}\right\|B,
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
&\left\|\Phi^{-1}\left(\rho\left(t_2\right)\right)\left(I+\mu\left(\rho\left(t_2\right)\right)A\right)^{-1}\right\| \\
&= \left\|e_1^{-1}\left(\rho\left(t_2\right),\vartheta^+\right)\begin{pmatrix}1 & -k\int_{\vartheta^+}^{\rho(t_2)}\left(1+\mu\left(\tau\right)\right)^{-1}\Delta\tau \\ 0 & 1\end{pmatrix}\right. \\
&\qquad\left.\times\begin{pmatrix}\left(1+\nu\left(t_2\right)\right)^{-1} & -k\nu\left(t_2\right)\left(1+\nu\left(t_2\right)\right)^{-2} \\ 0 & \left(1+\nu\left(t_2\right)\right)^{-1}\end{pmatrix}\right\| \\
&\le \left\|e_1^{-1}\left(\rho\left(t_2\right),\vartheta^+\right)\begin{pmatrix}1 & -k\int_{\vartheta^+}^{\rho(t_2)}\left(1+\mu\left(\tau\right)\right)^{-1}\Delta\tau \\ 0 & 1\end{pmatrix}\right\| \\
&\le \max\left\{e_1^{-1}\left(\rho\left(t_2\right),\vartheta^+\right),k\left(\rho\left(t_2\right)-\vartheta^+\right)e_1^{-1}\left(\rho\left(t_2\right),\vartheta^+\right)\right\}.
\end{aligned}
\tag{50}
$$

Since $t^{\Delta}/e_1^{\Delta}(t,\vartheta^+) = 1/e_1(t,\vartheta^+)$ for $t \in \mathbb{T}$ is decreasing and $e_1^{\Delta}(t,\vartheta^+) > 0$, then $(t-\vartheta^+)/(e_1(t,\vartheta^+)-e_1(\vartheta^+,\vartheta^+)) > 0$ is decreasing by Theorem 2.1 in [29] and therefore uniformly bounded on (ϑ^+,∞). Thus $(t-\vartheta^+)/e_1(t,\vartheta^+)$ is bounded as follows. By the boundedness of $\nu(t_2)$, P, $\Phi^{-1}(\rho(t_2))(I+\mu(\rho(t_2))A)^{-1}$ and absolute continuity of Φ, there is a $\delta_4' > 0$ such that $t_1 - t_2 < \delta_4'$ implies that

$$
\left\|\int_{\rho(t_2)}^{t_2}\left(\Phi\left(t_1\right)-\Phi\left(t_2\right)\right)P\Phi^{-1}\left(\sigma\left(s\right)\right)F\left(s,X\left(s\right)\right)\Delta s\right\| < \frac{\varepsilon}{8}.
\tag{51}
$$

Set $\delta_2 = \min\{\delta_3,\delta_4,\delta_3',\delta_4'\}$; then $\|J_1\|,\|J_4\| < \varepsilon/4$ for $0 < t_1 - t_2 < \delta_2$.

Therefore, for $0 < t_1 - t_2 < \min\{\delta_1,\delta_2\}$,

$$
\left\|\Gamma X\left(t_1\right)-\Gamma X\left(t_2\right)\right\| < \varepsilon.
\tag{52}
$$

As the method in the above proof, we can show that for $\varepsilon > 0$ there is a $\delta > 0$ such that if $t_1 \le \vartheta^+ \le t_2$ or $\vartheta^+ > t_1 > t_2$ with $t_1 - t_2 < \delta$, the inequality (52) holds. Thus, Γ is equicontinuous.

According to Arzela-Ascoli theorem and the continuity of Γ, Γ is completely continuous. Therefore, by Schauder's fixed point theorem there exists a fixed point $X_* = (x_*, x_*^{\Delta})^{\tau} \in \mathrm{AP}(\mathbb{T},\mathbb{R}^2)$ of Γ such that $\Gamma X_* = X_*$. Furthermore, x_* is almost periodic solution of (25). □

Theorem 32. *Suppose that*

(H1) $f \in UAP(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$ *is Lipschitz continuous in* $(x,y) \in \mathbb{R}^2$ *uniformly on* $\mathbb{T}$*; that is, there is a* $L > 0$ *such that*

$$
\left\|f\left(t,x,y\right)-f\left(t,\overline{x},\overline{y}\right)\right\| \le L\left(\left\|x-\overline{x}\right\|+\left\|y-\overline{y}\right\|\right),
\tag{53}
$$

for $t \in \mathbb{T}$*, and* $x, y, \overline{x}, \overline{y} \in \mathbb{R}$*;*

(H2) there exists $k \in \mathbb{R}^+$ *with* $\sup_{t\in\mathbb{T}}\{k^2\mu(t)+2k\} \le 2/3$ *such that*

$$
L\left(1+\frac{2}{k}\right)\left(\frac{1+M\alpha_1}{\alpha_1}+\frac{1}{\alpha_2}\right) < 1.
\tag{54}
$$

Then (25) *admits a unique almost periodic solution.*

Proof. Let $X = (x_1, x_2)^\tau, Y = (y_1, y_2)^\tau \in \mathbb{R}^2$. Note

$$\begin{aligned}
&\|F(t,X) - F(t,Y)\| \\
&\quad = \frac{1}{k}\left\|f(t, x_1, x_1 + kx_2) - f(t, y_1, y_1 + ky_2)\right\| \\
&\quad \le \frac{L}{k}\left(\|x_1 - y_1\| + \|x_1 + kx_2 - y_1 - ky_2\|\right) \\
&\quad \le \frac{L}{k}\left(2\|x_1 - y_1\| + k\|x_2 - y_2\|\right) \\
&\quad \le L\left(1 + \frac{2}{k}\right)\|X - Y\|.
\end{aligned} \tag{55}$$

Set $\Gamma : \mathrm{AP}(\mathbb{T}, \mathbb{R}^2) \to \mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$ as (38). Then

$$\begin{aligned}
&\|\Gamma X - \Gamma Y\| \\
&\quad \le \int_{-\infty}^{t} \left\|\Phi(t) P \Phi^{-1}(\sigma(s))\right\| \|F(s, X(s)) - F(s, Y(s))\| \Delta s \\
&\qquad + \int_{t}^{\infty} \left\|\Phi(t)(I - P)\Phi^{-1}(\sigma(s))\right\| \\
&\qquad\qquad \times \|F(s, X(s)) - F(s, Y(s))\| \Delta s \\
&\quad \le L\left(1 + \frac{2}{k}\right)\left(\int_{-\infty}^{t} K_1 e_{\ominus\alpha_1}(t, \sigma(s)) \Delta s \right. \\
&\qquad\qquad \left. + \int_{t}^{\infty} K_2 e_{\ominus\alpha_2}(\sigma(s), t) \Delta s\right) \\
&\quad \le L\left(1 + \frac{2}{k}\right)\left(\frac{1 + M\alpha_1}{\alpha_1} + \frac{1}{\alpha_2}\right)\|X - Y\|,
\end{aligned} \tag{56}$$

which implies that Γ is continuous. And by (H2), Γ is a contraction. Therefore, by contraction mapping principle, there exists a unique fixed point $X^* \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$ such that $\Gamma X^* = X^*$. By Lemma 27, $X^* = (x_1^*, x_2^*)^\tau \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$ implies that $x_1^* \in \mathrm{AP}^1(\mathbb{T}, \mathbb{R})$ is the solution of (25). □

Set $f(t, x, y) = g(t, x, y) - x$. Then $g \in \mathrm{UAP}(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ yields that $f \in \mathrm{UAP}(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$.

Theorem 33. *Suppose that there is a $k > 0$ such that $k^2\mu(t) + 2k \le 2/3$ for all $t \in \mathbb{T}$. Assume $g \in UAP(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ takes bounded sets into bounded sets. Then* (1) *admits a almost periodic solution.*

Theorem 34. *Let $\mathbb{T}$ be a two-way translation invariant time scale. Suppose that*

(H1) $g \in UAP(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ is Lipschitz continuous in $(x, y) \in \mathbb{R}^2$ uniformly on $\mathbb{T}$; that is, there is a $L > 0$ such that

$$\left\|g(t, x, y) - g(t, \overline{x}, \overline{y})\right\| \le L\left(\|x - \overline{x}\| + \|y - \overline{y}\|\right), \tag{57}$$

for $t \in \mathbb{T}$, and $x, y, \overline{x}, \overline{y} \in \mathbb{R}$;

(H2) there exists $k \in \mathbb{R}^+$ with $\sup_{t\in\mathbb{T}}\{k^2\mu(t) + 2k\} \le 2/3$ such that

$$\left(L + \frac{2L + 1}{k}\right)\left(\frac{1 + M\alpha_1}{\alpha_1} + \frac{1}{\alpha_2}\right) < 1. \tag{58}$$

Then (25) *admits a unique almost periodic solution.*

5. Alternative Theorem for Second Order Dynamic Equations on Time Scales

In this section, we consider the following equation:

$$x^{\Delta\Delta} = x + f\left(t, x, x^\Delta\right) + \lambda g\left(t, x, x^\Delta\right). \tag{59}$$

Theorem 35. *Assume the following conditions hold:*

(i) $f, g \in UAP(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ take bounded sets into bounded sets;

(ii) there exists an open bounded subset $\Omega \subset AP(\mathbb{T}, \mathbb{R}^2)$ such that every possible solution x of (59) *satisfies $(x, x^\Delta)^\tau \notin \partial\Omega$;*

(iii) $\deg(I - \Gamma, \Omega, 0) \ne 0$.

Then (59) *has an almost periodic solution x with $(x, x^\Delta)^\tau \in \Omega$.*

Proof. Let $X(t) = (x(t), x^\Delta(t))^\tau$, and let $G(t, X) = (0, g(t, x, x + kx^\Delta)/k)^\tau$. Then $X \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$ yields that $\widetilde{G}(t) = G(t, X(t)) \in \mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$.

For all $(X, \lambda) \in \Omega \times [0, 1]$, set

$$\begin{aligned}
H(X, \lambda) &= \Gamma(X) + \lambda \int_{-\infty}^{\cdot} \Phi(\cdot) P \Phi^{-1}(\sigma(s)) G(s, X(s)) \Delta s \\
&\quad + \lambda \int_{\cdot}^{\infty} \Phi(\cdot)(I - P)\Phi^{-1}(\sigma(s)) G(s, X(s)) \Delta s.
\end{aligned} \tag{60}$$

Obviously, H is a map from $\mathrm{AP}(\mathbb{T}, \mathbb{R}^2) \times [0, 1]$ to $\mathrm{AP}(\mathbb{T}, \mathbb{R}^2)$.

As the proof of Theorem 31, $H(\cdot, \lambda)$ for all $\lambda \in [0, 1]$ is completely continuous. Thus

$$\deg(I - H(\cdot, \lambda), \Omega, 0) = \deg(I - \Gamma, \Omega, 0) \ne 0; \tag{61}$$

that is, (59) has an almost periodic solution x with $(x, x^\Delta)^\tau \in \Omega$. □

Theorem 36. *Consider the following problems:*

$$x^{\Delta\Delta} = x + f(t), \quad t \in \mathbb{R}, \tag{62}$$

$$x^{\Delta\Delta} = x + f(t) + \lambda g\left(t, x, x^\Delta\right) \quad \text{for } t \in \mathbb{R},\ \lambda \in [0, 1], \tag{63}$$

where $f \in AP(\mathbb{T}, \mathbb{R})$ and $g \in UAP(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ takes bounded sets into bounded sets. Assume the following conditions hold.

(i) There exists an open bounded subset $\Omega \subset AP(\mathbb{T}, \mathbb{R}^2)$ such that every possible solution u of (63) *satisfies $u \notin \partial\Omega$.*

(ii) There exists a unique almost periodic solution x_* of (62) such that $(x_*, x_*^{\Delta})^{\tau} \in \Omega$.

Then (63) has an almost periodic solution x with $(x, x^{\Delta})^{\tau} \in \Omega$.

Proof. Set

$$\widetilde{H}(X,\lambda) = X_*(\cdot) + \lambda \int_{-\infty}^{\cdot} \Phi(\cdot) P\Phi^{-1}(\sigma(s)) G(s, X(s)) \Delta s + \lambda \int_{\cdot}^{\infty} \Phi(\cdot)(I-P)\Phi^{-1}(\sigma(s)) G(s, X(s)) \Delta s, \quad \forall (u,\lambda) \in \Omega \times [0,1], \tag{64}$$

where $X_*(t)$ is the almost periodic solution of (62). As the proof of Theorem 31, $\widetilde{H}(\cdot,\lambda) : \Omega \to \mathrm{AP}(\mathbb{T} \times \mathbb{R}^2, \mathbb{R}^2)$ for all $\lambda \in [0,1]$ is completely continuous. Note (see [30])

$$\deg(I, \Omega, X_*) = 1. \tag{65}$$

Therefore,

$$\begin{aligned} &\deg(I - H(\cdot,\lambda), \Omega, 0) \\ &\quad = \deg(I - H(\cdot,0), \Omega, 0) = \deg(I, \Omega, X_*) = 1; \end{aligned} \tag{66}$$

that is, (63) has an almost periodic solution x with $(x, x^{\Delta})^{\tau} \in \Omega$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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