Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 910189, 6 pages
http://dx.doi.org/10.1155/2013/910189

Research Article

A Fractional-Order Chaotic System with an Infinite Number of Equilibrium Points

Ping Zhou,[1,2] Kun Huang,[2] and Chun-de Yang[1]

[1] *Center of System Theory and Its Applications, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*
[2] *Key Laboratory of Network Control and Intelligent Instrument of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Correspondence should be addressed to Ping Zhou; zhouping@cqupt.edu.cn

Received 18 February 2013; Accepted 21 March 2013

Academic Editor: Xiao-Song Yang

A new 4D fractional-order chaotic system, which has an infinite number of equilibrium points, is introduced. There is no-chaotic behavior for its corresponded integer-order system. We obtain that the largest Lyapunov exponent of this 4D fractional-order chaotic system is 0.8939 and yield the chaotic attractor. A chaotic synchronization scheme is presented for this 4D fractional-order chaotic system. Numerical simulations is verified the effectiveness of the proposed scheme.

1. Introduction

Because the chaotic (hyperchaotic) signal can be used in electrical engineering, telecommunications, information processing, material engineering, and so forth much attention has been paid to effectively generating chaotic and hyperchaotic systems. Many chaotic (hyperchaotic) systems and fractional-order chaotic (hyperchaotic) systems are reported in recent years [1–14], such as Lorenz chaotic (hyperchaotic) system and its corresponded fractional-order system, and integer-order and fractional-order Chen chaotic (hyperchaotic) system, integer-order and fractional-order Lü chaotic (hyperchaotic) system.

However, for all the previous integer-order and fractional-order chaotic (hyperchaotic), many systems have a finite number of equilibrium points. For example, some chaotic systems have one equilibrium point [15–17], some chaotic systems have two equilibrium points [18], and some chaotic systems have three equilibrium points [1, 2, 5, 6, 9, 10], so a natural and interesting question is can we construct a chaotic (hyperchaotic) system which has an infinite number of equilibrium points? Moreover, many fractional-order chaotic and hyperchaotic systems also possess chaotic attractor for its corresponded integer-order system, so the other question is as follows: are the fractional-order chaotic and hyperchaotic systems no-chaotic behavior for its corresponded integer-order system? To the best of our knowledge, few results on the above mentioned two questions are reported.

Motivated by the above discussions, a new 4D fractional-order chaotic system is presented in this paper. This new 4D fractional-order chaotic system has an infinite number of equilibrium points, and no-chaotic behavior for its corresponded integer-order system. The largest Lyapunov exponent and chaotic attractor are yielded for the new 4D fractional-order chaotic system. A chaotic synchronization scheme is presented for this new 4D fractional-order chaotic system.

2. A New 4D Fractional-Order Chaotic System

Now, a new 4D fractional-order chaotic system is constructed, which is described as follows:

$$\frac{d^q x_1}{dt^q} = 10\left(x_2 - x_1\right) + x_4$$

$$\frac{d^q x_2}{dt^q} = 15x_1 - x_1 x_3$$

$$\frac{d^q x_3}{dt^q} = -2.5x_3 + 4{x_1}^2$$
$$\frac{d^q x_4}{dt^q} = -10x_2 - x_4, \tag{1}$$

where $q = 0.95$ is the fractional-order, and $x_i\,(i = 1, 2, 3, 4)$ are real state variables.

The real equilibrium points of system (1) is calculated by

$$\begin{aligned} 10\left(x_2 - x_1\right) + x_4 &= 0 \\ 15x_1 - x_1x_3 &= 0 \\ -2.5x_3 + 4{x_1}^2 &= 0 \\ -10x_2 - x_4 &= 0. \end{aligned} \tag{2}$$

Obviously, $(x_1, x_2, x_3, x_4) = (0, x_2, 0, -10x_2)$ is the real equilibrium points of system (1), where x_2 is a any real numbers, so system (1) has an infinite number of real equilibrium points. To the best of our knowledge, this result is different from all the previous fractional-order chaotic and hyperchaotic systems. It implies that we yield a new 4D fractional-order system, which has an infinite number of real equilibrium points.

The Jacobian J at all equilibrium points is

$$J = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 15 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 0 \\ 0 & -10 & 0 & -1 \end{pmatrix} \tag{3}$$

and its eigenvalues are $\lambda_1 = -18.548$, $\lambda_2 = -2.5$, $\lambda_3 = 0$, and $\lambda_4 = 7.548$ for all x_2. Therefore, all the equilibrium points in system (1) are unstable.

The dynamical behaviors of system (1) for its corresponded integer-order system ($q = 1$) can be characterized by its Lyapunov exponents. The Lyapunov exponents for its corresponded integer-order system are 0, 0, −0.779, and −12.724, respectively. Therefore, the fractional-order system (1) no-chaotic behaviors for $q = 1$, and which is periodic orbit for its corresponded integer-order system. Figure 1 shows the periodic orbit of fractional-order system (1) for its corresponded integer-order system ($q = 1$).

Now, we discuss the numerical solution of fractional differential equations. It is well known that there are direct time domain approximation (the improved version of Adams-Bashforth-Moulton numerical algorithm) and frequency domain approximation for nonlinear fractional-order system [6]. However, frequency domain approximation may result in wrong consequences [19], so the direct time domain approximation [6] numerical simulation is used to solve the fractional-order system in this paper. Let $h = T/N$, $t_n = nh\,(n = 0, 1, 2 \ldots, N)$, and initial condition $(x_1(0), x_2(0), x_3(0), x_4(0))$, so the fractional-order chaotic system (1) can be discretized as follows:

$$\begin{aligned}
x_1(n+1) &= x_1(0) + \frac{h^q}{\Gamma(q+2)} \times \left[10\left(x_2^p(n+1) - x_1^p(n+1)\right) + x_4^p(n+1) + \sum_{j=0}^{n} \alpha_{1,j,n+1} \times \left(10\left(x_2(j) - x_1(j)\right) + x_4(j)\right) \right] \\
x_2(n+1) &= x_2(0) + \frac{h^q}{\Gamma(q+2)} \times \left[15x_1^p(n+1) - x_1^p(n+1)\, x_3^p(n+1) + \sum_{j=0}^{n} \alpha_{2,j,n+1} \left(15x_1(j) - x_1(j)\, x_3(j)\right) \right] \\
x_3(n+1) &= x_3(0) + \frac{h^q}{\Gamma(q+2)} \times \left[4\left(x_1^p(n+1)\right)^2 - 2.5x_3^p(n+1) + \sum_{j=0}^{n} \alpha_{3,j,n+1} \left(4\left(x_1(j)\right)^2 - 2.5x_3(j)\right) \right] \\
x_4(n+1) &= x_4(0) + \frac{h^q}{\Gamma(q+2)} \times \left[-10x_2^p(n+1) - x_4^p(n+1) + \sum_{j=0}^{n} \alpha_{4,j,n+1} \times \left(-10x_2(j) - x_4(j)\right) \right],
\end{aligned} \tag{4}$$

where

$$x_1^p(n+1) = x_1(0) + \frac{1}{\Gamma(q)} \times \sum_{j=0}^{n} b_{1,j,n+1} \times \left[10\left(x_2(j) - x_1(j)\right) + x_4(j)\right]$$

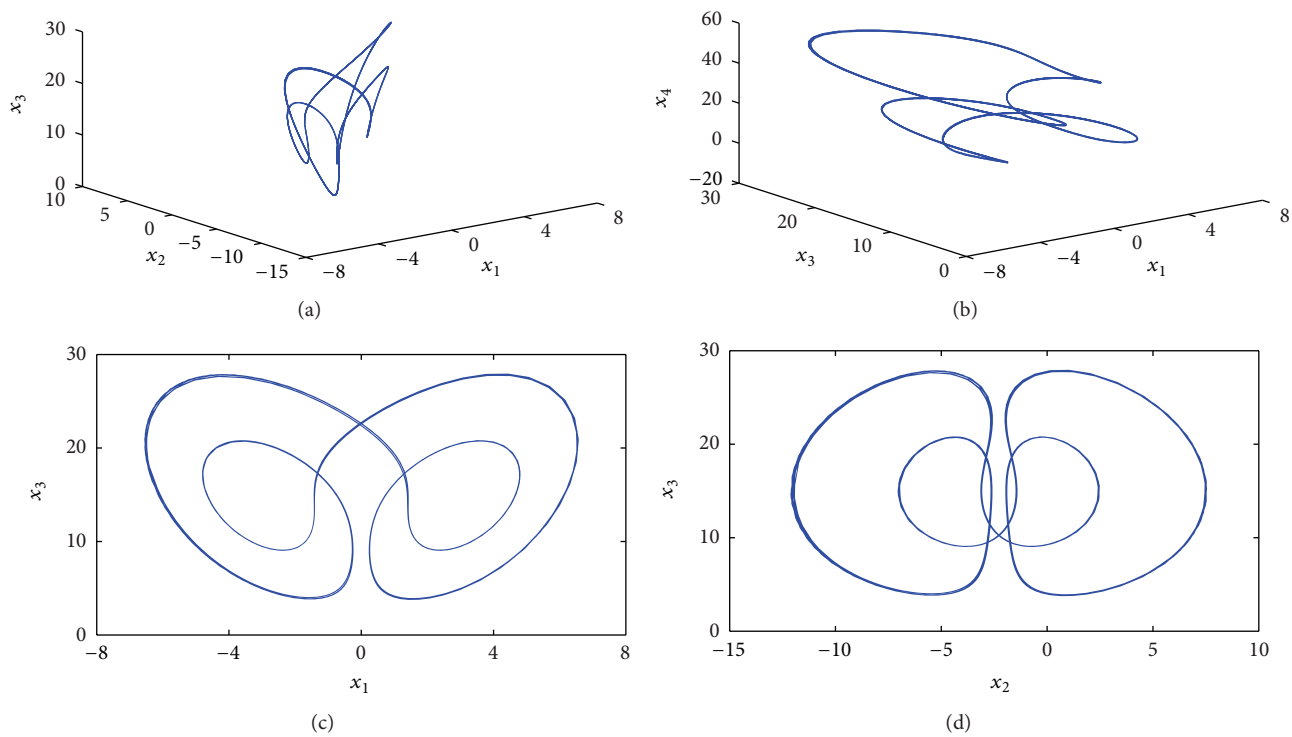

Figure 1: The periodic orbit of fractional-order system (1) for its corresponded integer-order system ($q = 1$).

$$
\begin{aligned}
x_2^p(n+1) &= x_2(0) + \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{j=0}^{n} b_{2,j,n+1}\left[15x_1(j) - x_1(j)x_3(j)\right] \\
x_3^p(n+1) &= x_3(0) + \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{j=0}^{n} b_{3,j,n+1}\left[4(x_1(j))^2 - 2.5x_3(j)\right] \\
x_4^p(n+1) &= x_4(0) + \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{j=0}^{n} b_{4,j,n+1} \times \left[-10x_2(j) - x_4(j)\right],
\end{aligned} \tag{5}
$$

and for $i = 1, 2, 3, 4$

$$
\begin{aligned}
&\alpha_{i,j,n+1} \\
&= \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1, \end{cases} \\
&b_{i,j,n+1} = \frac{h^q}{q}\left[(n-j+1)^q - (n-j)^q\right], \quad 0 \le j \le n.
\end{aligned} \tag{6}
$$

The error of this approximation is described as follows:

$$
\left|x_i(t_n) - x_i(n)\right| = o\left(h^p\right), \qquad p = \min(2, 1+q). \tag{7}
$$

The dynamical behaviors of 4D fractional-order system (1) can be characterized by its largest Lyapunov exponent. By computer simulation, we can obtain that the largest Lyapunov exponent of fractional-order system (1) is 0.8939, so the 4D fractional-order system (1) is chaotic. The chaotic attractor is shown in Figure 2.

According to the above mentioned, we obtain a new 4D fractional-order chaotic system, which has an infinite number of real equilibrium points. Moreover, the 4D fractional-order chaotic system is no-chaotic behaviors for its corresponded integer-order system ($q = 1$). The result in our paper is different from all the previous fractional-order chaotic and hyperchaotic systems.

3. Chaotic Synchronization for the New 4D Fractional-Order Chaotic System

In this section, the chaotic synchronization for the new 4D fractional-order chaotic system (1) is considered. Based on the stability theory of nonlinear fractional-order systems [20–24], one synchronization scheme is proposed, and some numerical simulations are performed.

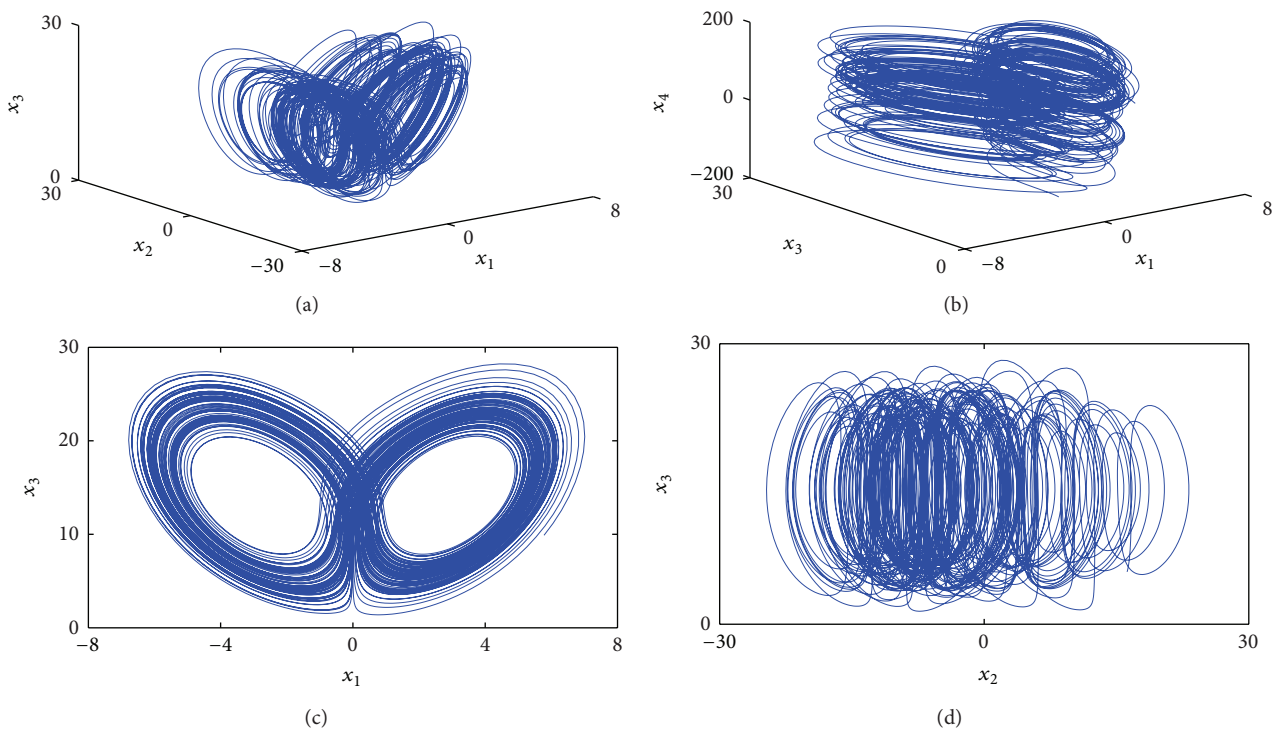

Figure 2: The chaotic attractor of 4D fractional-order system (1).

Now, the response fractional-order chaotic system is considered as

$$\begin{aligned} D^q y_1 &= 10(y_2 - y_1) + y_4 + u_1 \\ D^q y_2 &= 15y_1 - y_1 y_3 + u_2 \\ D^q y_3 &= -2.5y_3 + 4{y_1}^2 + u_3 \\ D^q y_4 &= -10y_2 - y_4 + u_4, \end{aligned} \tag{8}$$

where u_i $(i = 1, 2, 3, 4)$ is the feedback controller, and y_i $(i = 1, 2, 3, 4)$ are real state variables. Our goal is to choose suitable u_i $(i = 1, 2, 3, 4)$ such that drive system (1) and response system (8) can be achieved with chaotic synchronization.

Definition the synchronization errors are $e_i = y_i - x_i$ $(i = 1, 2, 3, 4)$. The following Theorem 1 is given in order to achieve the chaotic synchronization between the fractional-order chaotic system (1) and the fractional-order chaotic system (8).

Theorem 1. *If the feedback controllers are*

$$\begin{aligned} u_1 &= (y_3 - 25)e_2 - 4(y_1 + x_1)e_3 \\ u_2 &= 10e_4 \\ u_3 &= x_1 e_2 \\ u_4 &= -e_1, \end{aligned} \tag{9}$$

then the chaotic synchronization between fractional-order chaotic system (1) *and fractional-order chaotic system* (8) *can be arrived.*

Proof. Combining the fractional-order chaotic system (1), fractional-order chaotic system (8), and the feedback controller (9), we can obtain the following error system

$$\begin{aligned} &\begin{pmatrix} D^q e_1 \\ D^q e_2 \\ D^q e_3 \\ D^q e_4 \end{pmatrix} \\ &= \begin{pmatrix} -10 & -15 + y_3 & -4(y_1 + x_1) & 1 \\ 15 - y_3 & 0 & -x_1 & 10 \\ 4(y_1 + x_1) & x_1 & -2.5 & 0 \\ -1 & -10 & 0 & -1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \\ &\triangleq A(x, y) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}, \end{aligned} \tag{10}$$

where matrix

$$A(x, y) \triangleq \begin{pmatrix} -10 & -15 + y_3 & -4(y_1 + x_1) & 1 \\ 15 - y_3 & 0 & -x_1 & 10 \\ 4(y_1 + x_1) & x_1 & -2.5 & 0 \\ -1 & -10 & 0 & -1 \end{pmatrix}, \tag{11}$$

$\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4)^{\mathrm{T}}$, $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4)^{\mathrm{T}}$.

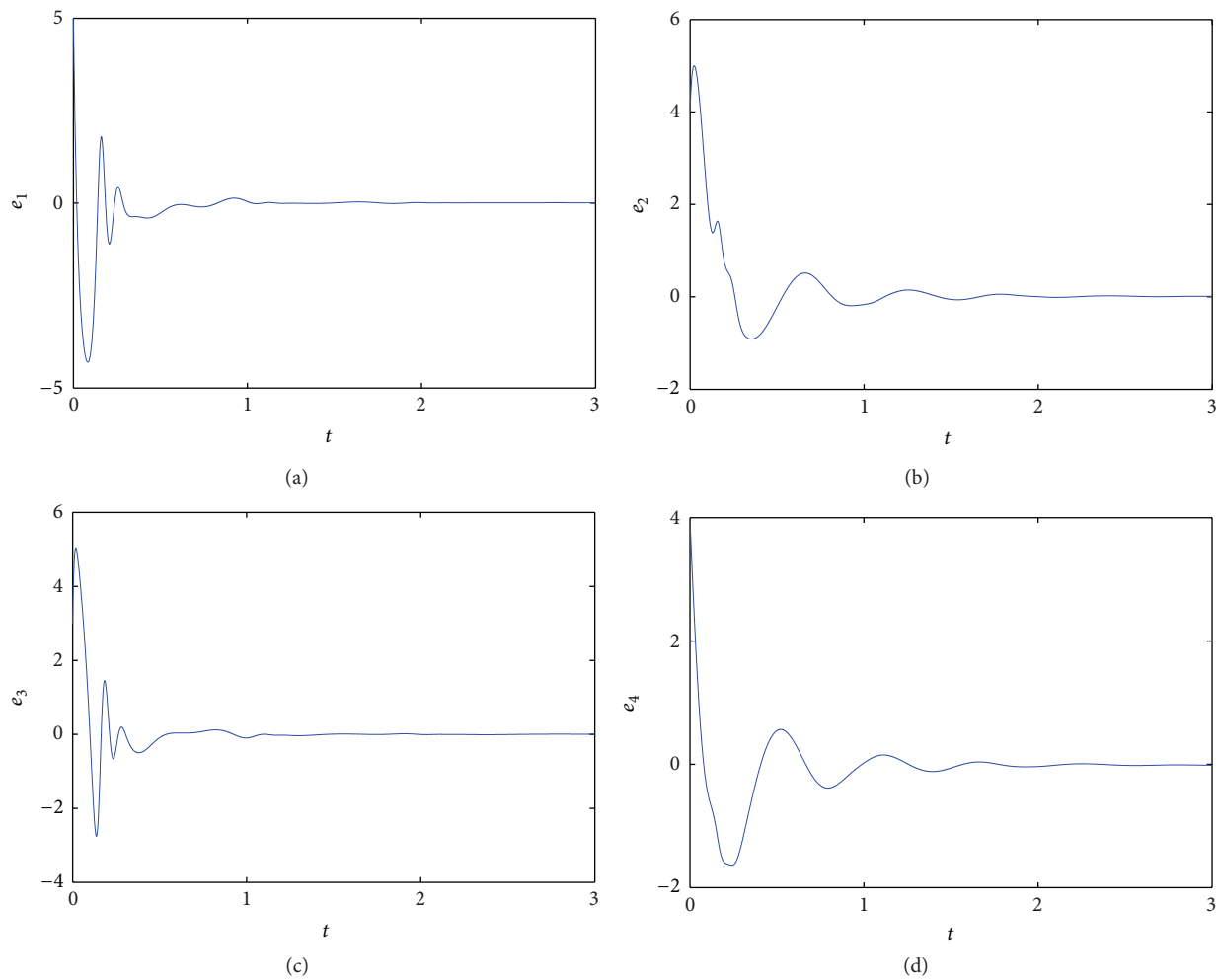

Figure 3: The synchronization errors between the drive systems (1) and the response system (8).

Let λ be one of the eigenvalues of $A(x, y)$ and ψ is the corresponding nonzero eigenvector, so

$$A(x, y)\psi = \lambda\psi,$$
$$\psi^{\mathbf{H}} A(x, y)^{\mathbf{H}} = \overline{\lambda}\psi^{\mathbf{H}}, \tag{12}$$

where $\mathbf{H}$ is conjugate transpose, and $\overline{\lambda}$ is the conjugate for eigenvalues λ.

According to (12), one can obtain

$$\psi^{\mathbf{H}} A(x, y)\psi + \psi^{\mathbf{H}} A(x, y)^{\mathbf{H}}\psi = \psi^{\mathbf{H}}\lambda\psi + \overline{\lambda}\psi^{\mathbf{H}}\psi. \tag{13}$$

Therefore

$$\lambda + \overline{\lambda} = \frac{\psi^{\mathbf{H}}[\operatorname{diag}(-20, 0, -5, -2)]\psi}{\psi^{\mathbf{H}}\psi}, \tag{14}$$

so

$$\lambda + \overline{\lambda} \le 0. \tag{15}$$

That is

$$\arg|\lambda| \ge \frac{\pi}{2} > \frac{q\pi}{2}. \tag{16}$$

Using the stability theory of nonlinear fractional-order systems, one can yield that the error system (10) is asymptotically stable, so

$$\lim_{t \to +\infty} e_i = 0 \quad (i = 1, 2, 3, 4). \tag{17}$$

Equation (17) indicates that the chaotic synchronization between fractional-order chaotic system (1) and fractional-order chaotic system (8) can be achieved. The proof is completed. □

Now, numerical simulations are considered. The numerical results are shown as Figure 3, in which the initial conditions are $\mathbf{x} = (3, 3, 1, 2)^{\mathbf{T}}$ for drive system (1), and $\mathbf{y} = (8, 7, 4, 6)^{\mathbf{T}}$ for response system (8), respectively.

4. Conclusions

In this paper, we obtain a new 4D fractional-order chaotic system, which has an infinite number of equilibrium points and no-chaotic behavior for its corresponded integer-order system. We yield the largest Lyapunov exponent of the new

4D fractional-order system and the Lyapunov exponents for its corresponded integer-order system. The chaotic attractor for the new 4D fractional-order chaotic system and the periodic orbit for its corresponded integer-order system are given. Finally, we realize the chaotic synchronization for the new 4D fractional-order chaotic system, and some numerical simulations are performed.

Acknowledgments

This work was supported by the Foundation of Science and Technology Project of Chongqing Education Commission (KJ110525, KJ100513) and by the National Natural Science Foundation of China (61104150).

References

[1] E. N. Lorenz, "Deterministic nonperodic flow," *Journal of the Atmospheric Sciences*, vol. 20, no. 2, pp. 130–141, 1963.

[2] I. Grigorenko and E. Grigorenko, "Chaotic dynamics of the fractional Lorenz system," *Physical Review Letters*, vol. 91, no. 3, Article ID 034101, 4 pages, 2003.

[3] Q. Jia, "Hyperchaos generated from the Lorenz chaotic system and its control," *Physics Letters A*, vol. 366, no. 3, pp. 217–222, 2007.

[4] X. Y. Wang and J. M. Song, "Synchronization of the fractional order hyperchaos Lorenz systems with activation feedback control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3351–3357, 2009.

[5] G. Chen and T. Ueta, "Yet another chaotic attractor," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 9, no. 7, pp. 1465–1466, 1999.

[6] C. Li and G. Peng, "Chaos in Chen's system with a fractional order," *Chaos, Solitons & Fractals*, vol. 22, no. 2, pp. 443–450, 2004.

[7] Z. Y. Yan, "Controlling hyperchaos in the new hyperchaotic Chen system," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1239–1250, 2005.

[8] X. J. Wu and Y. Lu, "Generalized projective synchronization of the fractional-order Chen hyperchaotic system," *Nonlinear Dynamics*, vol. 57, no. 1-2, pp. 25–35, 2009.

[9] J. Lü and G. Chen, "A new chaotic attractor coined," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 12, no. 3, pp. 659–661, 2002.

[10] J. G. Lu, "Chaotic dynamics of the fractional-order Lü system and its synchronization," *Physics Letters A*, vol. 354, no. 4, pp. 305–311, 2006.

[11] P. Zhou, R. Ding, and Y. X. Cao, "Multi drive-one response synchronization for fractional-order chaotic systems," *Nonlinear Dynamics*, vol. 70, no. 2, pp. 1263–1271, 2012.

[12] Q. Li and X.-S. Yang, "New walking dynamics in the simplest passive bipedal walking model," *Applied Mathematical Modelling*, vol. 36, no. 11, pp. 5262–5271, 2012.

[13] Q. Li, "A topological horseshoe in the hyperchaotic Rössler attractor," *Physics Letters A*, vol. 372, no. 17, pp. 2989–2994, 2008.

[14] Q. Li, X.-S. Yang, and S. Chen, "Hyperchaos in a spacecraft power system," *International Journal of Bifurcation and Chaos*, vol. 21, no. 6, pp. 1719–1726, 2011.

[15] S. Dadras, H. R. Momeni, G. Qi, and Z.-l. Wang, "Four-wing hyperchaotic attractor generated from a new 4D system with one equilibrium and its fractional-order form," *Nonlinear Dynamics*, vol. 67, no. 2, pp. 1161–1173, 2012.

[16] Z. Ping, C. Yu-Xia, and C. Xue-Feng, "A new hyperchaos system and its circuit simulation by EWB," *Chinese Physics B*, vol. 18, no. 4, pp. 1394–1398, 2009.

[17] D. R. Zhu, C. X. Liu, and B. N. Yan, "Controlling and synchronization of hyperchaotic system based on passive control," *Chinese Physics B*, vol. 21, no. 9, Article ID 090509, 7 pages, 2012.

[18] J. C. Sprott, "Some simple chaotic flows," *Physical Review E*, vol. 50, no. 2, pp. R647–R650, 1994.

[19] M. S. Tavazoei and M. Haeri, "A necessary condition for double scroll attractor existence in fractional-order systems," *Physics Letters A*, vol. 367, no. 1-2, pp. 102–113, 2007.

[20] M. S. Tavazoei and M. Haeri, "Chaos control via a simple fractional-order controller," *Physics Letters A*, vol. 372, no. 6, pp. 798–807, 2008.

[21] X.-Y. Wang, Y.-J. He, and M.-J. Wang, "Chaos control of a fractional order modified coupled dynamos system," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 71, no. 12, pp. 6126–6134, 2009.

[22] C.-M. Chang and H.-K. Chen, "Chaos and hybrid projective synchronization of commensurate and incommensurate fractional-order Chen-Lee systems," *Nonlinear Dynamics*, vol. 62, no. 4, pp. 851–858, 2010.

[23] Z. M. Odibat, "Adaptive feedback control and synchronization of non-identical chaotic fractional order systems," *Nonlinear Dynamics*, vol. 60, no. 4, pp. 479–487, 2010.

[24] Z. Odibat, "A note on phase synchronization in coupled chaotic fractional order systems," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 13, no. 2, pp. 779–789, 2012.