Research Article

The Solutions of Second-Order Linear Matrix Equations on Time Scales

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This paper studies the solutions of second-order linear matrix equations on time scales. Firstly, the necessary and sufficient conditions for the existence of a solution of characteristic equation are introduced; then two diverse solutions of characteristic equation are applied to express general solution of the matrix equations on time scales.

1. Introduction

In this paper, we consider the solutions for the following second-order linear matrix equations:

$$[P(t) X^\Delta(t)]^\Delta + QX^\Delta(t) + R(t) X^\sigma(t) = 0, \quad t \in \mathbb{T}, \quad (1)$$

where $P(t) = P\mu(t)$, $R(t) = R/\mu(t)$, $P, Q, R, X(t) \in \mathbb{R}^{m \times m}$, X^Δ is the delta derivative, $\sigma(t)$ is the forward jump operators, $X^\sigma(t) := X(\sigma(t))$, $\mu(t)$ is the graininess function, and $\mathbb{T}$ is an infinite isolated time scale, which is given by

$$\mathbb{T} := \{t_0, t_1, t_2, \dots, t_n, \dots\}. \quad (2)$$

As a tool for establishing a unified framework for continuous and discrete analysis, a theory of dynamic equations on measure chains was introduced by Hilger in his Ph.D. thesis [1] in 1988. In many cases, it is necessary to study a special case of measure chains-time scales. In the last decade the investigation of dynamic systems on time scales has involved much interesting, including quite a few fields, such as the theory of calculus, the oscillation of the dynamic systems, the eigenvalue problems and boundary value problems, and partial differential equations on time scales, and so forth [2–5]. Up to now, there are few results about matrix equations on time scales. In 1998, Agarwal and Bohner [6] studied the quadratic functionals for second order matrix equations on time scales; in 2002, Erbe and Peterson [7] obtained oscillation criteria for a second-order self-adjoint matrix differential equation on time scales in terms of the eigenvalues

of the coefficient matrices and the graininess function. The theory of dynamic systems on time scales is of very important theoretical significance and has a wide range of applications.

Based on the related literatures, the researches of solutions of matrix difference or differential equations have few results. In 1999, Barkatou [8] proposed an algorithm for the rational solutions of special matrix difference equations and discussed their applications; in 2003, Freiling and Hochhaus [9] investigated some properties of the solution for rational matrix difference equations; in 2004, Xu and Zhang [10] studied the representation of general solution for second order homogeneous matrix difference equations; in 2011, Wu and Zhou [11] obtained the particular solutions of one kind of second order matrix differential equations. Since the continuous case and the discrete case are two special cases of time scales, so, we study the solutions of second order linear matrix equations on time scales.

This paper is organized as follows. Section 2 introduces some basic concepts and fundamental theory about time scales. By using the characteristic equation of the matrix equation the solutions of (1) are obtained, which will be given in Section 3.

2. Preliminaries

In this section, some basic concepts and some fundamental results on time scales are introduced.

Let $\mathbb{T} \subset \mathbf{R}$ be a nonempty closed subset. Define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \quad (3)$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$. We put $\mathbb{T}^k = \mathbb{T}$ if $\mathbb{T}$ is unbounded above and $\mathbb{T}^k = \mathbb{T} \setminus (\rho(\max \mathbb{T}), \max \mathbb{T}]$ otherwise. The graininess functions $\nu, \mu : \mathbb{T} \rightarrow [0, \infty)$ are defined by

$$\mu(t) = \sigma(t) - t, \quad \nu(t) = t - \rho(t). \quad (4)$$

Let f be a function defined on $\mathbb{T}$. f is said to be (delta) differentiable at $t \in \mathbb{T}^k$ provided that there exists a constant a such that, for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) with

$$|f(\sigma(t)) - f(s) - a(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \quad (5)$$

In this case, denote $f^\Delta(t) := a$. If f is (delta) differentiable for every $t \in \mathbb{T}^k$, then f is said to be (delta) differentiable on $\mathbb{T}$. If f is differentiable at $t \in \mathbb{T}^k$, then

$$f^\Delta(t) = \begin{cases} \lim_{\substack{s \rightarrow t \\ s \in \mathbb{T}}} \frac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0, \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases} \quad (6)$$

For convenience, we introduce the following results ([3, Lemma 1] and [4, Chapter 1]), which are useful in this paper.

Lemma 1. Let $f, g : \mathbb{T} \rightarrow \mathbf{R}$ and $t \in \mathbb{T}^k$. If f and g are differentiable at t , then fg is differentiable at t and

$$\begin{aligned} (fg)^\Delta(t) &= f^\sigma(t) g^\Delta(t) + f^\Delta(t) g(t) \\ &= f^\Delta(t) g^\sigma(t) + f(t) g^\Delta(t). \end{aligned} \quad (7)$$

3. Main Results

We assume throughout this paper that P is an invertible matrix. It follows from Lemma 1, (6), and (2) that the matrix equation (1) can be written in the form

$$X^{\sigma^2}(t) - BX^\sigma(t) - AX(t) = 0, \quad t \in \mathbb{T}, \quad (8)$$

where $X^{\sigma^2}(t) = X^\sigma(\sigma(t))$, $B = 2E_m - P^{-1}Q - P^{-1}R$, $A = P^{-1}Q - E_m$, and E_m is the identity matrix.

Definition 2. The equation

$$F(\Lambda) = \Lambda^2 - B\Lambda - A = 0, \quad (9)$$

where $\Lambda \in \mathbf{R}^{m \times m}$, is called the characteristic equation of (8).

Definition 3. The functions

$$F(\lambda) = \lambda^2 - B\lambda - A, \quad h(\lambda) = \det F(\lambda) \quad (10)$$

are called the eigenmatrix and eigenpolynomial of (8). Such a λ which satisfies $h(\lambda) = 0$ is called an eigenvalue of $F(\lambda)$.

In order to study the solutions of matrix equation (8), we now introduce some results about the characteristic equation (9).

Theorem 4. If $AB = BA$ and there exists $C \in \mathbf{R}^{m \times m}$ such that $C^2 = B^2 + 4A$, $BC = CB$, then the characteristic equation (9) has solutions.

Proof. Let $\Lambda_1 = (1/2)(B + C)$, $\Lambda_2 = (1/2)(B - C)$. Then Λ_1 and Λ_2 are solutions of (9). $\square$

Remark 5. Generally, if $AB = BA$ and $C^2 = B^2 + 4A$, we cannot easily get $BC = CB$.

Example 6. Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}. \quad (11)$$

Then

$$\begin{aligned} C &= \begin{pmatrix} \frac{\sqrt{3}}{1 + \sqrt{3}} & 0 \\ 1 & 1 \end{pmatrix}, \quad BC = \begin{pmatrix} \frac{\sqrt{3}}{5 + 2\sqrt{3}} & 0 \\ \frac{5 + 2\sqrt{3}}{1 + \sqrt{3}} & -1 \end{pmatrix}, \\ CB &= \begin{pmatrix} \frac{\sqrt{3}}{3 + 2\sqrt{3}} & 0 \\ \frac{3 + 2\sqrt{3}}{1 + \sqrt{3}} & -1 \end{pmatrix}. \end{aligned} \quad (12)$$

Obviously, $BC \neq CB$.

So, Huang and Chen [12] and J. Huang and H. Huang [13] obtained the following results.

Definition 7. Let λ_0 be an eigenvalue of $F(\lambda)$. Then

$$V_{\lambda_0} = \{x \mid (\lambda_0^2 - B\lambda_0 - A)x = 0, x \in \mathbf{R}^{m \times 1}\} \quad (13)$$

is called a characteristic subspace of $F(\lambda)$ corresponding to λ_0 ; the nonzero vector x of V_{λ_0} is called the eigenvector corresponding to λ_0 .

Theorem 8. There exist the diagonalizable solutions of characteristic equation (9) if and only if the dimension of the sum of characteristic subspaces V_{λ_i} is n ; that is, $\dim\{\sum_{i=1}^h V_{\lambda_i}\} = m$, where λ_i ($i = 1, 2, \dots, h$) are the distinct eigenvalues of $F(\lambda)$.

Definition 9. Let λ_i , $i = 1, 2, \dots, s$, be eigenvalues of $F(\lambda)$. The extension vectors set of V_{λ_i} are as follows:

$$\begin{aligned} EV_{\lambda_i} = \{ & (x_0, x_1, \dots, x_k) \mid \\ & (\lambda_i^2 - B\lambda_i - A)x_0 = 0, \\ & (\lambda_i^2 - B\lambda_i - A)x_1 \\ & = -(2\lambda_i E_m - B)x_0, \quad k \leq k_i - 1, \\ & (\lambda_i^2 - B\lambda_i - A)x_k \\ & = -(2\lambda_i E_m - B)x_{k-1} - x_{k-2}E_m \}, \end{aligned} \quad (14)$$

where k_i is the multiplicity of λ_i , $(x_0, x_1, \dots, x_k)$ is called the component of EV_{λ_i} .

Theorem 10. The characteristic equation (9) has solutions if and only if there exist some components $(x_0, x_1, \dots, x_k), \dots, (y_0, y_1, \dots, y_q)$ of the set $\sum_{i=1}^s EV_{\lambda_i}$ in (14), and they are m linearly independent extension vectors.

In the following, we discuss the general solutions of the matrix equation (8) by using the solutions of the characteristic equation (9).

Theorem 11. If Λ_1 and Λ_2 are two diverse solutions of the characteristic equation (9), which satisfy $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$, and $\Lambda_1 - \Lambda_2$ is invertible, then

$$\begin{aligned} X(t_n) = & \left(\sum_{k=1}^n \Lambda_1^{k-1} \Lambda_2^{n-k} \right) X(t_1) \\ & - \left(\sum_{k=1}^{n-1} \Lambda_1^{n-k} \Lambda_2^k \right) X(t_0) \end{aligned} \quad (15)$$

is the general solution of the matrix equation (8).

Proof. Since Λ_1 and Λ_2 are two solutions of (9), then

$$\begin{aligned} \Lambda_1^2 - B\Lambda_1 - A &= 0, \\ \Lambda_2^2 - B\Lambda_2 - A &= 0, \end{aligned} \quad (16)$$

which deduce

$$(\Lambda_1 + \Lambda_2)(\Lambda_1 - \Lambda_2) = B(\Lambda_1 - \Lambda_2). \quad (17)$$

Hence, by the invertibility of $\Lambda_1 - \Lambda_2$, we have

$$\begin{aligned} B &= \Lambda_1 + \Lambda_2, \quad -A = \Lambda_1 \Lambda_2, \\ (\Lambda_1 - \Lambda_2)^2 &= B^2 + 4A. \end{aligned} \quad (18)$$

Let $C = \Lambda_1 - \Lambda_2$. It follows from $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ that

$$AB = BA, \quad AC = CA, \quad BC = CB. \quad (19)$$

Replacing B and A with $(1/2)(B + C) + (1/2)(B - C)$ and $-(1/4)(B + C)(B - C)$ in matrix equation (8), we obtain

$$\begin{aligned} 2x^{\sigma^2}(t) - (B - C)X^{\sigma}(t) &= \frac{1}{2}(B + C) \\ &\times [2X^{\sigma}(t) - (B - C)X(t)]. \end{aligned} \quad (20)$$

Let

$$Y(t) = 2X^{\sigma}(t) - (B - C)X(t). \quad (21)$$

Then (20) can be rewritten as

$$Y^{\sigma}(t) = \frac{1}{2}(B + C)Y(t). \quad (22)$$

Hence, the general solution of (21) is as follows:

$$Y(t_n) = \frac{1}{2^n}(B + C)^n Y(t_0), \quad (23)$$

where $Y(t_0) = 2X(t_1) - (B - C)X(t_0)$. Putting $Y(t_n)$ into (21), we can get the general solution of (8):

$$\begin{aligned} x(t_n) &= \frac{1}{2^{n-1}}(B - C)^{n-1}X(t_1) \\ &+ \sum_{k=1}^{n-1} \frac{1}{2^k}(B - C)^{k-1}Y(t_{n-k}) \\ &= -\frac{1}{2^n} \sum_{k=1}^{n-1} (B + C)^{n-k}(B - C)^k X(t_0) \\ &+ \frac{1}{2^{n-1}} \sum_{k=1}^n (B + C)^{k-1}(B - C)^{n-k} X(t_1) \\ &= \left(\sum_{k=1}^n \Lambda_1^{k-1} \Lambda_2^{n-k} \right) X(t_1) \\ &- \left(\sum_{k=1}^{n-1} \Lambda_1^{n-k} \Lambda_2^k \right) X(t_0). \end{aligned} \quad (24)$$

□

Theorem 12. If Λ_1 and Λ_2 are two diverse solutions of the characteristic equation (9) and $\Lambda_1 - \Lambda_2$ is invertible, then

$$X(t_n) = \Lambda_1^n C_1 + \Lambda_2^n C_2, \quad (25)$$

where $C_1 = (\Lambda_2 - \Lambda_1)^{-1}(\Lambda_2 X(t_0) - X(t_1))$ and $C_2 = (\Lambda_1 - \Lambda_2)^{-1}(\Lambda_1 X(t_0) - X(t_1))$, is the general solution of the matrix equation (8).

Proof. Firstly, we prove (25) is a solution of (8). Putting (25) into (8), we have

$$\begin{aligned} & (\Lambda_1^{n+2} C_1 + \Lambda_2^{n+2} C_2) - B(\Lambda_1^{n+1} C_1 + \Lambda_2^{n+1} C_2) \\ & - A(\Lambda_1^n C_1 + \Lambda_2^n C_2) \\ & = (\Lambda_1^2 - B\Lambda_1 - A)\Lambda_1^n C_1 + (\Lambda_2^2 - B\Lambda_2 - A)\Lambda_2^n C_2 \\ & \times \Lambda_2^n C_2 = 0. \end{aligned} \quad (26)$$

By using $X(t_0) = C_1 + C_2$, $X(t_1) = \Lambda_1 C_1 + \Lambda_2 C_2$, and $\Lambda_1 - \Lambda_2$ is invertible, then we get

$$\begin{aligned} C_1 &= (\Lambda_2 - \Lambda_1)^{-1} (\Lambda_2 X(t_0) - X(t_1)), \\ C_2 &= (\Lambda_1 - \Lambda_2)^{-1} (\Lambda_1 X(t_0) - X(t_1)). \end{aligned} \quad (27)$$

Next, we will prove all the solutions of (8) can be written into the form of (25).

Obviously, by $C_1 = (\Lambda_2 - \Lambda_1)^{-1} (\Lambda_2 X(t_0) - X(t_1))$ and $C_2 = (\Lambda_1 - \Lambda_2)^{-1} (\Lambda_1 X(t_0) - X(t_1))$, we have $X(t_0) = C_1 + C_2$, $X(t_1) = \Lambda_1 C_1 + \Lambda_2 C_2$; then $X(t_0)$ and $X(t_1)$ are in the form of (25). Assume the solution of (8) can be written in the form of (25) for $n \leq k$; that is,

$$X(t_k) = \Lambda_1^k C_1 + \Lambda_2^k C_2 \quad \text{for } n \leq k. \quad (28)$$

Then, it follows from (8) and Λ_1, Λ_2 being solutions of (9) that

$$\begin{aligned} x(t_{k+3}) &= X^{\sigma^2}(t_{k+1}) = BX^{\sigma}(t_{k+1}) + AX(t_{k+1}) \\ &= BX(t_{k+2}) + AX(t_{k+1}) \\ &= B(\Lambda_1^{k+2} C_1 + \Lambda_2^{k+2} C_2) + A(\Lambda_1^{k+1} C_1 + \Lambda_2^{k+1} C_2) \\ &= (B\Lambda_1 + A)\Lambda_1^{k+1} C_1 + (B\Lambda_2 + A)\Lambda_2^{k+1} C_2 \\ &= \Lambda_1^2 \Lambda_1^{k+1} C_1 + \Lambda_2^2 \Lambda_2^{k+1} C_2 \\ &= \Lambda_1^{k+3} C_1 + \Lambda_2^{k+3} C_2. \end{aligned} \quad (29)$$

By using the method of mathematical induction, we have all the solutions of (8) can be written in the form of (25). $\square$

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