Research Article

Finite-Time Stability and Stabilization of Networked Control Systems with Bounded Markovian Packet Dropout

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The finite-time stability and stabilization problems of a class of networked control systems (NCSs) with bounded Markovian packet dropout are investigated. The main results provided in the paper are sufficient conditions for finite-time stability and stabilization via state feedback. An iterative approach is proposed to model NCSs with bounded packet dropout as jump linear systems (JLSs). Based on Lyapunov stability theory and JLSs theory, the sufficient conditions for finite-time stability and stabilization of the underlying systems are derived via linear matrix inequalities (LMIs) formulation. Lastly, an illustrative example is given to demonstrate the effectiveness of the proposed results.

1. Introduction

The concept of Lyapunov asymptotic stability is largely known to the control community; see [1] and the references therein. However, often Lyapunov asymptotic stability is not enough for practical applications, because there are some cases where large values of state variables are not acceptable. For this purpose, the concept of finite-time stability (FTS) can be used. Some early results on FTS can be found in [2]; more recently the concept of FTS has been revisited in the light of recent results coming from linear matrix inequalities (LMIs) theory, which has made it possible to find less conservative conditions for guaranteeing FTS and finite time stabilization of discrete-time and continuous-time systems [3, 4]. In [5], finite-time stabilization of linear time-varying systems has been studied. In [6–8], finite-time control problem for the impulsive systems is discussed. In [9], sufficient conditions for finite-time stability and stabilization of a class of nonlinear quadratic systems are also presented. For more analysis and synthesis results of finite control problem, the readers are referred to the literature [10, 11] and the references therein. In [12], sufficient conditions for finite-time boundness and stability of switched system are proposed, and static state and dynamic output feedback controllers are designed to finite-time stabilise switched linear systems. In [13], a new design approach is proposed for robust finite-time $H_{\infty}$ control of a class of Markov jump systems with partially known information. In [14], finite-time reliable guaranteed cost fuzzy control for discrete-time nonlinear systems with actuator faults is investigated. In [15], the finite-time stochastic synchronization problem for complex networks with stochastic noise perturbations is studied. In [16], finite-time synchronization of the singular hybrid coupled networks is investigated.

On the other hand, NCSs are feedback control systems with control loops closed via digital communication channels. Compared with the traditional point-to-point wiring, the use of the communication channels can reduce the costs of cables and power, simplify the installation and maintenance of the whole system, and increase the reliability. Because of these attractive benefits, many industrial companies and institutes have shown interest in applying networks for remote industrial control purposes and factory automation [17]. However, the insertion of communication networks in feedback control loops makes the NCSs analysis and synthesis complex; see [18–20] and the references therein, where much attention has been paid to the delayed data packets of an NCSs due to network transmissions. In fact, data packets through networks suffer not only transmission delays, but also, possibly, packet dropout [21, 22]; the latter is a potential source of instability and poor performance in
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2. Problem Formulation and Preliminaries

The framework of NCSs considered in this paper is depicted in Figure 1, and the plant to be controlled is described by the following linear discrete-time systems:

\[ x(k+1) = Ax(k) + Bu(k), \]

where \( x(k) \in \mathbb{R}^n \) is the state and \( u(k) \in \mathbb{R}^m \) is the control input. \( A \) and \( B \) are known real constant matrices with appropriate dimensions. We assume that \( (A, B) \) is controllable.

Let \( \mathcal{J} = \{i_1, i_2, \ldots \} \), which is a subsequence of \( \mathbb{N} = \{1, 2, \ldots \} \), denote the sequence of time points of successful date transmission from sensor to actuator. The state feedback controller law is

\[ u(k) = Kx(k), \]

where \( K \in \mathbb{R}^{m \times n} \) is to be designed. The control input is held at the previous value by the zero-order hold during the two successively successful transmitted instant; that is

\[ u(l) = u(i_k) = Kx(i_k), \quad i_k \leq l \leq i_{k+1} - 1. \]  \hspace{1cm} (3)

Thus the closed-loop system is

\[ x(l + 1) = Ax(l) + BKx(i_k), \quad i_k \leq l \leq i_{k+1} - 1. \]  \hspace{1cm} (4)

Applying iteratively (4), we can obtain

\[ x(i_{k+1}) = \left( A^{i_{k+1} - i_k} + \sum_{r=0}^{i_{k+1} - i_k - 1} A^r BK \right) x(i_k), \quad i_k \in \mathcal{J}. \]  \hspace{1cm} (5)

Define the packet dropout process as follows:

\[ r(i_k) = i_{k+1} - i_k, \]  \hspace{1cm} (6)

which takes values in the finite state space \( \mathcal{S} = \{1, 2, \ldots, s\} \), where \( s \) is defined as

\[ s = \max_{i_k \in \mathcal{J}} (i_{k+1} - i_k). \]  \hspace{1cm} (7)

Then the closed-loop system (5) can be rewritten as jump linear systems:

\[ x(i_{k+1}) = \left( A^{r(i_k)} + \sum_{r=0}^{r(i_k) - 1} A^r BK \right) x(i_k), \quad i_k \in \mathcal{J}. \]  \hspace{1cm} (8)

The packet dropout process \( \{r(i_k)\} \) is described by a discrete-time homogeneous Markov chain with mode transition probabilities:

\[ \pi_{ij} = \mathbb{P}(r(i_k + 1) = j \mid r(i_k) = i) \geq 0, \quad \forall i, j \in \mathcal{S}. \]  \hspace{1cm} (9)

The corresponding transition probabilities is defined as

\[ \Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1s} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{s1} & \pi_{s2} & \cdots & \pi_{ss} \end{bmatrix}. \]  \hspace{1cm} (10)

Remark 1. It is worth pointing out that the packet dropout process \( \{r(i_k), \ i_k \geq 0\} \) includes both sensor-to-controller and controller-to-actuator packet dropouts.

The main aim of this paper is to find some sufficient conditions which guarantee that the system given (8) is stable over a finite-time interval. This concept can be formalized through the following definition.

Definition 2. System (8) is said to be finite-time stable with respect to \( (\alpha, \beta, R, N) \), where \( R \) is a positive-definite matrix, \( 0 \leq \alpha < \beta \), if

\[ x^T(i_0) Rx(i_0) \leq \alpha^2 \implies E\left\{ x^T(i_k) Rx(i_k) \right\} \leq \beta^2, \]

\[ k \in \{1, \ldots, N\}. \]  \hspace{1cm} (11)
To this end, the following lemmas will be essential for the proofs in the next section and theirs proofs can be found in the cited reference.

**Lemma 3** (see [26]). For any matrices $U \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{n \times q}$, if the matrix $V > 0$, then one has

$$U^T + U - V \leq U^T V^{-1} U.$$  (12)

**Lemma 4** (Schur complement lemma; see [26]). For a given symmetric matrix

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix},$$  (13)

where $W_{11} \in \mathbb{R}^{p \times p}$, $W_{22} \in \mathbb{R}^{q \times q}$, and $W_{12} \in \mathbb{R}^{p \times q}$, then the following three conditions are mutually equivalent:

1. $W < 0$,
2. $W_{11} < 0, W_{22} - W_{12}^T W_{11}^{-1} W_{12} < 0$,
3. $W_{22} < 0, W_{11} - W_{12} W_{22}^{-1} W_{12}^T < 0$.

### 3. Main Results

In this section, we will find a state feedback control matrix $K$, such that system (8) is finite-time stable with respect to $(\alpha, \beta, R, N)$. In order to solve the problem, the following theorem will be essential.

**Theorem 5.** For a constant $\gamma > 0$, system (8) is finite-time stable with respect to $(\alpha, \beta, R, N)$ if there exist matrix $P_i > 0$, $i \in \delta$, and two scalars $\lambda_1, \lambda_2$, such that the following conditions hold:

$$\lambda_1 I \leq P_i \leq \lambda_2 I,$$  (14)

$$(\gamma + 1)^N \alpha^2 \lambda_2 - \beta^2 \lambda_1 < 0,$$  (15)

$$\sum_{j=1}^{i} \pi_{ij} \left( A^j + B_j K \right)^T \tilde{P}_j \left( A^j + B_j K \right) - (\gamma + 1) \tilde{P}_i < 0,$$  (16)

where

$$\tilde{P}_i = R^{1/2} P_i R^{1/2}, \quad B_j = \sum_{r=0}^{i-1} A^r B.$$  (17)

**Proof.** Choose the Lyapunov function as

$$V(x(i_k), i) = x^T(i_k) \tilde{P}_i x(i_k).$$  (18)

Firstly, we will prove that conditions (14) combined with the following condition

$$\zeta V (x(i_k), r(i_k)) < \gamma V (x(i_k), r(i_k))$$  (19)

imply that (8) is finite-time stable with respect to $(\alpha, \beta, R, N)$, where $\zeta$ is difference operator and defined as follows:

$$\zeta V (x(i_k), r(i_k)) = E \{ V(x(i_{k+1}), r(i_{k+1})) - V(x(i_k), r(i_k)) \}.$$

Applying iteratively (18), we obtain

$$E \{ V(x(i_k), r(i_k)) \} < (\gamma + 1) E \{ V(x(i_{k-1}), r(i_{k-1})) \} < (\gamma + 1)^2 E \{ V(x(i_{k-2}), r(i_{k-2})) \} \ldots$$  (20)

$$< (\gamma + 1)^k E \{ V(x(i_0), r(i_0)) \}.$$  (21)

Note that

$$E \{ V(x(i_k), r(i_k)) \} = E \{ x^T(i_k) P_{r(i_k)} x(i_k) \} = E \{ x^T(i_k) R^{1/2} P_{r(i_k)} R^{1/2} x(i_k) \} \geq \lambda_{\min}(P_{r(i_k)}) E \{ x^T(i_k) R x(i_k) \},$$

$$(\gamma + 1)^k E \{ V(x(i_0), r(i_0)) \} = (\gamma + 1)^k E \{ x^T(i_0) P_{r(i_0)} x(i_0) \}.$$  (22)

Thus, we have

$$\lambda_{\min}(P_{r(i_k)}) E \{ x^T(i_k) R x(i_k) \} \leq (\gamma + 1)^k \lambda_{\max}(P_{r(i_k)}) x^T(i_k) R x(i_k) \leq \lambda_{\max}(P_{r(i_k)}) \lambda_{\min}(P_{r(i_k)}) \alpha^2.$$  (23)

From conditions (14), we can obtain

$$E \{ x^T(i_k) R x(i_k) \} \leq (\gamma + 1)^k \lambda_{\max}(P_{r(i_k)}) \leq \beta^2.$$  (24)

Secondly, we will prove that condition (15) is equivalent to condition (18). According to the definition of difference operator $\zeta$ and the transition probabilities of the packet dropout process $r(i_k)$, we can obtain

$$\zeta V (x(i_k), r(i_k)) = E \{ V(x(i_{k+1}), i) | x(i_k), i \} - V(x(i_k), i)$$

$$= x^T(i_k) \left( \sum_{j=1}^{i} \pi_{ij} \left( A^j + B_j K \right)^T \tilde{P}_j \left( A^j + B_j K \right) \right) x(i_k)$$

$$\leq \gamma x^T(i_k) \tilde{P}_i x(i_k),$$  (25)

which yields that

$$x^T(i_k) \left( \sum_{j=1}^{i} \pi_{ij} \left( A^j + B_j K \right)^T \tilde{P}_j \left( A^j + B_j K \right) - (\gamma + 1) \tilde{P}_i \right) x(i_k) < 0.$$  (26)
Now we turn back to our original problem, which is to find sufficient conditions which guarantee that the system (8) with the controller (2) is finite-time stable with respect to \((\alpha, \beta, R, N)\). The solution of this problem is given by the following theorem.

**Theorem 6.** For a constant \(\gamma > 0\), system (8) is finite-time stable with respect to \((\alpha, \beta, R, N)\) if there exist matrix \(\tilde{X}_i, i \in \mathcal{S}\), \(G \in \mathbb{R}^{m \times n}, Y \in \mathbb{R}^{m \times n}\), and two scalars \(\lambda_1, \lambda_2\), such that the following conditions hold:

\[
\lambda_2^{-1} R^{-1} \leq \tilde{X}_i \leq \lambda_1^{-1} R^{-1},
\]

\[
\lambda_1^{-1} (y + 1)^N \alpha^2 - \lambda_2^{-1} \beta^2 < 0,
\]

\[
\begin{bmatrix}
- G - G^T + (y + 1)^{-1} \tilde{X}_i & M_i \\
M_i^T & - \Lambda
\end{bmatrix} < 0,
\]

where

\[
M_i = \begin{bmatrix}
\sqrt{\pi_1} (A G + B_1 Y)^T \\
\sqrt{\pi_2} (A^2 G + B_2 Y)^T \\
\vdots \\
\sqrt{\pi_s} (A^s G + B_s Y)^T
\end{bmatrix},
\]

\[
\Lambda = \begin{bmatrix}
\tilde{X}_1 & 0 & 0 & 0 \\
0 & \tilde{X}_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \tilde{X}_s
\end{bmatrix}.
\]

Then the desired stabilizable control matrix is given by

\[
K = YG^{-1}.
\]

**Proof.** By Lemma 4, (27) is equivalent to

\[
\sum_{j=1}^{s} \pi_j (A^j G + B_j Y)^T \tilde{X}_i^{-1} (A^j G + B_j Y) - G - G^T + (y + 1)^{-1} \tilde{X}_i < 0.
\]

From Lemma 3, we can obtain

\[
- G - G^T + (y + 1)^{-1} \tilde{X}_i \geq -(y + 1) G^T \tilde{X}_i^{-1} G.
\]

Denote \(\tilde{P}_i = \tilde{X}_i^{-1}\). From (30) and (31), we have

\[
\sum_{j=1}^{s} \pi_j (A^j G + B_j Y)^T \tilde{P}_i (A^j G + B_j Y) - (y + 1) G^T \tilde{P}_i G < 0.
\]

Let \(Y = K G\). Then, we can obtain

\[
\sum_{j=1}^{s} \pi_j (A^j G + B_j K G)^T \tilde{P}_i (A^j G + B_j K G) - (y + 1) G^T \tilde{P}_i G < 0,
\]

which yields that

\[
G^T \left( \sum_{j=1}^{s} \pi_j (A^j + B_j K)^T \tilde{P}_i (A^j + B_j K) - (y + 1) \tilde{P}_i \right) G < 0,
\]

which is equivalent to

\[
\sum_{j=1}^{s} \pi_j (A^j + B_j K)^T \tilde{P}_i (A^j + B_j K) - (y + 1) \tilde{P}_i < 0.
\]

In view of Theorem 5, system (8) is finite-time stable with respect to \((\alpha, \beta, R, N)\). Moreover, the desired controller gain is given by (29).

**Remark 7.** For the case of \(i_k < l \leq i_{k+1} - 1\), the system transient performance can also be accommodated. In fact, denote \(h(i_k) = l - i_k \in \mathcal{S}\). Then we have

\[
x(l) = A^{h(i_k)} + \sum_{r=0}^{h(i_k)-1} A^r B K) x(i_k), \quad i_k \in \mathcal{J}.
\]

By Theorem 6, the system transient performance can be accommodated.

**4. Numerical Example**

In this section, a numerical example is given to illustrate the effectiveness of the proposed methods. Let us consider the continuous time system [27]:

\[
\dot{x}(t) = A x(t) + B u(t),
\]

where

\[
\begin{bmatrix}
-1 & 0 & -0.5 \\
1 & -0.5 & 0 \\
0 & 0 & 0.5
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0.3445 \\
0.7788
\end{bmatrix}, \quad
\begin{bmatrix}
0.5681
\end{bmatrix}
\]

When the plant is sampled with a sampling period \(T = 0.5s\), the corresponding discretized system is

\[
x(k + 1) = \begin{bmatrix}
0.6065 & 0 & -0.2258 \\
0.3445 & 0.7788 & -0.0536 \\
0 & 0 & 1.2840
\end{bmatrix} x(k) + \begin{bmatrix}
-0.0582 \\
-0.0093 \\
0.5681
\end{bmatrix} u(k).
\]

Both the continuous time system and discretized system are unstable because the eigenvalues of \(A\) are \(-0.5, -1, 0.5\) and the eigenvalues of \(A\) are \(0.7788, 0.6065, 1.2840\). Furthermore, we assume that the packet dropout upper bound is \(s = 4\) and the transition probabilities matrix is as follows:

\[
\begin{bmatrix}
0.3 & 0.2 & 0.1 & 0.4 \\
0.3 & 0.2 & 0.3 & 0.2 \\
0.5 & 0.1 & 0.1 & 0.3 \\
0.2 & 0.3 & 0.4 & 0.1
\end{bmatrix}.
\]
For given $\gamma = 1$, $\alpha = 1$, $\beta = 10$, $R = I$, $N = 5$, according to Theorem 6, the control matrix is given by

$$K = YG^{-1} = \begin{bmatrix} 0.0342 & 0.0155 & -0.9182 \end{bmatrix}. \quad (41)$$

To simulate, we take the initial state as $x_0 = \begin{bmatrix} -5 & 0 & 5 \end{bmatrix}^T$. Figure 2 depicts the trajectory of the system state. The numerical example and simulation demonstrate the effectiveness of the proposed results.

5. Conclusions

In this paper, we have considered the finite-time stabilization problems of a class of networked control systems (NCSs) with bounded Markovian packet dropout, based on the iterative approach the NCSs with bounded packet dropout as jump linear systems. The sufficient conditions for finite-time stabilization of the underlying systems are derived via linear matrix inequalities (LMIs) formulation. Lastly, an illustrative example is given to demonstrate the effectiveness of the proposed results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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