Research Article

Cluster Anticonsensus of Multiagent Systems Based on the Q-Theory

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We investigate the problem of cluster anticonsensus of multiagent systems. For multiagent continuous systems, a new control protocol is designed based on the Q-theory. Then by LaSalle's invariance principle we prove that if the graph is connected and bipartite, then the cluster anticonsensus is achieved by the proposed control protocol. On the other hand, a similar control protocol is designed for multiagent discrete-time systems. Then, sufficient conditions are given to guarantee the cluster anticonsensus of multiagent discrete-time systems by using the Q-theory and LaSalle's invariance principle. Numerical simulations show the effectiveness of our theoretical results.

1. Introduction

Recently, multiagent systems have attracted much attention in various disciplines, such as mathematical, physical, biological, and social sciences [1–4]. The multiagent systems can generate a desired collective behavior by local interaction among the agents, such as group consensus, group coordination, and oscillator synchronization [5–7]. Very recently, many researchers have investigated several consensus problems of multiagent systems, such as consensus over directed networks with fixed and switching topologies, consensus over networks with input delays, and stochastic consensus seeking with measurement noise [8–13]. In [14, 15], a kind of impulsive control protocol has been introduced for multiagent linear and nonlinear systems, respectively.

On the other hand, synchronization of chaotic systems and complex networks has sparked the interest of many researchers. Many different types of synchronization phenomena have been observed such as complete synchronization, generalized synchronization, lag synchronization, antisynchronization, and cluster synchronization [16–24]. Among them, antisynchronization is a noticeable phenomenon which has been observed in periodic oscillators and some biological systems. When the systems achieve antisynchronization, their states have the same absolute

values but opposite signs. Now antisynchronization has been investigated and many results have been published [19–22]. Besides, cluster synchronization often exists in biological science and communication engineering. When the systems achieve cluster synchronization, the nodes in the same group reach consensus with each other, but there is no synchronization between nodes in different groups. More recently cluster synchronization has been investigated intensively [23, 24]. To the best of our knowledge, however, there are very few results on cluster anticonsensus of multiagent systems, which motivates this study. When the systems achieve the cluster anticonsensus, the nodes can be partitioned into two disjoint groups. Besides, the nodes in the same group reach consensus with each other and the states of the nodes in different groups have the same absolute values but opposite signs. In [25], the authors studied the phenomenon of the synchronization in the array of the pendulum clocks hanging from an elastically fixed horizontal beam and observed the cluster antisynchronization phenomenon in the experiment of n Huygens' clocks, that is, antiphase synchronization in $n/2$ pairs of pendula for even n . Since Huygens' clocks are classical oscillators, we believe that cluster anticonsensus is useful for the array of oscillators and can be applied to secure communications, biological engineering, and so on in the future.

Very recently the signless Laplacian has attracted the attention of researchers. Several papers on the signless Laplacian spectrum have been reported since 2005 and a new spectral theory of graphs which is called the Q-theory is developing by many researchers [26–31]. Many real-world networks can be represented by bipartite graphs which have been extensively used in modern coding theory, concurrent systems, projective geometry, distributed systems, and so on. More recently, the cluster and community structure of bipartite networks has been studied by many researchers [32]. Therefore, the cluster anticonsensus problem of multiagent systems over the networks whose graphs are connected and bipartite is worth studying. In [33], the authors investigated the problem of impulsive cluster anticonsensus of discrete multiagent linear dynamic systems and sufficient conditions are given to guarantee the cluster anticonsensus of the discrete multiagent linear dynamic system based on the Q-theory.

In this paper, we investigate the problem of cluster anticonsensus of multiagent systems based on the Q-theory. For multiagent continuous systems, a new control protocol is designed and by LaSalle's invariance principle we prove that if the graph is connected and bipartite, then the cluster anticonsensus is achieved by the proposed control protocol. On the other hand, the cluster anticonsensus of multiagent discrete-time systems is considered and sufficient conditions are given to guarantee the cluster anticonsensus.

This paper is organized as follows. In Section 2, we provide some results in the Q-theory. In Section 3, we investigate the cluster anticonsensus problem for multiagent continuous systems. Section 4 considers the cluster anticonsensus of multiagent discrete-time systems. In Section 5, numerical simulations are included to show the effectiveness of our theoretical results. Some conclusions are drawn in Section 6.

Notation 1. Throughout this paper, the superscripts “−1” and “T” stand for the inverse and transpose of a matrix, respectively; $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$; $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. For real symmetric matrices X and Y , the notation $X \geq Y$ ($X > Y$, resp.) means that the matrix $X - Y$ is positive semidefinite (positive definite, resp.); $I_n \in \mathbb{R}^{n \times n}$ is an identity matrix.

2. Preliminaries

In this section, we provide some results in the Q-theory [26–31, 34].

An undirected graph $\mathcal{G}$ of order N consists of a vertex set $\mathcal{V} = \{1, 2, \dots, N\}$ and an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$. The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}, j \neq i\}$. A path between each distinct vertices i and j means a sequence of distinct edges of $\mathcal{G}$ of the form $(i, k_1), (k_1, k_2), \dots$, and (k_l, j) . A cycle is a path such that the start vertex and end vertex are the same. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is connected otherwise disconnected. A graph $\mathcal{G}$ is a bipartite graph if $\mathcal{V}(\mathcal{G})$ can be partitioned into two disjoint subsets U and W , called partite sets, such that every edge of $\mathcal{G}$ joins

a vertex of U and a vertex of W . A graph is bipartite if and only if it does not contain an odd cycle.

A weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} = 0$ and $a_{ij} = a_{ji} \geq 0, i \neq j$. $a_{ij} > 0$ if and only if there is an edge between vertex i and vertex j . For an unweighted graph $\mathcal{G}$, A is a 0-1 matrix. The outdegree of vertex i is defined as $\deg_{\text{out}}(i) = \sum_{j=1}^N a_{ij}$. Let D be the diagonal matrix with the outdegree of each vertex along the diagonal and call it the degree matrix of $\mathcal{G}$. The signless Laplacian matrix of the weighted graph is defined as $Q_{\mathcal{G}} = D + A$. For an unweighted graph $\mathcal{G}$,

$$Q_{\mathcal{G}} = [q_{ij}]_{N \times N}, \quad (1)$$

where

$$q_{ij} = \begin{cases} |\mathcal{N}_i|, & i = j, \\ 1, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

and here $|\mathcal{N}_i|$ denotes the cardinality of the set $\mathcal{N}_i$.

Let $\mathcal{G}$ be an undirected graph on N vertices, having m edges. Let R be its vertex edge incidence matrix which is a $N \times m$ matrix such that $r_{ij} = 1$ if the vertex i and edge e_j are incident and 0 if otherwise. The following relations are well known:

$$RR^T = D + A = Q. \quad (3)$$

From (3), the signless Laplacian is a positive semidefinite matrix; that is, all its eigenvalues are nonnegative. Let $\mathcal{G}$ be a graph with Q-eigenvalues $q_1, q_2, \dots, q_N$ ($q_1 \geq q_2 \geq \dots \geq q_N$). The largest eigenvalue q_1 is called the Q-index of $\mathcal{G}$.

Lemma 1 (see [26]). *The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.*

Lemma 2 (see [31]). *Let $\mathcal{G}$ be a graph on N vertices with vertex degrees $d_1, d_2, \dots, d_N$ and largest Q-eigenvalue q_1 . Then*

$$2 \min_{1 \leq i \leq N} d_i \leq q_1 \leq 2 \max_{1 \leq i \leq N} d_i. \quad (4)$$

For a connected graph $\mathcal{G}$, equality holds in either of these inequalities if and only if $\mathcal{G}$ is regular.

3. Cluster Anticonsensus of Multiagent Continuous Systems

In this section, we investigate the cluster anticonsensus problem for multiagent continuous systems.

Here, we consider a continuous system consisting of N agents indexed by $i = 1, 2, \dots, N$. The dynamics of each agent is

$$\dot{x}^i(t) = u^i(t), \quad i = 1, 2, \dots, N, \quad (5)$$

where $x^i(t) \in \mathbb{R}$ and $u^i(t) \in \mathbb{R}$ are the state and the control input of agent i at time t , respectively. $x^i(0)$ is the initial value of agent i .

Different from the traditional control protocol by the Laplacian matrix [5–7], a new control protocol is designed by the signless Laplacian matrix to achieve the cluster anticonsensus of multiagent systems. The control input of agent i is designed as

$$u^i(t) = - \sum_{j \in \mathcal{N}_i} (x^i(t) + x^j(t)), \quad i = 1, 2, \dots, N, \quad (6)$$

where $\mathcal{N}_i$ is the set of neighbors of agent i .

Then, under the control protocol (6), the dynamics of agent i satisfies the following equations:

$$\dot{x}^i(t) = - \sum_{j \in \mathcal{N}_i} (x^i(t) + x^j(t)), \quad i = 1, 2, \dots, N. \quad (7)$$

Let $x(t) = (x^1(t), x^2(t), \dots, x^N(t))^T$, then the system (7) can be described as

$$\dot{x}(t) = -Qx(t), \quad (8)$$

where Q is the signless Laplacian of the graph $\mathcal{G}$ of the network.

Definition 3. For the system (5), the cluster anticonsensus is said to be achieved under the control protocol (6) if

$$\begin{aligned} \lim_{i_1, i_2 \in U, t \rightarrow +\infty} |x^{i_1}(t) - x^{i_2}(t)| &= 0, \\ \lim_{j_1, j_2 \in W, t \rightarrow +\infty} |x^{j_1}(t) - x^{j_2}(t)| &= 0, \\ \lim_{i \in U, j \in W, t \rightarrow +\infty} |x^i(t) + x^j(t)| &= 0, \end{aligned} \quad (9)$$

where U and W are two nonempty subsets of $\mathcal{V}(\mathcal{G})$ and $U \cap W = \emptyset$ and $U \cup W = \mathcal{V}(\mathcal{G})$.

Remark 4. From Definition 3, if the system (5) achieves the cluster anticonsensus, the nodes can be partitioned into two disjoint groups. Besides, the nodes in the same group completely reach consensus with each other and the states of the nodes in different groups have the same absolute values but opposite signs. Since all state errors of the nodes in the same group converge to 0, then this cluster anticonsensus is complete cluster anticonsensus in fact.

Now we develop the cluster anticonsensus results of the system (5) in the following.

Theorem 5. Consider the system (5). If the graph $\mathcal{G}$ of the network is connected and bipartite, then the cluster anticonsensus is achieved under the control protocol (6). Moreover,

$$\lim_{i \in U, t \rightarrow +\infty} x^i(t) = - \lim_{j \in W, t \rightarrow +\infty} x^j(t) = c, \quad (10)$$

where U and W are two partite sets of the graph $\mathcal{G}$; $c = (\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))/N$.

Proof. Consider the Lyapunov function candidate

$$V(x(t)) = x^T(t) Qx(t) = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i, j < i} (x^i(t) + x^j(t))^2. \quad (11)$$

Taking the derivative of $V(x(t))$ with respect to t , we obtain

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}^T(t) Qx(t) + x^T(t) Q\dot{x}(t) \\ &= -2x^T(t) Q^T Qx(t) = -2(Qx(t))^T Qx(t) \leq 0. \end{aligned} \quad (12)$$

Obviously, $\dot{V}(x(t)) = 0$ if and only if $Qx(t) = 0$. Thus the set $E = \{x : Qx(t) = 0\}$. By LaSalle's invariance principle [35], we get $x^i(t) + x^j(t) \rightarrow 0$, $t \rightarrow \infty$, $j \in \mathcal{N}_i$, and $1 \leq i, j \leq N$. Let U and W be two partite sets of the bipartite $\mathcal{G}$. Since the graph $\mathcal{G}$ of the network is connected by Lemma 1 and [33, Lemma 2.1], it follows that

$$\begin{aligned} \lim_{i_1, i_2 \in U, t \rightarrow +\infty} |x^{i_1}(t) - x^{i_2}(t)| &= 0, \\ \lim_{j_1, j_2 \in W, t \rightarrow +\infty} |x^{j_1}(t) - x^{j_2}(t)| &= 0, \\ \lim_{i \in U, j \in W, t \rightarrow +\infty} |x^i(t) + x^j(t)| &= 0. \end{aligned} \quad (13)$$

Therefore, the cluster anticonsensus is achieved under the control protocol (6).

Observe that $\sum_{i \in U} \dot{x}^i(t) - \sum_{j \in W} \dot{x}^j(t) = 0$. This implies that $\sum_{i \in U} x^i(t) - \sum_{j \in W} x^j(t)$ is invariant. Then, we have

$$\lim_{i \in U, t \rightarrow +\infty} x^i(t) = - \lim_{j \in W, t \rightarrow +\infty} x^j(t) = c, \quad (14)$$

where $c = (\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))/N$. This completes the proof. $\square$

4. Cluster Anticonsensus of Multiagent Discrete-Time Systems

In this section, we consider the cluster anticonsensus of multiagent discrete-time systems.

Here, we consider a discrete-time system consisting of N agents indexed by $i = 1, 2, \dots, N$. The dynamics of each agent is

$$x^i(k+1) = x^i(k) + \epsilon u^i(k), \quad k \in \mathbb{N}, \quad i = 1, 2, \dots, N, \quad (15)$$

where $x^i(t) \in \mathbb{R}$ and $u^i(t) \in \mathbb{R}$ are the state and the control input of agent i at step k , respectively. $\epsilon > 0$ is a positive constant to be determined later. $x^i(0)$ is the initial value of agent i .

Similar to (6), the control input of agent i is designed as

$$u^i(k) = - \sum_{j \in \mathcal{N}_i} (x^i(k) + x^j(k)), \quad i = 1, 2, \dots, N, \quad (16)$$

where $\mathcal{N}_i$ is the set of neighbors of agent i .

Then, under the control protocol (16), the dynamics of agent i satisfies the following equations:

$$\dot{x}^i(k+1) = x^i(k) - \epsilon \sum_{j \in \mathcal{N}_i} (x^i(k) + x^j(k)), \quad i = 1, 2, \dots, N. \quad (17)$$

Let $x(k) = (x^1(k), x^2(k), \dots, x^N(k))^T$, then the system (17) can be described as

$$\dot{x}(k+1) = (I - \epsilon Q) x(k), \quad (18)$$

where Q is the signless Laplacian of the graph $\mathcal{G}$ of the network.

Definition 6. For the system (15), the cluster anticonsensus is said to be achieved under the control protocol (16) if

$$\begin{aligned} \lim_{i_1, i_2 \in U, k \rightarrow +\infty} |x^{i_1}(k) - x^{i_2}(k)| &= 0, \\ \lim_{j_1, j_2 \in W, k \rightarrow +\infty} |x^{j_1}(k) - x^{j_2}(k)| &= 0, \\ \lim_{i \in U, j \in W, k \rightarrow +\infty} |x^i(k) + x^j(k)| &= 0, \end{aligned} \quad (19)$$

where U and W are two nonempty subsets of $\mathcal{V}(\mathcal{G})$ and $U \cap W = \emptyset$ and $U \cup W = \mathcal{V}(\mathcal{G})$.

Now sufficient conditions which guarantee the cluster anticonsensus of the system (15) are proposed in the following theorem.

Theorem 7. Consider the system (15). Assume that the graph $\mathcal{G}$ of the network is connected and bipartite. If ϵ is chosen such that $0 < \epsilon < 1/d_{\max}$, where $d_{\max}$ is the maximum degree of the graph $\mathcal{G}$, then the cluster anticonsensus is achieved under the control protocol (16). Moreover,

$$\lim_{i \in U, k \rightarrow +\infty} x^i(k) = - \lim_{j \in W, k \rightarrow +\infty} x^j(k) = c, \quad (20)$$

where U and W are two partite sets of the graph $\mathcal{G}$; $c = (\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))/N$.

Proof. Consider the Lyapunov function candidate

$$V(x(k)) = x^T(k) Q x(k) = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i, j < i} (x^i(k) + x^j(k))^2. \quad (21)$$

Then, we have

$$\begin{aligned} \Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \\ &= x^T(k+1) Q x(k+1) - x^T(k) Q x(k) \\ &= x^T(k) (I - \epsilon Q)^T Q (I - \epsilon Q) x(k) - x^T(k) Q x(k) \\ &= x^T(k) (-2\epsilon Q^2 + \epsilon^2 Q^3) x(k). \end{aligned} \quad (22)$$

Let $q_1 \geq q_2 \geq \dots \geq q_N$ be the eigenvalues of Q and let P be a nonsingular matrix such that $P^{-1}QP = \text{diag}\{q_1, q_2, \dots, q_N\}$. Since the graph $\mathcal{G}$ of the network is connected and bipartite, by Lemma 1, it follows that $q_N = 0$ is a simple eigenvalue of Q . Then, $P^{-1}(-2\epsilon Q^2 + \epsilon^2 Q^3)P = \text{diag}\{-2\epsilon q_1^2 + \epsilon^2 q_1^3, \dots, -2\epsilon q_{N-1}^2 + \epsilon^2 q_{N-1}^3, 0\}$. By Lemma 2, we have $0 < q_i \leq 2d_{\max}$, $i = 1, \dots, N-1$, where $d_{\max}$ is the maximum degree of the graph $\mathcal{G}$. If ϵ is chosen such that $0 < \epsilon < 1/d_{\max}$, then $-2\epsilon q_i^2 + \epsilon^2 q_i^3 < 0$, $i = 1, \dots, N-1$. This implies that $-2\epsilon Q^2 + \epsilon^2 Q^3 \leq 0$ and 0 is a simple eigenvalue of $-2\epsilon Q^2 + \epsilon^2 Q^3$. Therefore, $\Delta V(x(k)) \leq 0$. Obviously, $\Delta V(x(k)) = 0$ if and only if $x^T(k)(-2\epsilon Q^2 + \epsilon^2 Q^3)x(k) = 0$; that is, $Qx(k) = 0$. Thus, the set $E = \{x : Qx(k) = 0\}$. By LaSalle's invariance principle [35], we get $x^i(k) + x^j(k) \rightarrow 0, k \rightarrow \infty, j \in \mathcal{N}_i$, and $1 \leq i, j \leq N$. Let U and W be two partite sets of the graph $\mathcal{G}$. Since the graph $\mathcal{G}$ of the network is connected, by Lemma 1 and [33, Lemma 2.1], it follows that

$$\begin{aligned} \lim_{i_1, i_2 \in U, k \rightarrow +\infty} |x^{i_1}(k) - x^{i_2}(k)| &= 0, \\ \lim_{j_1, j_2 \in W, k \rightarrow +\infty} |x^{j_1}(k) - x^{j_2}(k)| &= 0, \\ \lim_{i \in U, j \in W, k \rightarrow +\infty} |x^i(k) + x^j(k)| &= 0. \end{aligned} \quad (23)$$

Therefore, the cluster anticonsensus is achieved under the control protocol (16).

Observe that $\sum_{i \in U} (x^i(k+1) - x^i(k)) - \sum_{j \in W} (x^j(k+1) - x^j(k)) = 0$. This implies that $\sum_{i \in U} x^i(k) - \sum_{j \in W} x^j(k)$ is invariant. Then, we have

$$\lim_{i \in U, k \rightarrow +\infty} x^i(k) = - \lim_{j \in W, k \rightarrow +\infty} x^j(k) = c, \quad (24)$$

where $c = (\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))/N$. This completes the proof. $\square$

Remark 8. Since $I - \epsilon Q$ is not a stochastic matrix, the traditional method based on the properties of stochastic matrices is not applicable. Theorem 7 is proved by the Q-theory and LaSalle's invariance principle.

5. Simulations

In this section, two numerical examples are provided to show the effectiveness of our theoretical results.

Example 1. Here we consider a system consisting of 11 agents indexed by $i = 1, 2, \dots, 11$. The dynamics of each agent is

$$\dot{x}^i(t) = u^i(t), \quad i = 1, 2, \dots, N, \quad (25)$$

where $x^i(t) \in \mathbb{R}$ and $u^i(t) \in \mathbb{R}$ are the state and the control input of agent i at time t , respectively.

FIGURE 1: Herschel graph: original style.

FIGURE 2: Herschel graph: bipartite style.

The control input of agent i is designed as (6). Figures 1 and 2 show the Herschel graph which is a bipartite undirected graph with 11 vertices and 18 edges. From Figure 1, the signless Laplacian matrix of the Herschel graph is obtained as follows:

$$Q = \begin{bmatrix} 3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 4 \end{bmatrix}. \quad (26)$$

The signless Laplacian eigenvalues of the Herschel graph are $6.7321 \geq 5 \geq 5 \geq 4.7321 \geq 3.2679 \geq 3 \geq 3 \geq 2 \geq 2 \geq 1.2679 \geq 0$. The Herschel graph is connected and bipartite,

FIGURE 3: The time histories of $x^i(t)$, $i = 1, 2, \dots, 11$.

and thus the conditions in Theorem 5 are satisfied. In the simulation, the initial values are chosen as

$$x(0) = [2.1564, 0.7774, -0.6670, 3.8424, -1.0695, -3.2102, 1.3333, 1.2400, -1.7206, 3.0297, 4.9948]^T. \quad (27)$$

Simulation results are shown in Figure 3. The simulation results show that the cluster anticonsensus of the multiagent continuous system is achieved by the control protocol. Let $U = \{1, 3, 6, 7, 9, 10\}$ and $W = \{2, 4, 5, 8, 11\}$. Then, by a simple computation, we can verify that

$$\begin{aligned} \lim_{i \in U, t \rightarrow +\infty} x^i(t) &= - \lim_{j \in W, t \rightarrow +\infty} x^j(t) \\ &= \frac{(\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))}{11} \\ &= -0.8058. \end{aligned} \quad (28)$$

Example 2. Here we consider a system consisting of 30 agents indexed by $i = 1, 2, \dots, 30$. The dynamics of each agent is

$$x^i(k+1) = x^i(k) + \epsilon u^i(k), \quad k \in \mathbb{N}, \quad i = 1, 2, \dots, N, \quad (29)$$

where $x^i(t) \in \mathbb{R}$ and $u^i(t) \in \mathbb{R}$ are the state and the control input of agent i at step k , respectively.

The control input of agent i is designed as (16). Figures 4 and 5 show the Levi graph which is a bipartite undirected graph with 30 vertices and 45 edges. The Levi graph is connected. If we choose $\epsilon = 0.25$, then the conditions in Theorem 7 are satisfied. In the simulation, the initial values are randomly chosen in the interval $[-5, 5]$. Simulation results are shown in Figure 6. The simulation results show that the cluster anticonsensus of the multiagent discrete-time system is achieved by the control protocol.

FIGURE 4: Levi graph: original style.

FIGURE 5: Levi graph: bipartite style.

FIGURE 6: The histories of $x^i(k)$, $i = 1, 2, \dots, 30$.

Let $U = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29\}$ and $W = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$. Then, by a simple computation, we can also verify that

$$\lim_{i \in U, k \rightarrow +\infty} x^i(k) = - \lim_{j \in W, k \rightarrow +\infty} x^j(k) = c, \quad (30)$$

where $c = (\sum_{i \in U} x^i(0) - \sum_{j \in W} x^j(0))/30$.

6. Conclusions

In this paper, we investigate the problem of cluster anticonsensus of multiagent systems based on the Q-theory. For multiagent continuous systems, if the graph is connected and bipartite, then the cluster anticonsensus is achieved by the proposed control protocol. For multiagent discrete-time systems, sufficient conditions are given to guarantee the cluster anticonsensus by the proposed control protocol. Numerical simulations show the effectiveness of our theoretical results. However, the proposed methods cannot be used to study cluster anticonsensus based on other general graph topologies directly. The future work is to consider the cluster anticonsensus problem of multiagent systems under other general structures.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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