Research Article

Oscillation Criteria of Second-Order Nonlinear Differential Equations with Variable Coefficients

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Some new oscillation criteria are given for second-order nonlinear differential equations with variable coefficients. Our results generalize and extend some of the well-known results in the literatures. Some examples are considered to illustrate the main results.

1. Introduction

Since many problems in physics, chemically reacting systems, celestial mechanics, and others fields are modeled by second-order nonlinear differential equations, the oscillatory and asymptotic behaviors of solutions of such differential equations have been investigated by many authors [1–9]. Investigation of nonlinear differential equation in this paper is motivated by [1], where some of the conditions required in the theorems contain the unknown solution $x(t)$. It seems that any verification of such conditions is questionable.

In this paper, we consider the oscillation of second-order nonlinear forced differential equation

$$\begin{aligned} & \left(r(t) \psi(x(t)) f(x'(t)) \right)' + q(t) g(x(t)) \\ & = H(t, x'(t), x(t)), \quad t \in [t_0, \infty), \end{aligned} \quad (1)$$

where $r, q \in C([t_0, \infty), \mathbb{R})$ and $f, \psi, g \in C(\mathbb{R}, \mathbb{R})$ and H is a continuous function on $[t_0, \infty) \times \mathbb{R}^2$.

Consider the following with respect to (1):

$$(A_1) \quad r(t) > 0, \quad t \geq 0;$$

$$(A_2) \quad \psi(x) \in C(\mathbb{R}), \quad \psi(x) > 0 \text{ for all } x;$$

$$(A_3) \quad xg(x) > 0, \quad g \in C^1(\mathbb{R}) \text{ for } x \neq 0;$$

$$(A_4) \quad H(t, y, x)/g(x) \leq p(t) \text{ for all } t \in [t_0, \infty); \quad x, y \in \mathbb{R} \text{ and } x \neq 0.$$

We say that a nonzero function $x : [t_0, t_1) \rightarrow (-\infty, \infty)$, $t_1 > t_0$ is called a nontrivial solution of (1) if $x(t)$ satisfies (1) for all $t \in [t_0, t_1)$. A solution $x(t)$ of (1) is called oscillatory if there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of points in the interval $[t_0, \infty)$, so that $\lim_{n \rightarrow \infty} \lambda_n = \infty$ and $x(\lambda_n) = 0$, $n \in \mathbb{N}$; otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

We note that if $f(x'(t)) = x'(t)$ and $H(t, x(t), x'(t)) = 0$, then (1) becomes the differential equation of the form

$$\begin{aligned} & \left(r(t) \psi(x(t)) x'(t) \right)' + q(t) g(x(t)) = 0, \\ & t \in [t_0, \infty), \quad t_0 > 0. \end{aligned} \quad (2)$$

The oscillation of solutions of (2) has been studied by Elsheikh [1], Manojlovic [10], and Ohriska and Zulova [5] under different conditions. Also, if $\psi(x(t)) = 1$, $f(x'(t)) = x'(t)$ and $q(t)g(x(t)) = Q(t, x)$, then (1) becomes the differential equation of the form

$$\begin{aligned} & \left(r(t) x'(t) \right)' + Q(t, x) = H(t, x'(t), x(t)), \\ & t \in [t_0, \infty), \quad t_0 > 0. \end{aligned} \quad (3)$$

The oscillation of solutions of (3) has been studied by Remili [7, 8], where new results with additional suitable weighted function are found. Recently, Temtek and Tiriyaki

[9] obtained several new oscillation criteria for (3) with $x'(t) = |x'(t)|^{\alpha-1} x'(t)$ by using generalized Riccati transformation and well-known techniques, such as averaging technique.

The purpose of this research is to remove the above mentioned conditions which depend on the solution and extend some results presented in [1, 5, 7, 8].

Although Grace [2] considered the function H defined by $H(t, s) = \left(\int_{t_0}^t (ds/\Theta(s))\right)^\alpha$, $t \geq t_0 > 0$ for some constant $\alpha > 1$ and there is a positive continuous function Θ on $[t_0, \infty)$ such that $\int_{t_0}^\infty (ds/\Theta(s)) = \infty$, Kirane and Rogovchenko [11] and Tiryaki and Ayanlar [12] also mentioned that they can obtain new oscillation criteria with the a particular choice of the functions H and h with $H(t, s) = \left(\int_{t_0}^t (ds/\Theta(s))\right)^{\alpha-1}$, $t \geq t_0 > 0$ and $\alpha > 2$, $\int_{t_0}^\infty (ds/\Theta(s)) = \infty$, and Li [13] defined $H(t, s) = (R(t) - R(s))^\alpha$, $\alpha > 1$, $R(t) = \int_{t_0}^\infty (ds/r(s))$, $t \geq t_0$. However, according to Ohriska and Zulova [5], research which involves oscillatory criteria based on the weight function $H(t, s)$ has increased. These results have great theoretical value but they are less effective in applications. On the other hand, the results which contain the requirements only on the functions occurring in differential equation are usually better applicable. In particular, our weight function $H(t, s)$ is not strictly Philos-type function because in the present paper, we ignored Philos's conditions (see [2, 6, 10, 11, 14–17]). This paper contains only results of the latter kind by restricting only the functions in the differential equation (1). The relevance of the theorems in this paper is illustrated by examples.

2. Main Results

In this section we give some new sufficient conditions for all solutions of (1) to be oscillatory. We introduce the following notation:

$$\begin{aligned} R(t) &= \int_{t_0}^t \frac{ds}{r(s)}, \quad t \geq t_0, \\ Z(t) &= \rho(t) [q(t) - p(t)] - \frac{1}{4A} \frac{(\rho'(t))^2}{\rho(t)} r(t). \end{aligned} \quad (4)$$

By Theorems 1 and 3, the given conditions include Kamenev's integral criteria.

Theorem 1. *One assumes that there are positive constants k and l satisfying*

$$\begin{aligned} \frac{g'(x)}{\psi(x)} &\geq k > 0, \quad \forall x \neq 0, \\ f^2(y) &\leq lyf(y), \quad \forall y \in \mathbb{R}. \end{aligned} \quad (5)$$

Furthermore let, for some integer $m \geq 3$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \sup \frac{1}{R^{m-1}(t)} \int_{t_0}^t [R(t) - R(s)]^{m-1} (q(s) - p(s)) ds \\ = \infty. \end{aligned} \quad (6)$$

Then (1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1) has an eventually positive solution $x(t) > 0$, for all $T \geq t_0 > 0$. The case $x(t)$ being eventually negative can be similarly discussed. Set

$$w(t) = \frac{r(t) \psi(x(t)) f(x'(t))}{g(x(t))}, \quad t \geq T. \quad (7)$$

Then by (A_4) , (1), and (5), we have

$$\begin{aligned} w'(t) &= \frac{[r(t) \psi(x(t)) f(x'(t))]' }{g(x(t))} \\ &\quad - \frac{r(t) \psi(x(t)) f(x'(t)) x'(t) g'(x(t))}{g^2(x(t))}, \end{aligned} \quad (8)$$

$$w'(t) + q(t) - p(t) + \frac{k}{l} \frac{w^2(t)}{r(t)} \leq 0, \quad t \geq T.$$

Hence, for all $t \geq s \geq T$, it can be written as

$$\begin{aligned} &\int_T^t (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\ &\leq - \int_T^t (R(t) - R(s))^{m-1} w'(s) ds \\ &\quad - \frac{k}{l} \int_T^t (R(t) - R(s))^{m-1} \frac{w^2(s)}{r(s)} ds \\ &= w(T) (R(t) - R(T))^{m-1} \\ &\quad - \int_T^t \left[\frac{(m-1)}{r(s)} (R(t) - R(s))^{m-2} w(s) \right. \\ &\quad \left. + \frac{k(R(t) - R(s))^{m-1}}{lr(s)} w^2(s) \right] ds. \end{aligned} \quad (9)$$

Then (9) implies that

$$\begin{aligned}
& \int_T^t (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\
& \leq w(T) (R(t) - R(T))^{m-1} - \frac{l(m-1)^2}{4k} \\
& \quad \times \int_T^t (R(t) - R(s))^{m-3} \frac{1}{r(s)} ds \\
& \quad - \int_T^t \frac{(R(t) - R(s))^{m-3}}{r(s)} \\
& \quad \times \left\{ \sqrt{\frac{k}{l}} (R(t) - R(s)) w(s) + \frac{m-1}{2\sqrt{k/l}} \right\}^2 ds.
\end{aligned} \tag{10}$$

Then we have

$$\begin{aligned}
& \frac{1}{R^{m-1}(t)} \int_T^t (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\
& \leq \left[1 - \frac{R(T)}{R(t)} \right]^{m-1} w(T) \\
& \quad + \frac{l(m-1)^2}{4k(m-2)R(t)} \left[1 - \frac{R(T)}{R(t)} \right]^{m-2}, \quad t \geq T.
\end{aligned} \tag{11}$$

It is clear that, for $t \in [t_0, T]$, it holds

$$\begin{aligned}
& \frac{1}{R^{m-1}(t)} \int_{t_0}^T (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\
& \leq \frac{1}{R^{m-1}(t)} \int_{t_0}^T R^{m-1}(t) [q(s) - p(s)] ds \\
& = \int_{t_0}^T [q(s) - p(s)] ds.
\end{aligned} \tag{12}$$

Combining (11) and (12), it follows that

$$\begin{aligned}
& \frac{1}{R^{m-1}(t)} \int_{t_0}^t (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\
& \leq \left[1 - \frac{R(T)}{R(t)} \right]^{m-1} w(T) \\
& \quad + \frac{l(m-1)^2}{4k(m-2)R(t)} \left[1 - \frac{R(T)}{R(t)} \right]^{m-2} \\
& \quad + \int_{t_0}^T [q(s) - p(s)] ds, \quad t \geq t_0.
\end{aligned} \tag{13}$$

It is observable that $\lim_{t \rightarrow \infty} (1/R(t)) = N \in [0, \infty)$. Thus

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sup \frac{1}{R^{m-1}(t)} \int_{t_0}^t (R(t) - R(s))^{m-1} [q(s) - p(s)] ds \\
& \leq [1 - NR(T)]^{m-1} w(T) \\
& \quad + \frac{Nl(m-1)^2}{4k(m-2)R(t)} [1 - NR(T)]^{m-2} \\
& \quad + \int_{t_0}^T [q(s) - p(s)] ds < \infty,
\end{aligned} \tag{14}$$

which contradicts condition (6). The proof of the theorem is completed. $\square$

Example 2. Consider the differential equation

$$\begin{aligned}
& \left[\frac{1}{t} \left(\frac{1}{x^2(t) + 1} \right) x'(t) \right]' + 3x(t) \left(2 + \frac{1}{1 + x^2(t)} \right) \\
& = \frac{x(t)}{4} \left(2 + \frac{1}{1 + x^2(t)} \right) \sin \frac{(x'(t))^2}{(x'(t))^2 + 1}.
\end{aligned} \tag{15}$$

Clearly,

$$\frac{g'(x)}{\psi(x)} = 3 + \frac{2x^4}{1 + x^2} \geq 3 = k,$$

$$f^2(y) = y^2 + \frac{2y^2}{1 + y^2} + \frac{y^2}{(1 + y^2)^2} \leq 4y^2,$$

$$\frac{1}{4} f^2(y) \leq y^2 \leq y^2 + \frac{y^2}{y^2 + 1} = yf(y) \quad \forall y \in \mathbb{R},$$

$$\frac{H(t, x'(t), x(t))}{g(x(t))} = \frac{x(t)}{4} \left(2 + \frac{1}{1 + x^2(t)} \right) \sin \frac{(x'(t))^2}{(x'(t))^2 + 1}$$

$$\begin{aligned}
& \times \frac{1}{x(t)(1 + x^2(t))^2} \\
& \leq \frac{1}{4} = p(t),
\end{aligned}$$

$$\forall x' \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t \geq t_0 > 0. \tag{16}$$

and, for any integer $m \geq 3$, we have

$$\lim_{t \rightarrow \infty} \sup \frac{1}{R^{m-1}(t)} \int_{t_0}^t [R(t) - R(s)]^{m-1} (q(s) - p(s)) ds$$

$$= 4 \lim_{t \rightarrow \infty} \sup \frac{1}{t^4} \int_{t_0}^t \left[\frac{t^2}{2} - \frac{s^2}{2} \right]^2 \left(3 - \frac{1}{4} \right) ds = \infty. \quad (17)$$

By Theorem 1, this equation is oscillatory.

Theorem 3. Assume that (5) holds and suppose that

$$\lim_{t \rightarrow \infty} \sup \frac{1}{t^m} \int_{t_0}^t (t-s)^m (q(s) - p(s)) - \frac{lm^2}{4k} (t-s)^{m-2} r(s) ds = \infty, \quad (18)$$

for some $m \geq 2$. Then (1) is oscillatory.

Proof. Assume that (1) has an eventually positive solution $x(t) > 0$, for all $T \geq t_0 > 0$. Taking $w(t)$ as it is defined in (7), we obtain (8).

Hence by (A₃), for $t \geq s \geq T$, we can write

$$\begin{aligned} & \int_T^t (t-s)^m [q(s) - p(s)] ds \\ & \leq - \int_T^t (t-s)^m w'(s) ds - \frac{k}{l} \int_T^t (t-s)^m \frac{w^2(s)}{r(s)} ds. \end{aligned} \quad (19)$$

That is,

$$\begin{aligned} & \int_T^t (t-s)^m [q(s) - p(s)] ds \\ & \leq (t-T)^m w(T) - m \int_T^t (t-s)^{m-1} w(s) ds \\ & \quad - \frac{k}{l} \int_T^t (t-s)^m \frac{w^2(s)}{r(s)} ds = (t-T)^m w(T) \\ & \quad - \int_T^t \left[\sqrt{\frac{k(t-s)^m}{lr(s)}} w(s) + \frac{m}{2} \sqrt{\frac{lr(s)(t-s)^{m-2}}{k}} \right]^2 ds \\ & \quad + \frac{lm^2}{4k} \int_T^t r(s) (t-s)^{m-2} ds. \end{aligned} \quad (20)$$

Hence it follows that

$$\begin{aligned} & \int_T^t \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds \\ & \leq (t-T)^m w(T) \leq (t-t_0)^m w(T), \quad t \geq T. \end{aligned} \quad (21)$$

Using the inequality (21), for $t \geq t_0$, we obtained

$$\begin{aligned} & \int_T^t \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds \\ & = \int_{t_0}^T \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds \\ & \quad + \int_T^t \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds, \\ & \int_{t_0}^t \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds \\ & \leq (t-t_0)^m w(T) \int_{t_0}^t (q(s) - p(s)) ds + (t-t_0)^m w(T). \end{aligned} \quad (22)$$

Thus,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sup \frac{1}{t^m} \int_{t_0}^t \left((t-s)^m [q(s) - p(s)] - \frac{lm^2}{4k} r(s) (t-s)^{m-2} \right) ds \\ & \leq \int_{t_0}^t (q(s) - p(s)) ds + w(T) < \infty, \end{aligned} \quad (23)$$

and it contradicts (18). $\square$

Example 4. Consider the differential equation

$$\begin{aligned} & \left[\left(\frac{1 + (x(t))^2}{2 + (x(t))^2} \right) x'(t) \left(\frac{1 + 2(x'(t))^2}{4 + 2(x'(t))^2} \right) \right]' + (1 + \cos t) x(t) \\ & = \frac{x^5(t)}{x^2(t) + 1} \cos t \frac{(x'(t))^2}{(x'(t))^2 + 1}, \quad t > 0, \end{aligned} \quad (24)$$

which satisfies the conditions of Theorem 3, since we have

$$\begin{aligned} & \frac{g'(x)}{\psi(x)} = \frac{2 + (x(t))^2}{1 + (x(t))^2} \geq 2 = k \quad \forall x \neq 0, \\ & yf(y) = y^2 \frac{1 + 2y^2}{4 + 2y^2} > 0, \quad f^2(y) \leq Lyf(y), \quad L = 1, \\ & \frac{H(t, x'(t), x(t))}{g(x(t))} = \frac{x^5(t)}{x^2(t) + 1} \cos t \frac{(x'(t))^2}{(x'(t))^2 + 1} \times \frac{1}{x^3(t)} \\ & \leq \cos t = p(t), \quad \forall x' \in \mathbb{R}, x \in \mathbb{R}, t \geq t_0; \end{aligned} \quad (25)$$

and, for any integer $m \geq 2$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \sup \frac{1}{t^m} \int_{t_0}^t \left((t-s)^m [1 + \cos(s) - \cos(s)] \right. \\ \left. - \frac{m^2}{8} (t-s)^{m-2} \right) ds \\ = \lim_{t \rightarrow \infty} \sup \frac{1}{t^m} \left[\frac{t}{(m+1)} - \frac{m^2}{8(m-1)t} \right] = \infty. \end{aligned} \quad (26)$$

It thus follows that every solution of the given equation oscillates.

Theorem 5. Suppose that

$$\begin{aligned} 0 < k_2 \leq \psi(x(t)) \leq k_3, \quad \forall x \in \mathbb{R}, \\ 0 < k_4 \leq \frac{f(y)}{y} \leq k_5, \quad \forall y = x'(t) \neq 0. \end{aligned} \quad (27)$$

Let ρ be a positive continuously differentiable function over $[T, \infty)$ such that $\rho'(t) \geq 0$ over $[T, \infty)$,

$$\lim_{t \rightarrow \infty} \int_{T_0}^t \frac{1}{\rho(s)r(s)} ds = \infty, \quad (28)$$

$$\lim_{t \rightarrow \infty} \int_{T_0}^t Z(s) ds = \infty. \quad (29)$$

Then all solutions of (1) are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution on $[T, \infty)$, $T \geq T_0$ of (1). We assume that $x(t)$ is positive on $[T, \infty)$. A similar argument holds for the case when $x(t)$ is negative. Let

$$w(t) = \frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))}. \quad (30)$$

Then

$$w'(t) = \left[\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} \right]'. \quad (31)$$

Using (1), we obtain

$$\begin{aligned} w'(t) &= \frac{H(t, x'(t), x(t))}{g(x(t))} \\ &\quad - q(t) - \frac{r(t)\psi(x(t))f(x'(t))g'(x(t))x'(t)}{g^2(x(t))}. \end{aligned} \quad (32)$$

Since $g'(x(t)) > k$ and condition (A_4) we obtained,

$$\begin{aligned} w'(t) &\leq p(t) - q(t) \\ &\quad - \frac{k}{r(t)\psi(x(t))} \frac{r^2(t)\psi^2(x(t))(f(x'(t)))^2}{g^2(x(t))} \\ &\quad \times \frac{x'(t)}{f(x'(t))}. \end{aligned} \quad (33)$$

From (27) and (30),

$$w'(t) \leq -[q(t) - p(t)] - \frac{k}{k_3 k_4} \frac{1}{r(t)} w^2(t). \quad (34)$$

Therefore,

$$\begin{aligned} \left[\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} \right]' &\leq -[q(t) - p(t)] \\ &\quad - \frac{k}{k_3 k_4} \frac{1}{r(t)} w^2(t). \end{aligned} \quad (35)$$

Multiplying by $\rho(t)$ and integrating from T to t , we obtained

$$\begin{aligned} \rho(t) \left[\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} \right]' &\leq -\rho(t)[q(t) - p(t)] \\ &\quad - A \frac{\rho(t)}{r(t)} w^2(t); \quad A = \frac{k}{k_3 k_4}, \\ \int_T^t \rho(s) \left[\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))} \right]' ds &\leq \int_T^t -\rho(s)[q(s) - p(s)] ds \\ &\quad - \int_T^t A \frac{\rho(s)}{r(s)} w^2(s) ds. \end{aligned} \quad (36)$$

Let

$$C_T = \frac{\rho(T)r(T)\psi(x(T))f(x'(T))}{g(x(T))}. \quad (37)$$

By integrating by parts, we obtained

$$\begin{aligned} \frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))} &\leq C_T - \int_T^t \rho(s)[q(s) - p(s)] ds \\ &\quad - \int_T^t A \frac{\rho(s)}{r(s)} w^2(s) ds \\ &\quad + \int_T^t \rho'(s) \frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))} ds \\ &= C_T - \int_T^t \rho(s)[q(s) - p(s)] ds \\ &\quad - \int_T^t A \frac{\rho(s)}{r(s)} w^2(s) ds + \int_T^t \rho'(s) w(s) ds \\ &= C_T - \int_T^t \rho(s)[q(s) - p(s)] ds \\ &\quad + \int_T^t \left[-A \frac{\rho(s)}{r(s)} w^2(s) + \rho'(s) w(s) \right] ds. \end{aligned} \quad (38)$$

The inequality presented by (38) can be written as

$$\begin{aligned} & \frac{\rho(t) r(t) \psi(x(t)) f(x'(t))}{g(x(t))} \\ & \leq C_T - \int_T^t \rho(s) [q(s) - p(s)] ds \\ & \quad + \int_T^t \frac{-A\rho(s)}{r(s)} \\ & \quad \times \left[\left(w(s) - \frac{\rho'(s) r(s)}{2A\rho(s)} \right)^2 - \frac{(\rho'(s))^2 r^2(s)}{4A^2 \rho^2(s)} \right] ds. \end{aligned} \quad (39)$$

Let $w(s) - ((\rho'(s)r(s))/(2A\rho(s))) = W(s)$,

$$\begin{aligned} & \frac{\rho(t) r(t) \psi(x(t)) f(x'(t))}{g(x(t))} \\ & \leq C_T - \int_T^t \rho(s) [q(s) - p(s)] ds \\ & \quad - \int_T^t \frac{A\rho(s)}{r(s)} \left[W^2(s) - \left(\frac{\rho'(s)r(s)}{2A\rho(s)} \right)^2 \right] ds \\ & = C_T - \int_T^t \rho(s) [q(s) - p(s)] ds \\ & \quad - \int_T^t \left[\frac{A\rho(s)}{r(s)} W^2(s) - \frac{\rho'^2(s) r(s)}{4A\rho(s)} \right] ds \\ & = C_T - \int_T^t \rho(s) [q(s) - p(s)] - \frac{1}{4A} \frac{(\rho'(s))^2}{\rho(s)} r(s) ds \\ & \quad - \int_T^t \frac{A\rho(s)}{r(s)} W^2(s) ds, \\ & \frac{\rho(t) r(t) \psi(x(t)) f(x'(t))}{g(x(t))} \leq C_T - \int_T^t Z(s) ds. \end{aligned} \quad (40)$$

Taking the limit for both sides of (40) and using (29),

$$\lim_{t \rightarrow \infty} \frac{\rho(t) r(t) \psi(x(t)) f(x'(t))}{g(x(t))} \rightarrow -\infty. \quad (41)$$

Hence, there exists $T_1 \geq T$ such that

$$\begin{aligned} f(x'(t)) & < 0, \quad \forall t \geq T_1, \\ x'(t) & < 0, \quad \forall t \geq T_1. \end{aligned} \quad (42)$$

Condition (29) also implies that $\int_T^\infty \rho(s)[q(s) - p(s)]ds = \infty$, and there exists $T_2 \geq T_1$ such that

$$\int_{T_1}^{T_2} \rho(s) [q(s) - p(s)] ds = 0, \quad (43)$$

$$\int_{T_2}^t \rho(s) [q(s) - p(s)] ds \geq 0, \quad \forall t \geq T_2.$$

Multiplying (1) by $\rho(t)$ and integrating by parts on $[T_2, t]$, we obtained

$$\begin{aligned} & \rho(t) [r(t) \psi(x(t)) f(x'(t))] + \rho(t) q(t) g(x(t)) \\ & = \rho(t) H(t, x'(t), x(t)), \end{aligned} \quad (44)$$

which implies

$$\begin{aligned} & \frac{\rho(t) [r(t) \psi(x(t)) f(x'(t))]}{g(x(t))} \leq -\rho(t) [q(t) - p(t)], \\ & \rho(t) [r(t) \psi(x(t)) f(x'(t))] \\ & \leq -\rho(t) g(x(t)) [q(t) - p(t)]. \end{aligned} \quad (45)$$

Now, integrating by parts, we get

$$\begin{aligned} & \rho(t) [r(t) \psi(x(t)) f(x'(t))] \\ & \quad - \int_{T_2}^t \rho'(s) r(s) \psi(x(s)) f(x'(s)) ds \\ & \leq C_{T_2} - \int_{T_2}^t \rho(s) g(x(s)) [q(s) - p(s)] ds \\ & = C_{T_2} - g(x(t)) \int_{T_2}^t \rho(s) [q(s) - p(s)] ds \\ & \quad + \int_{T_2}^t x'(s) g'(x(s)) \int_{T_2}^s \rho(u) [q(u) - p(u)] du ds \\ & \quad + \int_{T_2}^t \rho'(s) r(s) \psi(x(s)) f(x'(s)) ds \\ & \leq C_{T_2}, \quad \forall t \geq T_1, \end{aligned} \quad (46)$$

where $C_{T_2} = (\rho(T_2)r(T_2)\psi(x(T_2))f(x'(T_2)))/(g(x(T_2))) < 0$. Therefore,

$$\rho(t) r(t) \psi(x(t)) f(x'(t)) \leq C_{T_2}. \quad (47)$$

From (27), we have

$$x'(t) \leq \frac{C_{T_2}}{k_2 k_4 r(t) \rho(t)}, \quad x(t) \leq \frac{C_{T_2}}{k_2 k_4} \int_{T_2}^t \frac{1}{r(s) \rho(s)} ds. \quad (48)$$

Finally, from (28), $x(t) \rightarrow -\infty$ as $t \rightarrow \infty$, which is a contradiction. $\square$

Example 6. Let us consider the following equation:

$$\left[\frac{1}{t} \left(5 + \frac{x^6(t)}{x^6(t)+1} \right) \left(13x'(t) + \frac{x'(t)}{(x'(t))^2+1} \right) \right]' + \left(t + \frac{\sin t}{t} \right) x^5(t) = \frac{2x^{12} \sin t \cos(x'(t)+1)}{(x^7+1)t^3}, \quad t \geq \frac{\pi}{2}. \quad (49)$$

We note that

$$\frac{H(t, x'(t), x(t))}{g(x(t))} = \frac{2x^{12} \sin t \cos(x'(t)+1)}{(x^7+1)t^3} \times \frac{1}{x^5(t)} \leq \frac{2}{t^3} = p(t), \quad (50)$$

$$\forall x' \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t \geq t_0.$$

Hence,

$$r(t) = \frac{1}{t} > 0, \quad t > \frac{\pi}{2},$$

$$5 \leq 5 + \frac{x^6(t)}{x^6(t)+1} < 6, \quad \forall x \in \mathbb{R}, \quad (51)$$

$$13 < 13 + \frac{1}{(x'(t))^2+1} \leq 14, \quad \forall y \neq 0.$$

Let

$$\rho(t) = t \implies \rho'(t) = 1,$$

$$\int_t^\infty \frac{ds}{r(s)\rho(s)} = \int_t^\infty ds = \infty, \quad (52)$$

$$Z(s) = \rho(s) [q(s) - p(s)] - \frac{1}{4A} \frac{(\rho'(s))^2}{\rho(s)} r(s)$$

$$= s \left[s + \frac{\sin s}{s} - \frac{2}{s^3} \right] - \frac{1}{4A} \frac{1}{s^2}.$$

For every $t \geq T = \pi/2$, we obtained

$$\int_{t_0}^\infty Z(s) ds = \int_{t_0}^\infty s^2 + \sin s - \frac{2}{s^2} - \frac{1}{4As^2} ds = \infty. \quad (53)$$

Then, Theorem 5 ensures that every solution in this example is oscillatory.

Theorem 7. If conditions (27)–(29) hold and

$$\int_{T_0}^\infty \rho(s) [q(s) - p(s)] ds < \infty, \quad (54)$$

$$\liminf_{t \rightarrow \infty} \left[\int_T^t Z(s) ds \right] \geq 0, \quad \forall \text{ large } T, \quad (55)$$

$$\lim_{t \rightarrow \infty} \int_{T_0}^t \frac{1}{\rho(s)r(s)} \int_s^\infty Z(u) du ds = \infty, \quad (56)$$

$$\int_\varepsilon^\infty \frac{dy}{g(y)} < \infty, \quad \int_{-\varepsilon}^\infty \frac{dy}{g(y)} < \infty \quad \text{for every } \varepsilon > 0, \quad (57)$$

thus all solutions of (1) are oscillatory.

Remark 8. Condition (55) implies that $\int_T^\infty Z(s) ds \geq 0$ and $\liminf_{t \rightarrow \infty} \int_T^\infty Z(s) ds = \int_T^\infty Z(s) ds$; hence, (56) takes the form of $\int_T^\infty Z(s) ds \geq 0$, for all large T .

Proof. Let $x(t)$ be a nonoscillatory solution on $[T, \infty)$, $T \geq T_0$ of (1). Let us assume that $x(t)$ is positive on $[T, \infty)$ and consider the following three cases for the behavior of $x'(t)$.

Case 1. Consider $x'(t) > 0$ for $T_1 \geq T$ for some $t \geq T_1$; then, from (40), we have

$$\int_{T_1}^t R(s) ds \leq \frac{r(T_1)\rho(T_1)\psi(x(T_1))f(x'(T_1))}{g(x(T_1))} - \frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))},$$

$$k_4 k_2 \frac{r(t)\rho(t)x'(t)}{g(x(t))} \leq \frac{r(T_1)\rho(T_1)\psi(x(T_1))f(x'(T_1))}{g(x(T_1))} - \int_{T_1}^t R(s) ds. \quad (58)$$

From (27), we obtained

$$\int_{T_1}^t Z(s) ds \leq \frac{r(T_1)\rho(T_1)\psi(x(T_1))f(x'(T_1))}{g(x(T_1))} - k_4 k_2 \frac{r(t)\rho(t)x'(t)}{g(x(t))}. \quad (59)$$

Hence, for all $t \geq T_1$,

$$\int_t^\infty R(s) ds \leq k_4 k_2 \frac{r(t)\rho(t)x'(t)}{g(x(t))},$$

$$\int_{T_1}^t \frac{1}{r(s)\rho(s)} \int_s^\infty Z(u) du ds \leq k_4 k_2 \int_{T_1}^\infty \frac{x'(s)}{g(x(s))} ds, \quad (60)$$

$$\int_{T_1}^t \frac{1}{r(s)\rho(s)} \int_s^\infty Z(u) du ds \leq k_4 k_2 \int_{T_1(s)}^\infty \frac{dy}{g(y)}.$$

Using (57), we obtained

$$\int_{T_1}^t \frac{1}{r(s)\rho(s)} \int_s^\infty Z(u) du ds < \infty. \quad (61)$$

This contradicts condition (56).

Case 2. If $x'(t)$ is oscillatory, then there exists a sequence $\{\alpha_n\} \rightarrow \infty$ on $[T, \infty)$ such that $x'(\alpha_n) < 0$. Let us assume that N is sufficiently large, so that

$$\int_{\alpha_N}^\infty Z(s) ds \geq 0. \quad (62)$$

Then, from (27) and (40), we have

$$k_3 k_5 \frac{\rho(t) r(t) x'(t)}{g(x(t))} \leq C_{\alpha_N} - \int_{\alpha_N}^t Z(s) ds. \quad (63)$$

Thus,

$$\begin{aligned} k_3 k_5 \limsup_{t \rightarrow \infty} \frac{\rho(t) r(t) x'(t)}{g(x(t))} &\leq C_{\alpha_N} + \limsup_{t \rightarrow \infty} \left[- \int_{\alpha_N}^t Z(s) ds \right] \\ &= C_{\alpha_N} - \liminf_{t \rightarrow \infty} \left[\int_{\alpha_N}^t Z(s) ds \right]. \end{aligned} \quad (64)$$

By (55), we obtained

$$k_3 k_5 \limsup_{t \rightarrow \infty} \frac{\rho(t) r(t) x'(t)}{g(x(t))} < 0, \quad (65)$$

which contradicts the fact that $x'(t)$ oscillates.

Case 3. Let $x'(t) < 0$ for $t \geq T$ for some $T_1 \geq T$; then, for any $t_0 \geq T_0$, there exists $t_1 \geq t_0$ such that $\int_{t_1}^{\infty} \rho(s)[q(s) - p(s)] ds \geq 0$ for all $t \geq t_1$. Choosing $t_1 \geq T_1$, and multiplying (1) by $\rho(t)$ and integrating by parts, we obtained

$$\begin{aligned} &\rho(t) [r(t) \psi(x(t)) f(x'(t))] \\ &\quad - \int_{t_1}^t \rho'(s) r(s) \psi(x(s)) f(x'(s)) ds \\ &\leq C_{t_1} - \int_{t_1}^t \rho(s) g(x(s)) [q(s) - p(s)] ds \\ &= C_{t_1} - g(x(t)) \int_{t_1}^t \rho(s) [q(s) - p(s)] ds \\ &\quad + \int_{t_1}^t x'(s) g'(x(s)) \int_{t_1}^s \rho(u) [q(u) - p(u)] du ds \\ &\quad + \int_{t_1}^t \rho'(s) r(s) \psi(x(s)) f(x'(s)) ds, \end{aligned} \quad (66)$$

where $C_{t_1} = \rho(t_1) r(t_1) \psi(x(t_1)) f(x'(t_1)) < 0$.

Thus,

$$\rho(t) r(t) \psi(x(t)) f(x'(t)) \leq C_{t_1}. \quad (67)$$

From (27), we obtained

$$\begin{aligned} x'(t) &\leq \frac{C_{t_1}}{k_2 k_4} \frac{1}{r(t) \rho(t)}, \\ x(t) &\leq \frac{C_{t_1}}{k_2 k_4} \int_{T_2}^t \frac{1}{r(s) \rho(s)} ds. \end{aligned} \quad (68)$$

From (28), it follows that $x(t) \rightarrow -\infty$, as $t \rightarrow \infty$, which is a contradiction. $\square$

Example 9. Consider the following equation:

$$\begin{aligned} &\left[t \left(8 + \frac{x^4(t)}{x^4(t) + 1} \right) \left(7x'(t) + \frac{(x'(t))^5}{(x'(t))^4 + 1} \right) \right]' \\ &\quad + \frac{1}{t^3} x^3(t) = \frac{x^3 \cos x \sin 2x'(t)}{t^4}, \quad t > 1. \end{aligned} \quad (69)$$

We note that

$$\begin{aligned} &\frac{H(t, x'(t), x(t))}{g(x(t))} \\ &= \frac{x^3 \cos x \sin 2x'(t)}{t^4} \times \frac{1}{x^3(t)} \leq \frac{1}{t^4} = p(t), \end{aligned} \quad (70)$$

$$\forall x' \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t \geq t_0,$$

hence,

$$r(t) = t > 0,$$

$$8 \leq \psi(x(t)) = 8 + \frac{x^4(t)}{x^4(t) + 1} < 9, \quad \forall x \in \mathbb{R}, \quad (71)$$

$$7 \leq \frac{f(y)}{y} 2 = 7 + \frac{(x'(t))^4}{(x'(t))^4 + 1} < 8, \quad \forall y \neq 0.$$

Let $\rho(t) = t \Rightarrow \rho'(t) = 1$; then,

$$\lim_{t \rightarrow \infty} \int_{T_0}^t \frac{1}{r(s) \rho(s)} ds = \lim_{t \rightarrow \infty} \int_{T_0}^t ds = \infty, \quad (72)$$

$$\int_{T_0}^t \rho(s) [q(s) - p(s)] ds = \int_{T_0}^t \left(\frac{1}{s^2} - \frac{1}{s^3} \right) ds < \infty.$$

Because $Z(s) = \rho(s)[q(s) - p(s)] - (1/4A)((\rho'(s))^2/\rho(s))r(s) = (1/s^2) - (1/s^3) - (1/4A)$,

$$\lim_{t \rightarrow \infty} \inf \int_T^t Z(s) ds = \lim_{t \rightarrow \infty} \inf \int_T^t \left(\frac{1}{s^2} - \frac{1}{s^3} - \frac{1}{4A} \right) ds = 0,$$

$$\lim_{t \rightarrow \infty} \int_{T_0}^t \frac{1}{r(s) \rho(s)} \int_s^\infty Z(u) du ds$$

$$= \lim_{t \rightarrow \infty} \int_{T_0}^t \frac{1}{s^2} \int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^3} - \frac{1}{4A} \right) du ds = \infty,$$

$$\int_\varepsilon^\infty \frac{dy}{f(y)} = \int_\varepsilon^\infty \frac{dy}{y^3} = \frac{-2}{y^2} \Big|_\varepsilon^\infty = \frac{2}{\varepsilon^2} < \infty,$$

$$\int_{-\varepsilon}^\infty \frac{dy}{f(y)} = \frac{2}{\varepsilon^2} < \infty.$$

(73)

Thus, from Theorem 7, it follows that the equation is oscillatory.

Remark 10. If we let $Z(t) = 1$ and $f(x'(t)) = x'(t)$ in our Theorems 5 and 7, we will get Theorems 1 and 2 of Remili [7] which extends the results of Graef et al. [3].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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