Research Article

Spectral Methods in Spatial Statistics

Kun Chen,¹ Lianmin Zhang,² and Maolin Pan³

¹ School of Statistics, Southwestern University of Finance and Economics, Chengdu 611130, China

² School of Management and Engineering, Nanjing University, Nanjing 210093, China

³ Department of Mathematics, Nanjing University, Nanjing 210093, China

Correspondence should be addressed to Lianmin Zhang; zhanglm@nju.edu.cn

Received 2 April 2014; Accepted 27 May 2014; Published 15 June 2014

Academic Editor: Xiaolin Xu

Copyright © 2014 Kun Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

When the spatial location area increases becoming extremely large, it is very difficult, if not possible, to evaluate the covariance matrix determined by the set of location distance even for gridded stationary Gaussian process. To alleviate the numerical challenges, we construct a nonparametric estimator called periodogram of spatial version to represent the sample property in frequency domain, because periodogram requires less computational operation by fast Fourier transform algorithm. Under some regularity conditions on the process, we investigate the asymptotic unbiasedness property of periodogram as estimator of the spectral density function and achieve the convergence rate.

1. Introduction

Recently, spatial statistics have attracted a number of researches from statisticians, geostatisticians, engineers, and econometricians [1–4]. The covariance clustering phenomena of neighborhood observations have always been the fundamental feature of spatial statistics, which means that closer observations have larger correlation and the correlation decays to zero for far separated ones. Therefore, the location information of spatial sited data plays a key role in constructing the probabilistic or statistical covariance function [5]. Using maximum likelihood estimation for the regularly spaced Gaussian data, the calculation of the covariance matrix inverse or determinant needs $O(n^3)$ operations with the sample size n . As n becomes large, it is difficult to implement statistical inference due to computational limitations. Fortunately, spectral method is a powerful tool to analyze this complex spatial structure using fewer calculations with $O(n \log n)$ operations [6, 7].

This spectral method is widely applied in time series analysis [8], but little research has been done in spatial analysis. The spectral analysis, also known as harmonic analysis, is mainly based on the study of the spectral density function, which represents the underlying process property analog to covariance function. By spectral representation theorem,

a positive definite covariance function in spatial domain is the Fourier transform of spectral density function in frequency domain. Operations performed on one domain have the corresponding operations on the other domain. In spatial domain, we usually use moment estimator for statistical inference. Under some regular assumptions, like ergodic or stationary, the nonparametric moment method provides asymptotically unbiased estimator. Because of the relationship between spatial and frequency domain, we can use the Fourier transform of moment estimator to make inference. The most important estimator in this sense is periodogram, which is asymptotically unbiased estimator of the spectral density function in time series analysis. Because the periodogram is the square of discrete Fourier transform (DFT), which decomposes the original sequence into components on different frequencies, in this case, the fast Fourier transform (FFT) provides a method to compute the DFT taking $O(n \log n)$ operations.

However, it is unknown whether the periodogram estimators have asymptotic unbiased property for spatial process. In this paper, based on the spatial version periodogram, we prove that the result still holds in spatial analysis and obtain that the convergence rate is a function of the sampling domain.

The periodogram method to study properties of regular stationary time series can be traced to Whittle [9], Ronsenblatt [10], and Priestly [11]. In some cases, the autocorrelation of time series decays very slowly, such that a pair of observations, when even separated far away, are still highly correlated. To overcome this difficulty, Heyde and Gay [12] introduced a smoothed version periodogram to estimate time series with long-range dependence. Geweke and Porter-Hudak [13] employed the log-periodogram as the response variable to construct a regression model for long memory time series.

The paper is organized as follows. Section 2 presents the basic definitions of periodogram and spectral density of spatial version. In Section 3, we give the main theorems on the asymptotic properties of periodogram. Concluding remarks and further research will be given in Section 4. Throughout the paper, the following notation will be adopted: $O(1)(O_p(1))$ denotes a term (a random variable) that is bounded (in probability); $o(1)(o_p(1))$ denotes a term (a random variable) that converges to zero (in probability); and $\mathbb{R}^d$ denotes the Euclidean space with dimension d .

2. Spectral Domain

We consider a weakly stationary Gaussian process with mean zero in which data are drawn from the regularly spaced grid on d -dimensional Euclidean space, $Z = \{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\} \in \mathbb{R}$ (see Figure 1). Here, we also call Z a random field on $\mathbb{R}^d$. A random field Z is weakly stationary if its mean is constant, and it has finite covariance function C , such that $C(\mathbf{h}) = \text{Cov}(Z(\mathbf{s} + \mathbf{h}), Z(\mathbf{s}))$, for any single location $\mathbf{h}$ belonging to domain $D \subset \mathbb{R}^d$. Let $X(\omega)$ be an orthogonal process on a regular gridded lattice, which is mean zero and has independent increments

$$E[(X(\omega_4) - X(\omega_3))(X(\omega_2) - X(\omega_1))] = 0, \quad (1)$$

for $\omega_4 > \omega_3 > \omega_2 > \omega_1$, with variance $E(|dX(\omega)|^2) = f(\omega)d\omega$, where $f(\omega)$ is the spectral density function. By spectral representation theorem [14], the real-valued weakly stationary random field $Z(\mathbf{s})$ with mean 0 has representation

$$Z(\mathbf{s}) = \int_{\mathbb{R}^d} \exp\{i\omega' \mathbf{s}\} dX(\omega). \quad (2)$$

Then the covariance function can be represented as

$$\begin{aligned} & \text{Cov}(Z(\mathbf{s}), Z(\mathbf{s} + \mathbf{h})) \\ &= \text{Cov}(Z(\mathbf{0}), Z(\mathbf{h})) \\ &= E \left[\int_{\mathbb{R}^d} \exp\{i\omega' \mathbf{0}\} dX(\omega) \int_{\mathbb{R}^d} \exp\{i\omega' \mathbf{h}\} dX(\omega) \right] \quad (3) \\ &= \int_{\mathbb{R}^d} \exp\{i\omega' \mathbf{h}\} E(|dX(\omega)|^2) \\ &= \int_{\mathbb{R}^d} \exp\{i\omega' \mathbf{h}\} f(\omega) d\omega. \end{aligned}$$

FIGURE 1: Simulation of regular spaced lattice data.

If the covariance function satisfies $\int_{\mathbb{R}^2} |C(\mathbf{x})| d\mathbf{x} < \infty$, the spectral density f is the Fourier transform of the covariance function

$$f(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\{-i\omega' \mathbf{h}\} C(\mathbf{h}) d\mathbf{h}. \quad (4)$$

If we further assume an additional assumption as in Fuentes [7],

$$(a) \sum_{\mathbf{s} \in D} [1 + \|\mathbf{s}\|] |C(\mathbf{s})| < \infty,$$

where $\|\cdot\|$ is the Euclidean norm given by $\|\mathbf{s}\| = \sqrt{s_1^2 + \dots + s_d^2}$, then the spectral density of Z has uniformly bounded first-order derivative. To simplify our discussion, we illustrate the idea by random fields on a two-dimensional grid; that is, $Z = \{Z(\mathbf{s}); \mathbf{s} = (s_1, s_2), s_1 = 1, \dots, n_1; s_2 = 1, \dots, n_2\}$, in what follows.

We estimate the spectral density function on the lattice by periodogram of spatial version

$$I_N(\omega) = \frac{1}{(2\pi)^2} \frac{1}{n_1 n_2} \left| \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} Z(\mathbf{s}) \exp\{-i\mathbf{s}' \omega\} \right|^2, \quad (5)$$

where $N = n_1 n_2$ is the total sample size; $i = \sqrt{-1}$; and $\omega = (\omega_1, \omega_2)$ is the spatial frequencies, computed at $\omega_1 = 2\pi l_1 / n_1$ and $\omega_2 = 2\pi l_2 / n_2$ for

$$\begin{aligned} (l_1, l_2) &= \left\{ \left\lfloor -\frac{n_1-1}{2} \right\rfloor, \dots, n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor \right\} \\ &\quad \times \left\{ \left\lfloor -\frac{n_2-1}{2} \right\rfloor, \dots, n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor \right\}. \end{aligned} \quad (6)$$

In time series study, under some regular conditions on the stationary process, the periodogram is asymptotically

unbiased estimator of spectral density; that is, $I_n(\omega) \rightarrow f(\omega)$ as sample size $n \rightarrow \infty$, where the bias is of order $O(n^{-1})$. However, the periodogram is not a consistent estimator of the spectral density [8].

3. Main Results

Analogous to one-dimensional equally spaced time series observations, define the DFT on two-dimensional grid as

$$\begin{aligned} J(\omega_1, \omega_2) &= \frac{1}{2\pi\sqrt{n_1 n_2}} \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} Z(s_1, s_2) \exp\{-i(\omega_1 s_1 + \omega_2 s_2)\} \quad (7) \\ &= J_{\mathcal{R}}(\omega_1, \omega_2) + iJ_{\phi}(\omega_1, \omega_2), \end{aligned}$$

where $J_{\mathcal{R}}(\omega_1, \omega_2)$ and $J_{\phi}(\omega_1, \omega_2)$ denote the real and imaginary parts, respectively, and

$$\begin{aligned} J_{\mathcal{R}}(\omega_1, \omega_2) &= \frac{1}{2\pi\sqrt{n_1 n_2}} \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} Z(s_1, s_2) \cos(\omega_1 s_1 + \omega_2 s_2), \\ J_{\phi}(\omega_1, \omega_2) &= \frac{1}{2\pi\sqrt{n_1 n_2}} \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} Z(s_1, s_2) \sin(\omega_1 s_1 + \omega_2 s_2). \end{aligned} \quad (8)$$

Since it is known that

$$E(I_N(\omega)) = E(J(\omega))^2 = E(J_{\mathcal{R}}(\omega))^2 + E(J_{\phi}(\omega))^2, \quad (9)$$

the expectation of periodogram is equal to the variance of real and imaginary parts of DFT. To achieve this aim, we first show some asymptotic properties of Dirichlet kernel $D_N(\omega) = \sin(N\omega/2)/\sin(\omega/2)$ in time series. By simple calculation, we have

$$\begin{aligned} \frac{1}{2\pi N} \int_{-\pi}^{\pi} D_N(u - \omega) D_N(v + \omega) d\omega &= 0, \\ &\text{for } 0 < |u - v| < 2\pi, \\ \frac{1}{2\pi N} \int_{-\pi}^{\pi} D_N(u - \omega) D_N(u + \omega) d\omega &= O\left(\frac{1}{N}\right), \\ &\text{for } 0 < |u| < \pi. \end{aligned} \quad (10)$$

Also, Anderson [15] shows that

$$\begin{aligned} \frac{1}{2\pi N} \int_{-\pi}^{\pi} D_N(u - \omega) D_N(u + \omega) f(\omega) d\omega &= o(1) \\ &\text{for } 0 < |u| < \pi. \end{aligned} \quad (11)$$

Also, the Fejér kernel defined as $K_N(\omega) = D_N^2(\omega)$ holds an equation

$$\frac{1}{2\pi N} \int_{-\pi}^{\pi} K_N(\omega) d\omega = 1. \quad (12)$$

Using the above formula (10), we obtain the next theorem on asymptotic properties of real and imaginary parts of DFT.

Theorem 1. Under Assumption (a), the variance value of real and imaginary parts of DFT is given by

$$\begin{aligned} E(J_{\mathcal{R}}^2(\omega_1, \omega_2)) &= A + \cos(\omega_1(n_1 + 1) + \omega_2(n_2 + 1)) B, \\ E(J_{\phi}^2(\omega_1, \omega_2)) &= A - \cos(\omega_1(n_1 + 1) + \omega_2(n_2 + 1)) B, \end{aligned} \quad (13)$$

where

$$\begin{aligned} A &= \frac{1}{2(2\pi)^2 N} \\ &\quad \times \iint_{\Pi^2} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \\ B &= \frac{1}{2(2\pi)^2 N} \\ &\quad \times \iint_{\Pi^2} \prod_{j=1}^2 D_{n_j}(\omega_j + \lambda_j) D_{n_j}(\omega_j - \lambda_j) \\ &\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \end{aligned} \quad (14)$$

where $\Pi^2 = (-\pi, \pi) \times (-\pi, \pi)$.

Proof. Consider

$$\begin{aligned} &E(J_{\mathcal{R}}^2(\omega_1, \omega_2)) + iEJ_{\mathcal{R}}(\omega_1, \omega_2) J_{\phi}(\omega_1, \omega_2) \\ &= EJ_{\mathcal{R}}^2(\omega_1, \omega_2) (J_{\mathcal{R}}(\omega_1, \omega_2) + iJ_{\phi}(\omega_1, \omega_2)) \\ &= \frac{1}{(2\pi)^2 N} E \left\{ \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} Z(s_1, s_2) Z(t_1, t_2) \right. \\ &\quad \times \cos(\omega_1 s_1 + \omega_2 s_2) \\ &\quad \times \cos(\omega_1 t_1 + \omega_2 t_2) \Big\} \\ &\quad + i \frac{1}{(2\pi)^2 N} E \left\{ \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} Z(s_1, s_2) Z(t_1, t_2) \right. \\ &\quad \times \sin(\omega_1 s_1 + \omega_2 s_2) \\ &\quad \times \sin(\omega_1 t_1 + \omega_2 t_2) \Big\} \\ &= \frac{1}{(2\pi)^2 N} \sum_{s_1, t_1, s_2, t_2} C(s_1 - t_1, s_2 - t_2) e^{i(\omega_1 t_1 + \omega_2 t_2)} \\ &\quad \times \cos(\omega_1 s_1 + \omega_2 s_2) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^2 N} \\
&\quad \times \iint_{\Pi^2} \sum_{s_1, t_1, s_2, t_2} f(\lambda_1, \lambda_2) e^{i[(s_1 - t_1)\lambda_1 + (s_2 - t_2)\lambda_2] + i(\omega_1 t_1 + \omega_2 t_2)} \\
&\quad \times \cos(\omega_1 s_1 + \omega_2 s_2) d\lambda_1 d\lambda_2 \\
&= \frac{1}{(2\pi)^2 N} \\
&\quad \times \iint_{\Pi^2} \sum_{s_1, s_2} \cos(\omega_1 s_1 + \omega_2 s_2) e^{i(s_1 \lambda_1 + s_2 \lambda_2)} \\
&\quad \times \sum_{t_1, t_2} e^{i(\omega_1 - \lambda_1)t_1 + i(\omega_2 - \lambda_2)t_2} f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&= \frac{1}{2(2\pi)^2 N} \iint_{\Pi^2} \sum_{s_1, s_2} (e^{i(\omega_1 s_1 + \omega_2 s_2)} + e^{-i(\omega_1 s_1 + \omega_2 s_2)}) \\
&\quad \times e^{i(s_1 \lambda_1 + s_2 \lambda_2)} \\
&\quad \times e^{i(n_1 + 1)(\omega_1 - \lambda_1)/2 + i(n_2 + 1)(\omega_2 - \lambda_2)/2} \\
&\quad \times \frac{\sin n_1 (\omega_1 - \lambda_1)/2}{\sin (\omega_1 - \lambda_1)/2} \frac{\sin n_2 (\omega_2 - \lambda_2)/2}{\sin (\omega_2 - \lambda_2)/2} \\
&\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&= \frac{1}{2(2\pi)^2 N} \\
&\quad \times \iint_{\Pi^2} \left\{ e^{i(n_1 + 1)\omega_1 + i(n_2 + 1)\omega_2} \right. \\
&\quad \times \prod_{j=1}^2 \frac{\sin n_j (\omega_j + \lambda_j)/2}{\sin (\omega_j + \lambda_j)/2} \frac{\sin n_j (\omega_j - \lambda_j)/2}{\sin (\omega_j - \lambda_j)/2} \\
&\quad + \left(\frac{\sin n_1 (\omega_1 - \lambda_1)/2}{\sin (\omega_1 - \lambda_1)/2} \right)^2 \\
&\quad \times \left(\frac{\sin n_2 (\omega_2 - \lambda_2)/2}{\sin (\omega_2 - \lambda_2)/2} \right)^2 \Big\} \\
&\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&= \frac{1}{2(2\pi)^2 N} \\
&\quad \times \iint_{\Pi^2} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) \\
&\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&\quad + \frac{1}{2(2\pi)^2 N} \cos[(n_1 + 1)\omega_1 + (n_2 + 1)\omega_2] \\
&\quad \times \iint_{\Pi^2} \prod_{j=1}^2 D_{n_j}(\omega_j + \lambda_j) D_{n_j}(\omega_j - \lambda_j) f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2
\end{aligned}$$

$$\begin{aligned}
&+ i \frac{1}{2(2\pi)^2 N} \sin[(n_1 + 1)\omega_1 + (n_2 + 1)\omega_2] \\
&\quad \times \iint_{\Pi^2} \prod_{j=1}^2 D_{n_j}(\omega_j + \lambda_j) D_{n_j}(\omega_j - \lambda_j) f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2.
\end{aligned} \tag{15}$$

Applying the similar calculation procedure to $E(J_\phi^2(\omega_1, \omega_2)) + iEJ_{\mathcal{R}}(\omega_1, \omega_2)J_\phi(\omega_1, \omega_2)$, we can get the result that $E(J_\phi^2(\omega_1, \omega_2)) = A - \cos(\omega_1(n_1 + 1) + \omega_2(n_2 + 1))B$. $\square$

By the conclusion of Theorem 1 and formula (9), it follows that

$$\begin{aligned}
&E(I_N(\omega)) \\
&= E(J_{\mathcal{R}}^2(\omega)) + E(J_\phi^2(\omega)) \\
&= \frac{1}{(2\pi)^2 N} \iint_{\Pi^2} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) \\
&\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2.
\end{aligned} \tag{16}$$

Therefore, the bias of periodogram $I_N(\omega)$ to spectral density is

$$\begin{aligned}
b(\omega) &= E[I_N(\omega) - f(\omega)] \\
&= \frac{1}{(2\pi)^2 N} \iint_{\Pi^2} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) \\
&\quad \times f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 - f(\omega).
\end{aligned} \tag{17}$$

The next theorem evaluates the bias in (17), such that we can prove the asymptotic unbiasedness property of periodogram for regular lattice random fields.

Theorem 2. Under Assumption (a), for $\omega \in \Pi^2 = (-\pi, \pi) \times (-\pi, \pi)$, we have

$$b(\omega_1, \omega_2) = \min \{O(n_1^{-1}), O(n_2^{-1})\}. \tag{18}$$

Proof. By formula (17), it follows that

$$\begin{aligned}
&b(\omega_1, \omega_2) \\
&= \frac{1}{(2\pi)^2 n_1 n_2} \iint_{-\pi}^{\pi} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) \\
&\quad \times (f(\lambda_1, \lambda_2) - f(\omega_1, \omega_2)) d\lambda_1 d\lambda_2 \\
&= \frac{1}{(2\pi)^2 n_1 n_2} \iint_{-\pi}^{\pi} K_{n_1}(\omega_1 - \lambda_1) K_{n_2}(\omega_2 - \lambda_2) \\
&\quad \times \left\{ (\lambda - \omega)^T \nabla f(\omega) \right. \\
&\quad \left. + \frac{1}{2} (\lambda - \omega)^T \nabla^2 f(\omega^*) (\lambda - \omega) \right\} d\lambda_1 d\lambda_2,
\end{aligned} \tag{19}$$

where $\nabla f(\omega) = (f_{\omega_1}, f_{\omega_2})^T$ and $\nabla^2 f(\omega^*)$ denote the first- and second-order derivatives of $f(\omega)$ at ω and ω^* , respectively, where $\|\omega^* - \omega\| < \|\lambda - \omega\|$. By the fact that $\|\nabla^2 f(\omega^*)\| < K$ and by integration range $\|\lambda - \omega\|^2 = (\lambda_1 - \omega_1)^2 + (\lambda_2 - \omega_2)^2 < K$, we have

$$\begin{aligned}
& \|b(\omega)\| \\
& \leq \frac{K}{(2\pi)^2 n_1 n_2} \int_{-\pi-\omega_1}^{\pi-\omega_1} \int_{-\pi-\omega_2}^{\pi-\omega_2} K_{n_1}(\lambda_1) K_{n_2}(\lambda_2) \\
& \quad \times (f_{\omega_1} \lambda_1 + f_{\omega_2} \lambda_2) d\lambda_1 d\lambda_2 \\
& \quad + \frac{K}{(2\pi)^2 n_1 n_2} \int_{-\pi-\omega_1}^{\pi-\omega_1} \int_{-\pi-\omega_2}^{\pi-\omega_2} K_{n_1}(\lambda_1) K_{n_2} \\
& \quad \times (\lambda_2) \|\lambda\|^2 d\lambda_1 d\lambda_2 \\
& \leq \frac{K}{(2\pi)^2 n_1 n_2} \\
& \quad \times \left\{ |f_{\omega_1}| \int_{-\pi-\omega_1}^{\pi-\omega_1} K_{n_1}(\lambda_1) \lambda_1 d\lambda_1 \int_{-\pi-\omega_2}^{\pi-\omega_2} K_{n_2}(\lambda_2) d\lambda_2 \right. \\
& \quad \left. + |f_{\omega_2}| \int_{-\pi-\omega_2}^{\pi-\omega_2} K_{n_2}(\lambda_2) \lambda_2 d\lambda_2 \int_{-\pi-\omega_1}^{\pi-\omega_1} K_{n_1}(\lambda_1) d\lambda_1 \right\} \\
& \quad + O\left(\frac{1}{n_1 n_2}\right) \\
& \leq \frac{K}{(2\pi) n_1} \left\{ \int_{-\pi-\omega_1}^{-\eta} + \int_{-\eta}^{\eta} + \int_{\eta}^{\pi-\omega_1} K_{n_1}(\lambda_1) \lambda_1 d\lambda_1 \right\} \\
& \quad + \frac{K}{(2\pi) n_2} \left\{ \int_{-\pi-\omega_2}^{-\eta} + \int_{-\eta}^{\eta} + \int_{\eta}^{\pi-\omega_2} K_{n_2}(\lambda_2) \lambda_2 d\lambda_2 \right\} \\
& \quad + O\left(\frac{1}{n_1 n_2}\right). \tag{20}
\end{aligned}$$

Also,

$$\begin{aligned}
& \frac{K}{(2\pi) n_1} \int_{-\eta}^{\eta} K_{n_1}(\lambda_1) \lambda_1 d\lambda_1 \\
& \quad + \frac{K}{(2\pi) n_2} \int_{-\eta}^{\eta} K_{n_2}(\lambda_2) \lambda_2 d\lambda_2 = 0, \tag{21}
\end{aligned}$$

because the integrand is odd, and the remaining terms hold as

$$\begin{aligned}
& \frac{K}{(2\pi) n_1} \left\{ \int_{-\pi-\omega_1}^{-\eta} + \int_{\eta}^{\pi-\omega_1} \right\} K_{n_1}(\lambda_1) \lambda_1 d\lambda_1 = O(n_1^{-1}), \\
& \frac{K}{(2\pi) n_2} \left\{ \int_{-\pi-\omega_2}^{-\eta} + \int_{\eta}^{\pi-\omega_2} \right\} K_{n_2}(\lambda_2) \lambda_2 d\lambda_2 = O(n_2^{-1}), \tag{22}
\end{aligned}$$

because $K_N(u) < 1/\sin^2(u/2)$, for $0 < u < \pi$. $\square$

The idea of Theorem 2 can be directly extended to random fields on d -dimensional regular gridded lattice.

In conclusion, the periodogram of spatial version is asymptotically unbiased estimator of the spectral density function, in which the convergence rate is the minimum of sampling dimension, that is, $\min\{n_1^{-1}, \dots, n_d^{-1}\}$.

4. Conclusions and Future Research

In this paper, we consider the nonparametric method to estimate the spectral density function. The spectral density function represents the population property and describes the behavior of underlying process. By constructing a moment estimator periodogram in frequency domain, which is a Fourier transform of covariance function in space domain, the spectral density can be estimated to be asymptotically unbiased. The spectral method outperforms traditional spatial method, because it provides significant computational advantages. In addition, we obtain the convergence rate as the minimum of spatial sampling dimension in the lattice. However, the consistency and asymptotic normality has not been established in this paper, which is important in statistical inference, hypothesis testing, and interval forecasts. On the other hand, by using the Whittle likelihood [9], we can approximate the maximum likelihood method [16] in spatial domain by Whittle estimator in frequency domain and thus estimate the specific parameter in the spectral density.

However, our asymptotic results are based on increasing domain asymptotics, allowing the fact that, as the sample size increases, the distance increases becoming sufficiently large, and thus the observations are nearly independent. However, if one is concerned to predict the value of a point in a bounded area, it is necessary to consider fixed domain asymptotics rather than increasing domain asymptotics. Specifically, even for regularly sited data, results from maximum likelihood estimate only exist in very limited cases. The difficulty under the fixed domain asymptotics is that there is at least one function of parameters that cannot be consistently estimated even for regularly gridded Gaussian process. This is indeed the case for Gaussian random fields with isotropic exponential covariance function $C(\mathbf{h}) = \sigma^2 \exp(-|\mathbf{h}|/\alpha)$, where σ^2 is the sill parameter and α is the range parameter. Ying [17] proved that when $d = 1$, the maximum likelihood estimator of σ^2/α is strongly consistent, but not possible for either σ^2 or α . Therefore, it needs more researches on the fixed domain asymptotics in spatial statistics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the guest editor and two anonymous referees for their helpful comments and suggestions. This work was partially supported by the National Natural Science Foundation of China (Grant no. 71301023), the Humanities and Social Sciences Foundation of Ministry of Education of China (Grant no. 13YJC630197), and the Basic Research Foundation (Natural Science) of Jiangsu Province (Grant no. BK20130582).

References

- [1] L. Anselin, *Spatial Econometrics: Methods and Models*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.
- [2] N. A. C. Cressie, *Statistics for Spatial Data*, Revised Edition, John Wiley & Sons, New York, NY, USA, 1993.
- [3] P. J. Diggle, J. A. Tawn, and R. A. Moyeed, "Model-based geostatistics," *Journal of the Royal Statistical Society C: Applied Statistics*, vol. 47, no. 3, pp. 299–350, 1998.
- [4] B. D. Ripley, *Spatial Statistics*, John Wiley & Sons, New York, NY, USA, 1981.
- [5] J. Besag, "Spatial interaction and the statistical analysis of lattice systems," *Journal of the Royal Statistical Society B: Methodological*, vol. 36, pp. 192–236, 1974.
- [6] P. Whittle, "On stationary processes in the plane," *Biometrika*, vol. 41, pp. 434–449, 1954.
- [7] M. Fuentes, "Approximate likelihood for large irregularly spaced spatial data," *Journal of the American Statistical Association*, vol. 102, no. 477, pp. 321–331, 2007.
- [8] D. R. Brillinger, *Time Series Data Analysis and Theory*, Holden-Day, San Francisco, Calif, USA, 2nd edition, 1981.
- [9] P. Whittle, "Estimation and information in stationary time series," *Arkiv för Matematik*, vol. 2, pp. 423–434, 1953.
- [10] M. R. Rosenblatt, *Stationary Sequences and Random Fields*, Birkhäuser, Boston, Mass, USA, 1985.
- [11] M. B. Priestley, *Spectral Analysis and Time Series*, Academic Press, 1981.
- [12] C. C. Heyde and R. Gay, "Smoothed periodogram asymptotics and estimation for processes and fields with possible long-range dependence," *Stochastic Processes and Their Applications*, vol. 45, no. 1, pp. 169–182, 1993.
- [13] J. Geweke and S. Porter-Hudak, "The estimation and application of long memory time series models," *Journal of Time Series Analysis C*, vol. 4, no. 4, pp. 221–238, 1983.
- [14] A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, Springer, New York, NY, USA, 1987.
- [15] T. W. Anderson, *The Statistical Analysis of Time Series*, John Wiley & Sons, New York, NY, USA, 1971.
- [16] K. V. Mardia and R. J. Marshall, "Maximum likelihood estimation of models for residual covariance in spatial regression," *Biometrika*, vol. 71, no. 1, pp. 135–146, 1984.
- [17] Z. Ying, "Asymptotic properties of a maximum likelihood estimator with data from a Gaussian process," *Journal of Multivariate Analysis*, vol. 36, no. 2, pp. 280–296, 1991.