Intuitionistic Fuzzy Planar Graphs

Noura Alshehri\textsuperscript{1} and Muhammad Akram\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Faculty of Sciences (Girls), King Abdulaziz University, Jeddah, Saudi Arabia
\textsuperscript{2} Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Correspondence should be addressed to Noura Alshehri; nalshehrie@kau.edu.sa

Received 21 April 2014; Accepted 7 August 2014; Published 24 August 2014

1. Introduction

Graph theory is now a very important research area due to its wide applications. There are many practical applications with a graph structure in which crossing between edges is nuisance including design problems for circuits, subways, and utility lines. Crossing of two connections normally means that the communication lines must be run at different heights. This is not a big issue for electrical wires, but it creates extra expenses for some types of lines. Circuits, in particular, are easier to manufacture if their connections can be constructed in fewer layers. These applications are designed by the concept of planar graphs. Circuits, where crossing of lines is necessary, can not be represented by planar graphs. Numerous computational challenges can be solved by means of cuts of planar graph. In the city planning, subway tunnels, pipelines, and metro lines are essential in twenty first century. Due to crossing, there is a chance for an accident. Also, the cost of crossing of routes in underground is high. But underground routes reduce the traffic jam. In a city planning, routes without crossing are perfect for safety. But due to lack of space, crossing of such lines is allowed. It is easy to observe that the crossing between one congested and one noncongested route is better than the crossing between two congested routes. The term “congested” has no definite meaning. We generally use “congested,” “very congested,” “highly congested” routes, and so forth. These terms are called linguistic terms and they have some membership values. A congested route may be referred to as strong route and low congested route may be called weak route. Thus crossing between strong route and weak route is more safe than the crossing between two strong routes. That is, crossing between strong route and weak route may be allowed in city planning with certain amount of safety. The terms strong route and weak route lead to strong edge and weak edge of a fuzzy graph, respectively. And the permission of crossing between strong and weak edges leads to the concept of fuzzy planar graph [1–3].

Presently, science and technology is featured with complex processes and phenomena for which complete information are not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models are based on an extension of the ordinary set theory, namely, fuzzy sets. The notion of fuzzy sets was introduced by Zadeh [4] as a method of representing uncertainty and vagueness. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines. In 1983, Atanassov [5] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Atanassov added in the definition of fuzzy set a new component which determines the degree of nonmembership. Fuzzy sets give the degree of
2. Preliminaries

In this section, we review some elementary concepts whose understanding is necessary for full benefit from this paper.

By a graph we mean a pair \( G^* = (V, E) \), where \( V \) is the set and \( E \) is a relation on \( V \). The elements of \( V \) are vertices of \( G^* \) and the elements of \( E \) are edges of \( G^* \). We write \((x, y) \in E\) to mean \((x, y) \in E\), and if \( e = xy \in E\), we say \( x \) and \( y \) are adjacent. Formally, given a graph \( G^* = (V, E) \), two vertices \( x\), \( y \in V \) are said to be neighbors, or adjacent nodes, if \( xy \in E \).

The number of vertices, the cardinality of \( V \), is called the order of graph and denoted by \(|V|\). The number of edges, the cardinality of \( E \), is called the size of graph and denoted by \(|E|\). A multigraph is a graph that may contain multiple edges between any two vertices, but it does not contain any self-loops. A graph can be drawn in many different ways. A graph may or may not be drawn on a plane without crossing of edges. A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding. A graph \( G^* \) is planar if it can be drawn in the plane with its edges only intersecting at vertices of \( G^* \). A crisp graph is called nonplanar graph if there is at least one crossing between the edges for all possible geometrical representations of the graph. A planar graph with cycles divides the plane into a set of regions are called faces. The length of a face in a plane graph \( G^* \) is the total length of the closed walk(s) in \( G^* \) bounding the face. The portion of the plane lying outside a graph embedded in a plane is infinite region. The dual graph of a plane graph \( G^* \) is a graph that has a vertex corresponding to each face of \( G^* \) and an edge joining two neighboring faces for each edge in \( G^* \). The term “dual” is used because this property is symmetric, meaning that if \( H^* \) is a dual of \( G^* \), then \( G^* \) is a dual of \( H^* \) (if \( G^* \) is connected).

Let \( X \) be a nonempty set. A fuzzy set \([4]\) A drawn from \( X \) is defined as \( A = \{(x : \mu(x)) : x \in X\} \), where \( \mu : X \rightarrow [0, 1] \) is the membership function of the fuzzy set \( A \). A fuzzy binary relation \([7]\) on \( X \) is a fuzzy subset \( \mu \) on \( X \times X \). By a fuzzy relation, we mean a fuzzy binary relation given by \( \mu : X \times X \rightarrow [0, 1] \). A fuzzy graph \([6]\) \( G = (V, \sigma, \mu) \) is a nonempty set \( V \) together with a pair of functions \( \sigma : V \rightarrow [0, 1] \) and \( \mu : V \times V \rightarrow [0, 1] \) such that, for all \( x, y \in V \), \( \mu(x, y) \leq \min(\sigma(x), \sigma(y)) \), where \( \sigma(x) \) and \( \mu(x, y) \) represent the membership values of the vertex \( x \) and of the edge \((x, y)\) in \( G \), respectively.

A loop at a vertex \( x \) in a fuzzy graph is represented by \( \mu(x, x) \neq 0 \). An edge is nontrivial if \( \mu(x, y) \neq 0 \). Let \( G \) be a fuzzy graph and, for a certain geometric representation, the graph has only one crossing between two fuzzy edges \((u, w), \mu(u, w))\) and \((v, z), \mu(v, z))\). If \( \mu(u, w) = 1 \) and \( \mu(v, z) = 0 \), then we say that the fuzzy graph has no crossing. Similarly, if \( \mu(u, w) \) has value near to 1 and \( \mu(v, z) \) has value near to 0, the crossing will not be important for the planarity. If \( \mu(v, z) \) has value near to 1 and \( \mu(u, w) \) has value near to 1, then the crossing becomes very important for the planarity.

Let \( X \) be a nonempty set. A fuzzy multiset \([28]\) A drawn from \( X \) is characterized by a function, “count membership” of \( A \) denoted by CM\(_{A} \) such that \( CM_{A} : X \rightarrow Q \), where \( Q \) is the set of all crisp multisets drawn from the unit interval \([0, 1]\). For each \( x \in X \), the value \( CM_{A}(x) \) is a crisp multiset drawn from \([0, 1]\). For each \( x \in X \), the membership sequence is defined as the decreasingly ordered sequence of elements in \( CM_{A}(x) \). It is denoted by \( (\mu_{1}^{A}(x), \mu_{2}^{A}(x), \mu_{3}^{A}(x), \ldots, \mu_{n}^{A}(x)) \) where \( \mu_{1}^{A}(x) \geq \mu_{2}^{A}(x) \geq \mu_{3}^{A}(x) \geq \cdots \geq \mu_{n}^{A}(x) \).

Let \( V \) be a nonempty set and \( \sigma : V \rightarrow [0, 1] \) a mapping and let \( \mu = \{(x, y), \mu(x, y)\} \), \( j = 1, 2, \ldots, p_{xy} \) for \((x, y) \in V \times V\) be a fuzzy multiset of \( V \times V \) such that \( \mu(x, y) \leq \min(\sigma(x), \sigma(y)) \) for all \( j = 1, 2, \ldots, p_{xy} \) where \( p_{xy} = \max(j \mid \mu(x, y) \neq 0) \). Then \( G = (V, \sigma, \mu) \) is denoted as fuzzy multigraph \([2]\) where \( \sigma(x) \) and \( \mu(x, y) \), represent...
the membership value of the vertex $x$ and the membership value of the edge $(x, y)$ in $G$, respectively. In 1983, Atanassov [5] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [4]. An intuitionistic fuzzy set (IFS, for short) on a universe $X$ is an object of the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \},$$

(1)

where $\mu_A(x) \in [0,1]$ is called degree of membership of $x$ in $A$ and $\nu_A(x) \in [0,1]$ is called degree of nonmembership of $x$ in $A$, and $\mu_A, \nu_A$ satisfies the following condition for all $x \in X, \mu_A(x) + \nu_A(x) \leq 1$. An intuitionistic fuzzy relation $R = (\mu_R(x, y), \nu_R(x, y))$ in a universe $X \times Y$ (for short) is an intuitionistic fuzzy set of the form

$$R = \{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \mid (x, y) \in X \times Y \},$$

(2)

where $\mu_R : X \times Y \to [0,1]$ and $\nu_R : X \times Y \to [0,1]$. The intuitionistic fuzzy relation $R$ satisfies $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for all $x, y \in X$. An intuitionistic fuzzy graph is a pair $G = (A, B)$ where $A = (V, \mu_A, \nu_A)$ is an intuitionistic fuzzy set in $V$ and $B = (V \times V, \mu_B, \nu_B)$ is an intuitionistic fuzzy relation on $V$ such that

$$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)),$$

(3)

$$\nu_B(xy) \leq \max(\nu_A(x), \nu_A(y))$$

(4)

such that $0 \leq \mu_B(xy) + \nu_B(xy) \leq 1$ for all $x, y \in V$.

Let $X$ be a nonempty set. An intuitionistic fuzzy multiset (IFMS) [29] $A$ drawn from $X$ is characterized by two functions: “count membership” of $A$ and “count nonmembership” of $A$ ($CM_A$ and $CN_A$) given by $CM_A : X \to Q$ and $CN_A : X \to Q$ where $Q$ is the set of all crisp multisets drawn from the unit interval $[0,1]$ such that, for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$ which is denoted by $(\mu_1(x), \mu_2(x), \mu_3(x), \ldots, \mu_p(x))$ where $\mu_1(x) \geq \mu_2(x) \geq \mu_3(x) \geq \cdots \geq \mu_p(x)$ and the corresponding nonmembership sequence will be denoted by $(\nu_1(x), \nu_2(x), \nu_3(x), \ldots, \nu_p(x))$ such that $\mu_i(x) + \nu_i(x) \leq 1$ for all $x \in X$ and $i = 1, 2, \ldots, p$. An IFMS $A$ is denoted by

$$\{ \langle x : (\mu_1(x), \mu_2(x), \mu_3(x), \ldots, \mu_p(x)), (\nu_1(x), \nu_2(x), \nu_3(x), \ldots, \nu_p(x)) \rangle \mid x \in X \}.$$ 

(5)

We arrange the membership sequence in decreasing order but the corresponding nonmembership sequence may not be in decreasing or increasing order.

### 3. Intuitionistic Fuzzy Planar Graphs

We first introduce the notion of an intuitionistic fuzzy multigraph using the concept of an intuitionistic fuzzy multiset.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
</tr>
<tr>
<td>$\mu_a$</td>
</tr>
<tr>
<td>$\nu_a$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_b$</td>
</tr>
<tr>
<td>$\nu_b$</td>
</tr>
</tbody>
</table>

![Figure 1: Intuitionistic fuzzy multigraph.](image)

**Definition 1.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on $V$ and let

$$B = \{(xy, \mu_B(xy), \nu_B(xy)) \mid xy \in V \times V\}$$

be an intuitionistic fuzzy multiset of $V \times V$ such that

$$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)),$$

(7)

$$\nu_B(xy) \leq \max(\nu_A(x), \nu_A(y))$$

for all $i = 1, 2, \ldots, m$. Then $G$ is called an intuitionistic fuzzy multigraph.

Note that there may be more than one edge between the vertices $x$ and $y$. $\mu_B(xy), \nu_B(xy)$ represent the membership value and nonmembership value of the edge $xy$ in $G$, respectively. $m$ denotes the number of edges between the vertices. In intuitionistic fuzzy multigraph $G$, $B$ is said to be intuitionistic fuzzy multiedge set.

**Example 2.** Consider a multigraph $G^* = (V, E)$ such that $V = \{a, b, c, d\}$, $E = \{ab, ab, ab, bc, bd\}$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of $V$ and let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy multisette set of $V \times V$ defined by Table 1 and Figure 1.

By routine computations, it is easy to see from Figure 1 that it is an intuitionistic fuzzy multigraph.
Table 2

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ₀</td>
<td>0.5</td>
<td>0.4</td>
<td>0.3</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>ν₀</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ₀</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>ν₀</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

(a) Let \( G \) be an intuitionistic fuzzy multigraph with vertex set \( V \) and edge set \( E \). Let \( a, b, c, d \) be the points of intersection between the edges \( ab \) and \( cd \), and \( (0.4, 0.3) \) be the geometrical representation of an intuitionistic fuzzy planar graph.

(b) The strength of the intuitionistic fuzzy edge \( ab \) is defined by

\[
I_{ab} = (M_{ab}, N_{ab})
\]

where \( I_{ab} \) is the strength of the intuitionistic fuzzy edge \( ab \) and \( M_{ab}, N_{ab} \) are the membership and non-membership values, respectively.

Example 4. In Example 2, degrees of the vertices \( a, b, c, d \) are \( \deg(a) = (0.5, 0.4), \deg(b) = (0.9, 0.9), \deg(c) = (0.3, 0.3) \), and \( \deg(d) = (0.1, 0.2) \).

Definition 5. Let \( B = \{(xy, \mu_B(xy)), \nu_B(xy))\}, i = 1, 2, \ldots, m \mid xy \in V \times V \) be an intuitionistic fuzzy multiedge set in intuitionistic fuzzy multigraph \( G \). A multiedge \( xy \) of \( G \) is strong if \( (1/2) \min(\mu_A(x), \mu_A(y)) \leq \mu_B(xy)_i \) and \( (1/2) \max(\nu_A(x), \nu_A(y)) \leq \nu_B(xy)_i \), for all \( i = 1, 2, \ldots, m \).

Example 7. Consider an intuitionistic fuzzy multigraph \( G \) as shown in Figure 2.

Figure 2: Intuitionistic fuzzy complete multigraph.

By routine computations, it is easy to see from Figure 2 that it is an intuitionistic fuzzy complete multigraph.

Definition 8. Strength of the intuitionistic fuzzy edge \( ab \) can be measured by the value

\[
I_{ab} = (M_{ab}, N_{ab})
\]

where \( I_{ab} \) is the strength of the intuitionistic fuzzy edge \( ab \) and \( M_{ab}, N_{ab} \) are the membership and non-membership values, respectively.

Definition 9. Let \( G \) be an intuitionistic fuzzy multigraph. An edge \( ab \) is said to be intuitionistic fuzzy strong if \( M_{ab} \geq 0.5 \) or \( N_{ab} \leq 0.5 \), otherwise weak.

Definition 10. Let \( G \) be an intuitionistic fuzzy multigraph and let \( B \) contain two edges \( (ab, \mu_B(ab)_i), (cd, \mu_B(cd)_j) \) and \( (cd, \mu_B(cd)_j), (ae, \mu_B(ae)_k) \) which are intersected at a point \( P \), where \( i, j, k \) are fixed integers. We define the intersecting value at the point \( P \) by

\[
J_P = (M_{ab}, N_{ab}) = \left( \frac{M_{ab} + N_{cd}}{2}, \frac{N_{ab} + N_{cd}}{2} \right).
\]

Example 12. Consider a multigraph \( G^* = (V, E) \) such that \( V = \{a, b, c, d, e\} \)

\[
E = \{ab, ac, ad, bc, bd, cd, ce, ae, de, be\}.
\]
The intuitionistic fuzzy multigraph as shown in Figure 3 has two points of intersections $P_1$ and $P_2$. $P_1$ is a point between the edges $(ad,0.2,0.1)$ and $(bc,0.2,0.1)$ and $P_2$ is between $(ad,0.3,0.1)$ and $(bc,0.2,0.1)$. For the edge $(ad,0.2,0.1)$, $I_{ad} = (0.4,0.5)$. For the edge $(ad,0.3,0.1)$, $I_{ad} = (0.6,0.5)$, and for the edge $(bc,0.2,0.1)$, $I_{bc} = (0.6667,1)$. For the first point of intersection $P_1$, intersecting value $J_{P_1}$ is (0.5335, 0.75) and that for the second point of intersection $P_2$, $J_{P_2} = (0.63335, 0.75)$. Therefore, the intuitionistic fuzzy planarity value for the intuitionistic fuzzy multigraph shown in Figure 3 is (0.461, 0.4).

Intuitionistic fuzzy planarity value for the intuitionistic fuzzy complete multigraph is calculated from Theorem 13.

**Theorem 13.** Let $G$ be an intuitionistic fuzzy complete multigraph. The intuitionistic fuzzy planarity value $f = (M_f, N_f)$ of $G$ is given by $M_f = 1/(1+n_p)$ and $N_f = 1/(1+n_p)$ such that $M_f + N_f \leq 1$, where $n_p$ is the number of points of intersections between the edges in $G$.

**Definition 14.** An intuitionistic fuzzy planar graph $G$ is called strong intuitionistic fuzzy planar graph if the intuitionistic fuzzy planarity value $f = (M_f, N_f)$ of the graph is $M_f \geq 0.5$, $N_f \leq 0.5$.

**Theorem 15.** Let $G$ be a strong intuitionistic fuzzy planar graph. The number of points of intersections between strong edges in $G$ is at most one.

**Proof.** Let $G$ be a strong intuitionistic fuzzy planar graph. Assume that $G$ has at least two points of intersections $P_1$ and $P_2$ between two strong edges in $G$. For any strong edge $(ab, \mu_{ab}(ab), \nu_{ab}(ab))$,

$$
\mu_{ab}(ab) = \frac{1}{2} \min \{\mu_A(a), \mu_A(b)\},
$$

$$
\nu_{ab}(ab) = \frac{1}{2} \max \{\nu_A(a), \nu_A(b)\}.
$$

These inequalities show that $M_{ab} \geq 0.5$ or $N_{ab} \leq 0.5$. Thus for two intersecting strong edges $(ab, \mu_{ab}(ab), \nu_{ab}(ab))$ and $(cd, \mu_{cd}(cd), \nu_{cd}(cd))$,

$$
\frac{M_{ab} + M_{cd}}{2} \geq 0.5, \quad \frac{N_{ab} + N_{cd}}{2} \leq 0.5.
$$

that is, $M_{P_1} \geq 0.5$, $N_{P_1} \leq 0.5$. Similarly, $M_{P_2} \geq 0.5$, $N_{P_2} \leq 0.5$. This implies that $1 + M_{P_1} + M_{P_2} \geq 2, 1 + N_{P_1} + N_{P_2} \leq 2$. Therefore, $f_M = 1/(1 + M_{P_1} + M_{P_2}) \leq 0.5$, $f_N = 1/(1 + N_{P_1} + N_{P_2}) \geq 0.5$. It contradicts the fact that the intuitionistic fuzzy graph is a strong intuitionistic fuzzy planar graph. Thus number of points of intersections between strong edges can not be two. Obviously, if the number of points of intersections of strong intuitionistic fuzzy edges increases, the intuitionistic fuzzy planarity value decreases. Similarly, if the number of points of intersection of strong edges is one, then the intuitionistic fuzzy planarity value $M_f > 0.5, N_f < 0.5$. Any intuitionistic fuzzy planar graph without any crossing between edges is a strong intuitionistic fuzzy planar graph. Thus, we conclude that the maximum number of points of intersections between the strong edges in $G$ is one.

**Theorem 16.** Let $G$ be an intuitionistic fuzzy planar graph with intuitionistic fuzzy planarity value $f = (M_f, N_f)$. If $M_f \geq 0.67, N_f < 0.33$, $G$ do not contain any point of intersection between two strong edges.

**Definition 17.** Let $G$ be an intuitionistic fuzzy graph. Let $0 < c < 0.5$ be a rational number. An edge $xy$ is said to be considerable edge if

$$
\frac{\mu_{xy}(x)}{\min\{\mu_A(x), \mu_A(y)\}} \geq c, \quad \frac{\nu_{xy}(x)}{\max\{\nu_A(x), \nu_A(y)\}} \leq c.
$$

(14)

If an edge is not considerable, it is called nonconsiderable edge. For intuitionistic fuzzy multigraph $G$, a multiedge $xy$ is said to be considerable edge if $M_{xy} \geq c, N_{xy} \leq c$, for each edge $xy$ in $G$.

**Remark 18.** Let $0 < c < 0.5$ be a rational number. If $(\mu_{xy}(x)/\min\{\mu_A(x), \mu_A(y)\}) \geq c, (\nu_{xy}(x)/\max\{\nu_A(x), \mu_A(y)\}) \leq c$ for all edges $xy$ of an intuitionistic fuzzy graph $G$, then the number $c$ is said to be considerable number of the intuitionistic fuzzy graph. Considerable number of an intuitionistic fuzzy graph may not be unique, but it is countable. Clearly, for a specific value of $c$, a set of considerable edges is obtained and for different values of $c$ one can obtain different sets of considerable edges. Actually, $c$ is a preassigned number for a specific application.

**Theorem 19.** Let $G$ be a strong intuitionistic fuzzy planar graph with considerable number $c$. The number of points of intersections between considerable edges in $G$ is at most $1/c$ (or $1/c - 1$).

**Proof.** Let $G$ be a strong intuitionistic fuzzy planar graph, where $E = \{(xy, \mu_{xy}(x), \nu_{xy}(x)), i = 1, 2, \ldots, m \mid xy \in V \times V\}$. Let $0 < c < 0.5$ be the considerable number and let $f = (M_f, N_f)$ be the intuitionistic fuzzy planarity value. For any considerable edge $(ab, \mu_{ab}(ab), \nu_{ab}(ab))$,

$$
\frac{\mu_{ab}(ab)}{\mu_A(a) \cup \mu_A(b)} \leq c \times \min\{\mu_A(a), \mu_A(b)\},
$$

$$
\nu_{ab}(ab) \leq c \times \max\{\nu_A(a), \nu_A(b)\}.
$$

(15)
This shows that $M_{ab} \geq c$, $N_{ab} \leq c$. Let $P_1, P_2, \ldots, P_n$ be the $n$ points of intersections between considerable edges. Then for the point $P_i$ between two intersecting considerable edges $(ab, \mu_B(ab), \nu_B(ab))$ and $(cd, \mu_D(cd), \nu_D(cd))$, $M_{Pi} = (M_{ab} + M_{cd})/2 \geq c$, $N_{Pi} = (N_{ab} + N_{cd})/2 \leq c$. So $\sum_{i=1}^{n} M_{Pi} \geq n \times c$, $\sum_{i=1}^{n} N_{Pi} \leq n \times c$. Hence $M_i \leq 1/(1+n\times c)$, $N_i \geq 1/(1+n\times c)$. As $G$ is strong intuitionistic fuzzy planar graph, $0.5 < M_i \leq 1/(1+n\times c)$, $0.5 > N_i \geq 1/(1+n\times c)$. Hence $0.5 < 1/(1+n\times c)$. This implies $n < 1/c$. This inequality will be satisfied for some integral values of $n$ which are obtained from the following expression:

$$n = \begin{cases} \frac{1}{c}, & \text{if } \frac{1}{c} \text{ is not an integer} \\ \frac{1}{c} \times c, & \text{if } \frac{1}{c} \text{ is an integer.} \end{cases}$$ (16)

Face of an intuitionistic fuzzy planar graph is an important parameter. Face of an intuitionistic fuzzy graph is a region bounded by intuitionistic fuzzy edges. Every intuitionistic fuzzy face is characterized by intuitionistic fuzzy edges in its boundary. If all the edges in the boundary of an intuitionistic fuzzy face have membership and nonmembership values 1 and 0, respectively, it becomes crisp face. If one of such edges is removed or has membership and nonmembership values 0 and 1, respectively, the intuitionistic fuzzy face does not exist. So the existence of an intuitionistic fuzzy face depends on the minimum value of strength of intuitionistic fuzzy edges in its boundary. A intuitionistic fuzzy face and its membership and nonmembership values of an intuitionistic fuzzy graph are defined below.

**Definition 20.** Let $G$ be an intuitionistic fuzzy planar graph and $B = \{(xy, \mu_B(xy), \nu_B(xy)), i = 1, 2, \ldots, m \mid xy \in V \times V\}$. An intuitionistic fuzzy face of $G$ is a region, bounded by the set of intuitionistic fuzzy edges $E' \subset E$, of a geometric representation of $G$. The membership and nonmembership values of the intuitionistic fuzzy face are

$$\min \left\{ \frac{\mu_B(xy)_i}{\min \{\mu_A(x), \mu_A(y)\}}, i = 1, 2, \ldots, m \mid xy \in E' \right\},$$

$$\max \left\{ \frac{\nu_B(xy)_i}{\max \{\nu_A(x), \nu_A(y)\}}, i = 1, 2, \ldots, m \mid xy \in E' \right\}. $$ (17)

**Definition 21.** An intuitionistic fuzzy face is called strong intuitionistic fuzzy face if its membership value is greater than 0.5 or nonmembership value is less than 0.5, and weak face otherwise. Every intuitionistic fuzzy planar graph has an infinite region which is called outer intuitionistic fuzzy face. Other faces are called inner intuitionistic fuzzy faces.

**Example 22.** Consider an intuitionistic fuzzy planar graph as shown in Figure 4. The intuitionistic fuzzy planar graph has the following faces:

(i) intuitionistic fuzzy face $F_1$ is bounded by the edges $(v_1, v_2, 0.5, 0.1), (v_2, v_3, 0.6, 0.1)$, and $(v_1, v_3, 0.5, 0.1);

(ii) outer intuitionistic fuzzy face $F_2$ is bounded by edges $(v_4, v_1, 0.5, 0.1), (v_4, v_2, 0.5, 0.1), (v_4, v_3, 0.6, 0.1)$, and $(v_2, v_3, 0.6, 0.1);

(iii) intuitionistic fuzzy face $F_3$ is bounded by the edges $(v_1, v_2, 0.5, 0.1), (v_2, v_4, 0.6, 0.1), (v_1, v_4, 0.5, 0.1)$. Clearly, the membership value and nonmembership value of an intuitionistic fuzzy face $F_1$ are 0.833 and 0.333, respectively. Thus $F_1$ and $F_3$ are strong intuitionistic fuzzy faces.

We now introduce dual of intuitionistic fuzzy planar graph. In intuitionistic fuzzy dual graph, vertices are corresponding to the strong intuitionistic fuzzy faces of the intuitionistic fuzzy planar graph and each intuitionistic fuzzy edge between two vertices is corresponding to each edge in the boundary between two faces of intuitionistic fuzzy planar graph. The formal definition is given below.

**Definition 23.** Let $G$ be an intuitionistic fuzzy planar graph and let

$$B = \{(xy, \mu_B(xy), \nu_B(xy)), i = 1, 2, \ldots, m \mid xy \in V \times V\}. $$ (18)

Let $F_1, F_2, \ldots, F_k$ be the strong intuitionistic fuzzy faces of $G$. The intuitionistic fuzzy dual graph of $G$ is an intuitionistic fuzzy planar graph $G' = (V', A', B')$, where $V' = \{x_i, i = 1, 2, \ldots, k\}$, and the vertex $x_i$ of $G'$ is considered for the face $F_i$ of $G$. The membership and nonmembership values of vertices are given by the mapping $\mu_{A'}(x_i) = \max\{\mu_E(uv), i = 1, 2, \ldots, p \mid uv \text{ is an edge of the boundary of the strong intuitionistic fuzzy face } F_i\}$, and $v_{A'}(x_i) = \min\{\nu_E(uv), i = 1, 2, \ldots, p \mid uv \text{ is an edge of the boundary of the strong intuitionistic fuzzy face } F_i\}$.

There may exist more than one common edge between two faces $F_i$ and $F_j$ of $G$. Thus there may be more than one edge between two vertices $x_i$ and $x_j$ in intuitionistic fuzzy
dual graph $G'$. Let $\mu_{F'}(x_i, x_j)$ denote the membership value of the $i$th edge between $x_i$ and $x_j$, and $\nu_{F'}(x_i, x_j)$ denote the nonmembership value of the $i$th edge between $x_i$ and $x_j$. The membership and nonmembership values of the intuitionistic fuzzy edges of the intuitionistic fuzzy dual graph are given by $\mu_{F'}(x_i, x_j) = \mu_{G}(uv)_i$, $\nu_{F'}(x_i, x_j) = \nu_{G}(uv)_i$, where $(uv)_i$ is an edge in the boundary between two strong intuitionistic fuzzy faces $F_i$ and $F_j$, and $i = 1, 2, \ldots, s$, where $s$ is the number of common edges in the boundary between $F_i$ and $F_j$ or the number of edges between $x_i$ and $x_j$. If there is any strong pendant edge in the intuitionistic fuzzy planar graph, then there will be a self-loop in $G'$ corresponding to this pendant edge. The edge membership and nonmembership value of the self-loop is equal to the membership and nonmembership value of the pendant edge. Intuitionistic fuzzy dual graph of intuitionistic fuzzy planar graph does not contain point of intersection of edges for a certain representation, so it is intuitionistic fuzzy planar graph with planarity value $(1, 1)$. Thus the intuitionistic fuzzy face of intuitionistic fuzzy dual graph can be similarly described as in intuitionistic fuzzy planar graphs.

Example 24. Consider an intuitionistic fuzzy planar graph $G = (V, A, B)$ as shown in Figure 5 such that $V = \{a, b, c, d\}$, $A = (a, 0.6, 0.2), (b, 0.7, 0.2), (c, 0.8, 0.2), (d, 0.9, 0.1)$, and $B = \{(ab, 0.5, 0.01), (ac, 0.4, 0.01), (ad, 0.55, 0.01), (bc, 0.45, 0.01), (bd, 0.6, 0.01), (cd, 0.7, 0.01)\}$

![Figure 5: Intuitionistic fuzzy dual graph.](image)

The intuitionistic fuzzy planar graph has the following faces:

(i) intuitionistic fuzzy face $F_1$ is bounded by $(ab, 0.5, 0.01), (ac, 0.4, 0.01), (bc, 0.45, 0.01),$ 
(ii) intuitionistic fuzzy face $F_2$ is bounded by $(ad, 0.55, 0.01), (cd, 0.7, 0.01), (ac, 0.4, 0.01),$ 
(iii) intuitionistic fuzzy face $F_3$ is bounded by $(bc, 0.45, 0.01), (bd, 0.6, 0.01),$ 
(iv) outer intuitionistic fuzzy face $F_4$ is surrounded by $(ab, 0.5, 0.01), (bc, 0.6, 0.01), (cd, 0.7, 0.01), (ad, 0.55, 0.01).$

Routine calculations show that all faces are strong intuitionistic fuzzy faces. For each strong intuitionistic fuzzy face, we consider a vertex for the intuitionistic fuzzy dual graph. So the vertex set $V' = \{x_1, x_2, x_3, x_4\}$, where the vertex $x_i$ is taken corresponding to the strong intuitionistic fuzzy face $F_i$, $i = 1, 2, 3, 4$. Thus

$$
\begin{align*}
\mu_{A'}(x_1) &= \max\{0.5, 0.4, 0.45\} = 0.5, \\
\mu_{A'}(x_2) &= \max\{0.55, 0.7, 0.4\} = 0.7, \\
\nu_{A'}(x_1) &= \min\{0.01, 0.01, 0.01\} = 0.01, \\
\nu_{A'}(x_2) &= \min\{0.01, 0.01, 0.01\} = 0.01, \\
\mu_{A'}(x_3) &= \max\{0.45, 0.6\} = 0.6, \\
\mu_{A'}(x_4) &= \max\{0.5, 0.6, 0.7, 0.55\} = 0.7, \\
\nu_{A'}(x_3) &= \min\{0.01, 0.01\} = 0.01, \\
\nu_{A'}(x_4) &= \min\{0.01, 0.01, 0.01\} = 0.01.
\end{align*}
$$

There are two common edges $ad$ and $cd$ between the faces $F_2$ and $F_4$ in $G$. Hence between the vertices $x_2$ and $x_4$, there exist two edges in the intuitionistic fuzzy dual graph of $G$. Membership and nonmembership values of these edges are given by

$$
\begin{align*}
\mu_{B'}(x_2x_4) &= \mu_{B}(cd) = 0.7, \\
\mu_{B'}(x_2x_4) &= \mu_{B}(ad) = 0.55, \\
\nu_{B'}(x_2x_4) &= \nu_{B}(cd) = 0.01, \\
\nu_{B'}(x_2x_4) &= \nu_{B}(ad) = 0.01.
\end{align*}
$$

The membership and nonmembership values of other edges of the intuitionistic fuzzy dual graph are calculated as

$$
\begin{align*}
\mu_{B'}(x_1x_3) &= \mu_{B}(bc) = 0.45, \\
\mu_{B'}(x_1x_3) &= \mu_{B}(ac) = 0.4, \\
\mu_{B'}(x_1x_4) &= \mu_{B}(ab) = 0.5, \\
\mu_{B'}(x_3x_4) &= \mu_{B'}(bc) = 0.6, \\
\nu_{B'}(x_1x_3) &= \nu_{B}(bc) = 0.01, \\
\nu_{B'}(x_1x_4) &= \nu_{B}(ac) = 0.01, \\
\nu_{B'}(x_3x_4) &= \nu_{B}(bc) = 0.01.
\end{align*}
$$

Thus the edge set of intuitionistic fuzzy dual graph is

$$
B' = \{(x_1x_3, 0.45, 0.01), (x_1x_2, 0.4, 0.01), (x_1x_4, 0.5, 0.01),
\quad (x_3x_4, 0.6, 0.01), (x_2x_4, 0.7, 0.01), (x_2x_4, 0.55, 0.01)\}.
$$

In Figure 5, the intuitionistic fuzzy dual graph $G' = (V', A', B')$ of $G$ is drawn by dotted line.

Weak edges in planar graphs are not considered for any calculation in intuitionistic fuzzy dual graphs. We state the following theorems without their proofs.
Theorem 25. Let \( G \) be an intuitionistic fuzzy planar graph whose number of vertices, number of intuitionistic fuzzy edges, and number of strong faces are denoted by \( n, p, m \), respectively. Let \( G' \) be the intuitionistic fuzzy dual graph of \( G \). Then

(i) the number of vertices of \( G' \) is equal to \( m \),
(ii) number of edges of \( G' \) is equal to \( p \),
(iii) number of intuitionistic fuzzy faces of \( G' \) is equal to \( n \).

Theorem 26. Let \( G = (V, A, B) \) be an intuitionistic fuzzy planar graph without weak edges and let the intuitionistic fuzzy dual graph of \( G \) be \( G' = (V', A', B') \). The membership and nonmembership values of intuitionistic fuzzy edges of \( G' \) are equal to membership and nonmembership values of the intuitionistic fuzzy edges of \( G \).

We now study isomorphism between intuitionistic fuzzy planar graphs.

Definition 27. Let \( G_1 \) and \( G_2 \) be intuitionistic fuzzy graphs. An isomorphism \( f : G_1 \rightarrow G_2 \) is a bijective mapping \( f : V_1 \rightarrow V_2 \) which satisfies the following conditions:

(c) \( \mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1)) \),
(d) \( \mu_{B_1}(x_1,y_1) = \mu_{B_2}(f(x_1),f(y_1)), \nu_{B_1}(x_1,y_1) = \nu_{B_2}(f(x_1),f(y_1)) \),
for all \( x_1 \in V_1, x_1 y_1 \in E_1 \).

Definition 28. Let \( G_1 \) and \( G_2 \) be intuitionistic fuzzy graphs. Then, a weak isomorphism \( f : G_1 \rightarrow G_2 \) is a bijective mapping \( f : V_1 \rightarrow V_2 \) which satisfies the following conditions:

(e) \( f \) is homomorphism,
(f) \( \mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1)) \),
for all \( x_1 \in V_1 \).

Definition 29. Let \( G_1 \) and \( G_2 \) be the intuitionistic fuzzy graphs. A co-weak isomorphism \( f : G_1 \rightarrow G_2 \) is a bijective mapping \( f : V_1 \rightarrow V_2 \) which satisfies that

(g) \( f \) is homomorphism,
(h) \( \mu_{B_1}(x_1,y_1) = \mu_{B_2}(f(x_1),f(y_1)), \nu_{B_1}(x_1,y_1) = \nu_{B_2}(f(x_1),f(y_1)) \),
for all \( x_1 y_1 \in V_1 \).

It is known that isomorphism between intuitionistic fuzzy graphs is an equivalence relation. If there is an isomorphism between two intuitionistic fuzzy graphs such that one is an intuitionistic fuzzy planar graph, then the other will be intuitionistic fuzzy graph. We state the following result without its proof.

Theorem 30. Let \( G \) be an intuitionistic fuzzy planar graph and let \( H \) be an intuitionistic fuzzy graph. If there exists an isomorphism \( f : G \rightarrow H \), \( H \) can be drawn as intuitionistic fuzzy planar graph with same planarity value of \( G \).


