Research Article

Hyers-Ulam Stability of Iterative Equation in the Class of Lipschitz Functions

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Hyers-Ulam stability is a basic sense of stability for functional equations. In the present paper we discuss the Hyers-Ulam stability of a kind of iterative equations in the class of Lipschitz functions. By the construction of a uniformly convergent sequence of functions we prove that, for every approximate solution of such an equation, there exists an exact solution near it.

1. Introduction

It is pointed out that the theory of Ulam's type stability (also quite often connected, e.g., with the names of Bourgin, Găvruta, Aoki, Hyers, and Rassias) is a very popular subject of investigations at the moment in [1]. For more details of Ulam's type stability, we refer the audience to [1–3]. In [1], the authors present a survey of some selected recent developments (results and methods) in the theory of Ulam's type stability, such as the Forti method and the methods of fixed points and stability in non-Archimedean spaces. These results and methods have not been treated at all or have been treated only marginally. The very book of Jung [2] covers and offers almost all classic results on the Hyers-Ulam-Rassias stability in an integrated and self-contained fashion. And the authors of [3] discussed Hyers-Ulam stability for functional equations in single variable, including the forms of linear functional equations, nonlinear functional equations, and iterative equations. And they also clarified the relation between Hyers-Ulam stability and other senses of stability which are used for functional equations.

Let $C(X, X)$ be the set of all continuous self-mappings on a topological space X . For any $f \in C(X, X)$, let f^m denote the m th iteration of f ; that is, $f^m = f \circ f^{m-1}$, $f^0 = \text{id}$, $m = 1, 2, \dots$. Equations having iterations as their main operation, that is, including iterations of the unknown mapping, are called iterative equations. It is one of the most interesting classes of functional equations [4–7]. As a natural

generalization of the problem of iterative roots, iterative equations of the following form:

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \dots + \lambda_n f^n(x) = F(x), \quad (1)$$

$$x \in I = [a, b],$$

are known as polynomial-like iterative equations. Here $n \geq 2$ is an integer, $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$), $F : I \rightarrow \mathbb{R}$ is a given mapping, and $f : I \rightarrow I$ is unknown. As mentioned in [8, 9], polynomial-like iterative equations are important not only in the study of functional equations but also in the study of dynamical systems.

In 1986, Zhang [10] constructed an interesting operator called “structural operator” for (1) and used the fixed point theory in Banach space to get the solutions of (1). From then on (1) and other types of equations were discussed extensively by employing this idea (see [8, 9, 11–16] and references therein). In 2002, by means of a modification of Zhang's method applied in [10], Kulczycki and Tabor [11] investigated the existence of Lipschitzian solutions of the iterative equation

$$\sum_{i=1}^{\infty} a_i f^i(x) = F(x), \quad x \in \mathbb{B}, \quad (2)$$

where $\mathbb{B}$ is a compact convex subset of $\mathbb{R}^N$ and $F : \mathbb{B} \rightarrow \mathbb{B}$ is a given Lipschitz function. They generalized the results of Zhang [10] to a more general case.

In 2002, Xu and Zhang [17] discussed the Hyers-Ulam stability for a nonlinear iterative equation which includes the polynomial-like iterative equation. In 2003, Agarwal et al. [3] investigated the Hyers-Ulam stability for a general form of iterative equations which include the polynomial-like iterative equation with variable coefficients. Motivated by the above results, in this paper, we will discuss the Hyers-Ulam stability of (2). As in [11], we give a result on the Hyers-Ulam stability of a functional equation with a more general form firstly. And, by means of this result, the Hyers-Ulam stability of (2) is investigated. In fact, we want to generalize the result of Xu and Zhang [17] to high-dimensional case.

2. Basic Assumptions, Definitions, and Lemmas

Let $\mathbb{B}$ be a compact convex subset of $\mathbb{R}^N$, $N \in \mathbb{Z}^+$, with nonempty interior. Let

$$C(\mathbb{B}, \mathbb{R}^N) = \{f : \mathbb{B} \longrightarrow \mathbb{R}^N \mid f \text{ is continuous}\}. \quad (3)$$

In $C(\mathbb{B}, \mathbb{R}^N)$, we use the supremum norm

$$\|f\|_{\mathbb{B}} = \sup_{x \in \mathbb{B}} \|f(x)\|, \quad \text{for } \forall f \in C(\mathbb{B}, \mathbb{R}^N), \quad (4)$$

where $\|\cdot\|$ denotes the usual metric of $\mathbb{R}^N$. Obviously, $C(\mathbb{B}, \mathbb{R}^N)$ is a complete metric space.

Definition 1 (see [11]). Let $\mathbb{B}$ be a convex compact subset of $\mathbb{R}^N$ with nonempty interior. For $m \in [0, 1]$, and $M \in [1, \infty]$, one defines

$$\begin{aligned} \text{Lip}_{\mathbb{B}}(m, M) \\ := \{f : \mathbb{B} \longrightarrow \mathbb{B} \mid f \text{ is continuous, } f|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}, \\ \forall x, y \in \mathbb{B}, \end{aligned} \quad (5)$$

$$m\|x - y\| \leq \|f(x) - f(y)\| \leq M\|x - y\|,$$

where $\partial\mathbb{B}$ denotes the boundary of $\mathbb{B}$.

For a given function $f \in C(\mathbb{B}, \mathbb{R}^N)$, we define its Lipschitz constant by

$$\begin{aligned} \text{lip}(f) \\ := \inf \{M \geq 0 \mid \|f(x) - f(y)\| \leq M\|x - y\| \forall x, y \in \mathbb{B}\}. \end{aligned} \quad (6)$$

Lemma 2 (see [11]). Let $k \in [0, 1]$, $K \in [1, \infty]$, and $f \in \text{Lip}_{\mathbb{B}}(k, K)$ be arbitrary. One assumes additionally that f is invertible. Then

$$f^{-1} \in \text{Lip}_{\mathbb{B}}\left(\frac{1}{K}, \frac{1}{k}\right). \quad (7)$$

Lemma 3 (see [11]). For every $m \in (0, 1]$, the mapping

$$\mathcal{J} : \text{Lip}_{\mathbb{B}}(m, \infty) \ni f \longrightarrow f^{-1} \in \text{Lip}_{\mathbb{B}}\left(0, \frac{1}{m}\right) \quad (8)$$

is well defined with Lipschitz constant $1/m$.

Lemma 4 (see [11]). For $K, M < \infty$, $F \in \text{Lip}_{\mathbb{B}}(0, K)$, the mapping defined by

$$\mathcal{S}_F : \text{Lip}_{\mathbb{B}}(0, M) \ni f \longrightarrow f \circ F \in \text{Lip}_{\mathbb{B}}(0, KM) \quad (9)$$

is Lipschitz with constant 1.

Lemma 5. Let $m \in [0, 1]$, and $M \in [1, \infty]$ be given. For any $f \in \text{Lip}_{\mathbb{B}}(m, M)$, $f : \mathbb{B} \rightarrow \mathbb{B}$ is surjective.

Proof. Note that $\mathbb{B}$ is homeomorphic to D^N , where $D^N = \{x \in \mathbb{R}^N : \|x\| \leq 1\}$ is the unit ball of $\mathbb{R}^N$. Without any loss of generality, let $\mathbb{B} = D^N$. Since $f|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}$, set $f|_{\partial\mathbb{B}} = f_0$; we have $\deg(f_0) = 1$, where $\deg(f_0)$ denotes the degree of f_0 . Suppose that f is not surjective. Let $x_0 \in \mathbb{B} \setminus f(\mathbb{B})$. Since $f|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}$, we know that $x_0 \notin \partial\mathbb{B}$; then there exists a retraction mapping $r : \mathbb{B} \setminus \{x_0\} \rightarrow \partial\mathbb{B}$. Thus $r \circ f : \mathbb{B} \rightarrow \partial\mathbb{B}$ is a continuous mapping and $f_0 = (r \circ f)|_{\partial\mathbb{B}}$. This means that $\deg(f_0) = 0$, a contradiction. So f is surjective. $\square$

Lemma 6. If f, g are homeomorphisms of $\mathbb{B}$ with Lipschitz constant M , then

$$\|f - g\|_{\mathbb{B}} \leq M\|f^{-1} - g^{-1}\|_{\mathbb{B}}. \quad (10)$$

Proof. Note that

$$\begin{aligned} 0 &= \|f^{-1} \circ f - g^{-1} \circ g\|_{\mathbb{B}} \\ &= \|f^{-1} \circ f - g^{-1} \circ f + g^{-1} \circ f - g^{-1} \circ g\|_{\mathbb{B}}. \end{aligned} \quad (11)$$

So we have

$$f^{-1} \circ f - g^{-1} \circ f = -g^{-1} \circ f + g^{-1} \circ g. \quad (12)$$

Furthermore

$$\begin{aligned} \|f^{-1} - g^{-1}\|_{\mathbb{B}} &= \|f^{-1} \circ f - g^{-1} \circ f\|_{\mathbb{B}} \\ &= \|g^{-1} \circ f - g^{-1} \circ g\|_{\mathbb{B}} \\ &\geq \frac{1}{M}\|f - g\|_{\mathbb{B}}. \end{aligned} \quad (13)$$

This means that $\|f - g\|_{\mathbb{B}} \leq M\|f^{-1} - g^{-1}\|_{\mathbb{B}}$. $\square$

Lemma 7. Suppose that $f, g : \mathbb{B} \rightarrow \mathbb{B}$ are continuous mappings and $\text{lip}(f) = M$, where M is a positive constant. Then

$$\|f^k - g^k\|_{\mathbb{B}} \leq \sum_{j=0}^{k-1} M^j \|f - g\|_{\mathbb{B}}, \quad \forall k = 1, 2, \dots \quad (14)$$

Proof. We use induction to prove this lemma. For $k = 1$, $\|f - g\|_{\mathbb{B}} \leq \|f - g\|_{\mathbb{B}}$. Suppose that this is true, for k ; then

$$\begin{aligned} \|f^{k+1} - g^{k+1}\|_{\mathbb{B}} &\leq \|f \circ f^k - f \circ g^k\|_{\mathbb{B}} + \|f \circ g^k - g \circ g^k\|_{\mathbb{B}} \\ &\leq M\|f^k - g^k\|_{\mathbb{B}} + \|f - g\|_{\mathbb{B}} \\ &\leq M \sum_{j=0}^{k-1} M^j \|f - g\|_{\mathbb{B}} + \|f - g\|_{\mathbb{B}} \\ &= \sum_{j=0}^k M^j \|f - g\|_{\mathbb{B}}. \end{aligned} \quad (15)$$

□

Let A be a subset of $\text{Lip}_{\mathbb{B}}(0, \infty)$ and satisfy

(A1): for all $f, g \in A$, $f \circ g \in A$;

(A2): if $f \in A$ is invertible, then $f^{-1} \in A$;

(A3): A is closed and convex.

And let

$$\mathfrak{A} = \{A \subset \text{Lip}_{\mathbb{B}}(0, \infty) \mid (A1), (A2), \text{ and } (A3) \text{ hold}\}. \quad (16)$$

It is easy to see that $\text{Lip}_{\mathbb{B}}(0, \infty) \in \mathfrak{A}$.

Let $A \in \mathfrak{A}$ and let $\mathcal{P} : A \rightarrow A$ be a map. For any $F \in A$ we will consider a functional equation of the following form:

$$\mathcal{P}(f) \circ f = F, \quad (17)$$

where $f \in A$ is unknown.

Definition 8. If, for every $g \in \text{Lip}_{\mathbb{B}}(0, \infty)$ such that $\|F - \mathcal{P}(g) \circ g\|_{\mathbb{B}} \leq \delta$, there exists a solution f of (17) such that $\|f - g\|_{\mathbb{B}} \leq K\delta$, where $K > 0$ is a constant and K does not depend on the choice of g , then (17) is said to have Hyers-Ulam stability.

3. Main Results

Lemma 9. Let $\mathbb{B}$ be a convex compact subset of $\mathbb{R}^N$ with nonempty interior. Let $A \in \mathfrak{A}$ and $F \in A$ with $\text{lip}(F) \leq M_0$. Let $\mathcal{P} : A \rightarrow A$ be given. If there exists a decreasing function $\alpha : [1, \infty) \rightarrow [0, 1]$ and a constant $M \in [1, \infty)$ such that

(P1): for any $f \in A$, $\mathcal{P}(f) \in \text{Lip}_{\mathbb{B}}(\alpha(\text{lip}(f)), \infty)$;

(P2): $M \cdot \alpha(M) \geq M_0$.

Then the following two facts hold:

(c1): for any $f \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$, $\mathcal{P}(f)$ is a homeomorphism with $\text{lip}((\mathcal{P}(f))^{-1}) \leq 1/\alpha(M)$;

(c2): for any given $f_0 \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$, the sequence

$$f_k = \mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P}(f_{k-1}), \quad k = 1, 2, \dots, \quad (18)$$

is well defined and each $f_k \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$ is surjective.

Proof. (c1): since $F|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}$, we get that $M_0 \geq 1$. By inequality (P2) and the range of the function α , we know that both $\alpha(M) \in (0, 1]$ and $M \geq M_0$ hold. Therefore $F \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$, which implies that $A \cap \text{Lip}_{\mathbb{B}}(0, M)$ is nonempty. For any $x_1 \neq x_2 \in \mathbb{B}$ and any $f \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$, since $\text{lip}(f) \leq M$ and α is decreasing, we have

$$\begin{aligned} \|\mathcal{P}(f)(x_1) - \mathcal{P}(f)(x_2)\| \\ \geq \alpha(\text{lip}(f)) \|x_1 - x_2\| \geq \alpha(M) \|x_1 - x_2\| > 0. \end{aligned} \quad (19)$$

This means that $\mathcal{P}(f)$ is injective. Since $\mathcal{P}(f) \in A$, by Lemma 5, we obtain that $\mathcal{P}(f)$ is surjective. Thus $\mathcal{P}(f)$ is a homeomorphism. And $\text{lip}((\mathcal{P}(f))^{-1}) \leq 1/\alpha(M)$ follows from Lemma 2.

(c2): by Lemma 3, the mapping

$$\mathcal{J} : \text{Lip}_{\mathbb{B}}(\alpha(M), \infty) \ni g \longrightarrow g^{-1} \in \text{Lip}_{\mathbb{B}}\left(0, \frac{1}{\alpha(M)}\right) \quad (20)$$

is continuous. By Lemma 4 and assumptions of F , the mapping

$$\mathcal{S}_F : \text{Lip}_{\mathbb{B}}\left(0, \frac{1}{\alpha(M)}\right) \ni g \longrightarrow g \circ F \in \text{Lip}_{\mathbb{B}}\left(0, \frac{M_0}{\alpha(M)}\right) \quad (21)$$

is well defined and continuous.

Now, by the assumption (P1) and the fact $\mathcal{P} : A \rightarrow A$, it is easy to see that, for every $g \in \text{Lip}_{\mathbb{B}}(0, M) \cap A$,

$$\mathcal{P}(g) \in \text{Lip}_{\mathbb{B}}(\alpha(M), \infty) \cap A. \quad (22)$$

Since (A2) holds and by Lemma 2, we know that all elements of $\text{Lip}_{\mathbb{B}}(\alpha(M), \infty)$ are invertible. Moreover we can get that

$$\mathcal{J} \circ \mathcal{P}(g) \in \text{Lip}_{\mathbb{B}}\left(0, \frac{1}{\alpha(M)}\right) \cap A, \quad (23)$$

and by means of (A1) we have

$$\mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P}(g) \in \text{Lip}_{\mathbb{B}}\left(0, \frac{M_0}{\alpha(M)}\right) \cap A. \quad (24)$$

Finally, by inequality (P2), we obtain that the mapping

$$\mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P} : \text{Lip}_{\mathbb{B}}(0, M) \cap A \longrightarrow \text{Lip}_{\mathbb{B}}(0, M) \cap A \quad (25)$$

is well defined.

Take any $f_0 \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$. By means of the mapping (25), it is easy to see that the sequence

$$f_k = \mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P}(f_{k-1}), \quad k = 1, 2, \dots, \quad (26)$$

is well defined and each $f_k \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$. By virtue of Lemma 5, it is easy to see that all f_k are surjective. □

Theorem 10. Let $\mathbb{B}$ be a convex compact subset of $\mathbb{R}^N$ with nonempty interior. Let $A \in \mathfrak{A}$ and $F \in A$ with $\text{lip}(F) \leq M_0$. Let $\mathcal{P} : A \rightarrow A$ be given and let there exist a decreasing function $\alpha : [1, \infty) \rightarrow [0, 1]$ and a constant $M \in [1, \infty)$ such that

(P1) and (P2) of Lemma 9 hold. Moreover, if there is a constant $0 < \eta < \alpha(M)$ such that

$$\begin{aligned} \|\mathcal{P}(g_1) - \mathcal{P}(g_2)\|_{\mathbb{B}} &\leq \eta \|g_1 - g_2\|_{\mathbb{B}}, \\ \forall g_1, g_2 &\in A \cap \text{Lip}_{\mathbb{B}}(0, M), \end{aligned} \quad (27)$$

then, for any $g \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$ with

$$\|F - \mathcal{P}(g) \circ g\|_{\mathbb{B}} \leq \delta, \quad (28)$$

where $\delta > 0$ is a constant, there exists a unique continuous solution $f \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$ of (17) such that

$$\|f - g\|_{\mathbb{B}} \leq K\delta, \quad (29)$$

where

$$K = \frac{1}{\alpha(M) - \eta}. \quad (30)$$

Proof. Since all the assumptions of Lemma 9 hold, so we can construct a sequence $\{f_k\}$ of functions as follows. We take

$$f_0 = g \quad (31)$$

first and then by means of equality (18) repeatedly, define

$$f_k = \mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P}(f_k), \quad k = 1, 2, \dots \quad (32)$$

By the conclusions (c1) and (c2) of Lemma 9, both $\mathcal{P}(f_{k-1})$ and f_k are well defined for all $k > 1$. And each $\mathcal{P}(f_k)$ is a homeomorphism of $\mathbb{B}$ onto itself with $\text{lip}((\mathcal{P}(f_k))^{-1}) \leq 1/\alpha(M)$.

Now we claim that both

$$\begin{aligned} \|f_k - f_{k-1}\|_{\mathbb{B}} &\leq \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^{k-1} \delta, \\ \|F - \mathcal{P}(f_k) \circ f_k\|_{\mathbb{B}} &\leq \left(\frac{\eta}{\alpha(M)} \right)^k \delta \end{aligned} \quad (33)$$

hold for $k = 1, 2, \dots$. We prove these two inequalities inductively.

For $k = 1$, by virtue of the definition of f_1 , we have

$$\begin{aligned} \|f_1 - f_0\|_{\mathbb{B}} &= \|\mathcal{S}_F \circ \mathcal{J} \circ \mathcal{P}(f_0) - f_0\|_{\mathbb{B}} \\ &= \|(\mathcal{P}(f_0))^{-1} \circ F - (\mathcal{P}(f_0))^{-1} \circ (\mathcal{P}(f_0)) \circ f_0\|_{\mathbb{B}} \\ &\leq \frac{1}{\alpha(M)} \|F - (\mathcal{P}(f_0)) \circ f_0\|_{\mathbb{B}} = \frac{1}{\alpha(M)} \delta. \end{aligned} \quad (34)$$

And noting that $F = \mathcal{P}(f_{k-1}) \circ f_k$ and $f_k : \mathbb{B} \rightarrow \mathbb{B}$ is surjective for $k = 1, 2, \dots$, we have

$$\begin{aligned} \|F - \mathcal{P}(f_1) \circ f_1\|_{\mathbb{B}} &= \|\mathcal{P}(f_0) \circ f_1 - \mathcal{P}(f_1) \circ f_1\|_{\mathbb{B}} \\ &= \|\mathcal{P}(f_0) - \mathcal{P}(f_1)\|_{\mathbb{B}} \\ &\leq \eta \|f_1 - f_0\|_{\mathbb{B}} = \frac{\eta}{\alpha(M)} \delta. \end{aligned} \quad (35)$$

Assume that they are true for integer k . Then, by the same arguments, we can get that

$$\begin{aligned} \|f_{k+1} - f_k\|_{\mathbb{B}} &= \|(\mathcal{P}(f_k))^{-1} \circ F - (\mathcal{P}(f_k))^{-1} \circ (\mathcal{P}(f_k)) \circ f_k\|_{\mathbb{B}} \\ &\leq \frac{1}{\alpha(M)} \|F - (\mathcal{P}(f_k)) \circ f_k\|_{\mathbb{B}} \leq \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^k \delta, \\ \|F - \mathcal{P}(f_{k+1}) \circ f_{k+1}\|_{\mathbb{B}} &= \|\mathcal{P}(f_k) \circ f_{k+1} - \mathcal{P}(f_{k+1}) \circ f_{k+1}\|_{\mathbb{B}} \\ &= \|\mathcal{P}(f_k) - \mathcal{P}(f_{k+1})\|_{\mathbb{B}} \leq \eta \|f_{k+1} - f_k\|_{\mathbb{B}} \\ &= \left(\frac{\eta}{\alpha(M)} \right)^{k+1} \delta. \end{aligned} \quad (36)$$

Thus (33) is proved by induction.

For any positive integers k and s with $k > s$, we have

$$\begin{aligned} \|f_k - f_s\|_{\mathbb{B}} &\leq \|f_k - f_{k-1}\|_{\mathbb{B}} + \|f_{k-1} - f_{k-2}\|_{\mathbb{B}} \\ &\quad + \dots + \|f_{s+1} - f_s\|_{\mathbb{B}} \\ &\leq \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^{k-1} \delta + \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^{k-2} \delta \\ &\quad + \dots + \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^s \delta \\ &= \frac{\delta}{\alpha(M)} \cdot \frac{(\eta/\alpha(M))^s - (\eta/\alpha(M))^k}{1 - \eta/\alpha(M)}. \end{aligned} \quad (37)$$

Since $\alpha(M) > \eta > 0$, it follows from inequality (37) that

$$\|f_k - f_s\|_{\mathbb{B}} \longrightarrow 0, \quad \text{as } k > s \longrightarrow \infty. \quad (38)$$

As a Cauchy sequence, $\{f_k\}_{k=0}^{\infty}$ converges uniformly in the Banach space $C(\mathbb{B}, \mathbb{R}^N)$. Let

$$\lim_{k \rightarrow \infty} f_k = f. \quad (39)$$

Clearly, $f \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$. From

$$\begin{aligned} \|F - \mathcal{P}(f) \circ f\|_{\mathbb{B}} &= \lim_{k \rightarrow \infty} \|F - \mathcal{P}(f_k) \circ f_k\|_{\mathbb{B}} \\ &= \lim_{k \rightarrow \infty} \left(\frac{\eta}{\alpha(M)} \right)^k \delta = 0, \end{aligned} \quad (40)$$

we know that f is a solution of (17). Furthermore,

$$\begin{aligned}
 \|f - g\|_{\mathbb{B}} &= \lim_{k \rightarrow \infty} \|f_k - f_0\|_{\mathbb{B}} \\
 &\leq \lim_{k \rightarrow \infty} \{\|f_k - f_{k-1}\|_{\mathbb{B}} + \dots + \|f_1 - f_0\|_{\mathbb{B}}\} \\
 &\leq \lim_{k \rightarrow \infty} \left\{ \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^{k-1} \delta + \frac{1}{\alpha(M)} \left(\frac{\eta}{\alpha(M)} \right)^{k-2} \delta \right. \\
 &\quad \left. + \dots + \frac{1}{\alpha(M)} \delta \right\} \\
 &= \frac{1}{\alpha(M) - \eta} \delta.
 \end{aligned} \tag{41}$$

This proves (29).

Concerning uniqueness, we assume that there is another continuous solution $\phi \in A \cap \text{Lip}_{\mathbb{B}}(0, M)$ for (17), such that

$$\|\phi - g\|_{\mathbb{B}} \leq \varepsilon, \tag{42}$$

where $\varepsilon > 0$ only depends on δ . By Lemma 5, we know that $F : \mathbb{B} \rightarrow \mathbb{B}$ is also surjective. By means of Lemma 6 and the assumption that $\mathcal{P}$ is Lipschitzian, it follows that

$$\begin{aligned}
 \|f - \phi\|_{\mathbb{B}} &= \|(\mathcal{P}(f))^{-1} \circ F - (\mathcal{P}(\phi))^{-1} \circ F\|_{\mathbb{B}} \\
 &= \|(\mathcal{P}(f))^{-1} - (\mathcal{P}(\phi))^{-1}\|_{\mathbb{B}} \\
 &\leq \frac{1}{\alpha(M)} \|\mathcal{P}(f) - \mathcal{P}(\phi)\|_{\mathbb{B}} \\
 &\leq \frac{\eta}{\alpha(M)} \|f - \phi\|_{\mathbb{B}}.
 \end{aligned} \tag{43}$$

This implies that

$$\left(1 - \frac{\eta}{\alpha(M)}\right) \|f - \phi\|_{\mathbb{B}} \leq 0. \tag{44}$$

However, $\alpha(M) > \eta > 0$; this implies that $\|f - \phi\|_{\mathbb{B}} = 0$; that is, $f = \phi$. $\square$

4. Iterative Equations in $\mathbb{R}^N$

Theorem 11. Let $\mathbb{B}$ be a convex compact subset of $\mathbb{R}^N$ with nonempty interior. Let $F \in \text{Lip}_{\mathbb{B}}(0, \infty)$ be arbitrary with $\text{lip}(F) \leq M_0$. Assume that $a_k \in [0, 1]$, $k = 0, 1, 2, \dots$, and there exists a constant $p \in [1, \infty]$ such that

- (a1): $\sum_{k=0}^{\infty} a_k = 1$;
- (a2): $a_0 p > a_0 p - \sum_{k=1}^{\infty} a_k p^{k+1} \geq M_0$;
- (a3): $\sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} p^j + \sum_{k=1}^{\infty} a_k p^k < a_0$

hold. Then the equation

$$\sum_{k=0}^{\infty} a_k f^{k+1}(x) = F(x), \quad x \in \mathbb{B}, \tag{45}$$

has Hyers-Ulam stability.

Proof. Let $A = \text{Lip}_{\mathbb{B}}(0, \infty)$. For $\forall f \in A$, define

$$\mathcal{P}(f) = \sum_{k=0}^{\infty} a_k f^k. \tag{46}$$

We claim that, for $\forall f \in A$, $\mathcal{P}(f) \in A$. For $\forall x \in \mathbb{B}$, by the compactness and convexity of $\mathbb{B}$ and the assumption (a1), we have

$$\sum_{k=0}^{\infty} a_k f^k(x) \in \overline{\text{co}\{x, f(x), f^2(x), \dots\}} \subset \mathbb{B}, \tag{47}$$

where $\text{co}\{x, f(x), f^2(x), \dots\}$ is the convex hull of the set $\{x, f(x), f^2(x), \dots\}$. Moreover, by means of the assumption (a1) and the definition of the mapping $\mathcal{P}$ and the set A , it is easy to see that $\mathcal{P}(f)|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}$. Now we prove the continuity of $\mathcal{P}(f)$. By the compactness of $\mathbb{B}$, we first denote

$$L = \max_{x \in \mathbb{B}} \{\|x\|\}. \tag{48}$$

By the assumption (a1), for any $\epsilon > 0$, there exists a positive integer l_0 such that

$$\sum_{k=l_0+1}^{\infty} a_k \leq \frac{\epsilon}{4L}. \tag{49}$$

Note that each f^k , $k = 0, 1, 2, \dots, l_0$, is continuous. Then, for any $x \in \mathbb{B}$, there is a $\delta > 0$ such that when $\|y - x\| < \delta$ the fact

$$\|f^k(y) - f^k(x)\| \leq \frac{\epsilon}{2(l_0 + 1)}, \quad k = 0, 1, 2, \dots, l_0, \tag{50}$$

holds. Thus we can get that

$$\begin{aligned}
 \|\mathcal{P}f(y) - \mathcal{P}f(x)\| &\leq \sum_{k=0}^{l_0} a_k \|f^k(y) - f^k(x)\| + \sum_{k=l_0+1}^{\infty} a_k \|f^k(y) - f^k(x)\| \\
 &\leq \sum_{k=0}^{l_0} \frac{\epsilon}{2(l_0 + 1)} + 2L \sum_{k=l_0+1}^{\infty} a_k = \epsilon.
 \end{aligned} \tag{51}$$

This means that the fact $\mathcal{P} : A \rightarrow A$ holds.

By virtue of the mapping $\mathcal{P}$, (45) can be rewritten as

$$\mathcal{P}(f) \circ f = F. \tag{52}$$

Since $F|_{\partial\mathbb{B}} = \text{id}|_{\partial\mathbb{B}}$, this implies that $M_0 \geq 1$. And by the assumption (a2), we see that $a_0 > 0$. For $x \in [1, \infty)$, define

$$\alpha(x) = \max \left\{ a_0 - \sum_{k=1}^{\infty} a_k x^k, 0 \right\}. \tag{53}$$

Since $\alpha(x) \leq a_0 \leq 1$ we obtain that $\alpha : [1, \infty) \rightarrow [0, 1]$ and it is easy to see that α is decreasing.

For any $x \in [1, \infty)$ and any $g \in \text{Lip}_{\mathbb{B}}(0, \infty)$ with $\text{lip}(g) = x$, we claim that $\mathcal{P}(g) \in \text{Lip}_{\mathbb{B}}(\alpha(x), \infty)$. It is enough to show that, for $\forall b, c \in \mathbb{B}$,

$$\alpha(x) \|b - c\| \leq \|\mathcal{P}(g)(b) - \mathcal{P}(g)(c)\|. \quad (54)$$

But

$$\begin{aligned} & \|\mathcal{P}(g)(b) - \mathcal{P}(g)(c)\| \\ &= \left\| \sum_{k=0}^{\infty} a_k (g^k(b) - g^k(c)) \right\| \\ &\geq a_0 \|b - c\| - \sum_{k=1}^{\infty} a_k \|g^k(b) - g^k(c)\| \\ &\geq a_0 \|b - c\| - \sum_{k=1}^{\infty} a_k x^k \|b - c\| \\ &= \left(a_0 - \sum_{k=1}^{\infty} a_k x^k \right) \|b - c\| = \alpha(x) \|b - c\|. \end{aligned} \quad (55)$$

It follows from the assumption (a2) that

$$p \cdot \alpha(p) = a_0 p - \sum_{k=1}^{\infty} a_k p^{k+1} \geq M_0. \quad (56)$$

Thus (P1) and (P2) of Lemma 9 also hold. Moreover, we can get that

$$\begin{aligned} \|\mathcal{P}(f) - \mathcal{P}(g)\|_{\mathbb{B}} &= \left\| \sum_{k=0}^{\infty} a_k (f^k - g^k) \right\|_{\mathbb{B}} \\ &\leq \sum_{k=0}^{\infty} a_k \|f^k - g^k\|_{\mathbb{B}} \\ &\leq \sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} p^j \|f - g\|_{\mathbb{B}} \end{aligned} \quad (57)$$

hold. Let $\eta = \sum_{k=1}^{\infty} a_k \sum_{j=0}^{k-1} p^j$. By the assumption (a2), we know that there must be a positive integer k_0 such that $a_{k_0} \neq 0$. So we know that $\eta > 0$. Furthermore, by the assumption (a3), we have $0 < \eta < \alpha(p)$. Then, by Theorem 10, we get the conclusion. $\square$

Example 12. For $F(x) = (x_1, x_2, x_3, \dots, x_N) + \frac{\text{dist}(x, S^{N-1})}{\sqrt{x_1^2 + x_2^2 + \dots + x_N^2}}(1, 0, 0, \dots, 0) = (1 + x_1 - \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}, x_2, \dots, x_N)$, where $x = (x_1, x_2, \dots, x_N) \in D^N$ and $\text{dist}(x, S^{N-1})$ denotes the distance of the point x from S^{N-1} , obviously, $F|_{S^{N-1}} = \text{id}|_{S^{N-1}}$. By simple calculation, we get that, for any $x, y \in D^N$, the following two inequalities:

$$\begin{aligned} & \|F(x) - F(y)\| \leq 2 \|x - y\|, \\ & \|F(x)\| \leq \|x\| + \text{dist}(x, S^{N-1}) = 1 \end{aligned} \quad (58)$$

hold. This implies that $F \in \text{Lip}_{D^N}(0, 2)$. Then we consider the equation

$$\frac{254}{255} f(x) + \sum_{i=1}^{\infty} \left(\frac{1}{256} \right)^i f^{i+1}(x) = F(x), \quad x \in D^N, \quad (59)$$

$$F : D^N \longrightarrow D^N, \quad F|_{S^{N-1}} = \text{id}|_{S^{N-1}}.$$

Denote $a_0 = 254/255$, $a_k = (1/256)^k$, $k = 1, 2, \dots$. Define $\alpha(x) = \max\{(254/255) - \sum_{i=1}^{\infty} (1/256)^i x^i, 0\}$. Let $M_0 = 2$ and $p = 3$. Then we have

$$\begin{aligned} \sum_{k=0}^{\infty} a_k &= 1, \quad \frac{254}{255} \times 3 > 3\alpha(3) = \frac{190491}{64515} > 2 \geq \text{lip}(F), \\ \frac{254}{255} &> \sum_{k=1}^{\infty} \left(\frac{1}{256} \right)^k \sum_{j=0}^{k-1} 3^j + \sum_{i=1}^{\infty} \left(\frac{3}{256} \right)^i = \frac{1021}{64515}. \end{aligned} \quad (60)$$

By Theorem 11, (59) has Hyers-Ulam stability.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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