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Research Article

Adaptive Synchronization between Fractional-Order Chaotic Real and Complex Systems with Unknown Parameters

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The complex modified projective synchronization (CMPS) between fractional-order chaotic real and complex systems is investigated for the first time. The parameters of both master and slave systems are assumed to be unknown in advance; moreover, the slave system is perturbed by unknown but bounded external disturbances. The master and slave systems that achieved CMPS can be synchronized up to a complex constant matrix. On the basis of frequency distributed model of fractional integrator and Lyapunov stability theory, a robust adaptive control law is designed to realize the CMPS for two different types of fractional-order chaotic systems. Meanwhile, to deal with these unknown parameters, some fractional-order type parametric update laws are provided. An example is given to demonstrate the effectiveness and feasibility of the proposed synchronization scheme.

1. Introduction

Although fractional calculus is a mathematical topic with more than 300-year history, its applications in the fields of physics and engineering have attracted lots of attentions only in the recent years. It was found that, with the help of fractional calculus, many systems in interdisciplinary fields can be described more accurately. That is to say, fractional calculus provides a superb instrument for the description of memory and hereditary properties of various materials and processes. Many literatures have proven that some fractional-order differential systems can behave chaotically [1–4].

The research on chaotic systems has grown significantly over the past decades and has become a popular topic. For example, Zhang and Yang [5] investigated a single driving variable approach to realize the adaptive stabilization of chaotic systems. Taghvafard and Erjaee [6] proposed an active control method to perform the phase and antiphase synchronization of two fractional-order chaotic systems. Aghababa [7] used finite-time theory to realize the finite-time synchronization of chaotic systems. Lu [8] developed a nonlinear observer to synchronize the chaotic systems. Chen et al. [9, 10] researched the chaos synchronization of fractional-order chaotic neural network. Recently, we [11] applied the sliding mode control strategy to stabilize a class of fractional-order chaotic systems with input nonlinearity.

However, all of the aforementioned researches only focus on the systems with real variables, while chaotic complex systems are not involved. Actually, chaotic complex systems can be widely used to describe a variety of physical phenomena, such as population inversion [12], detuned laser systems [13], and thermal convections of liquid flows [14]. At present, some control schemes have been proposed to synchronize two chaotic systems with complex variables [15–19]. But, it should be noted that all of chaotic complex nonlinear systems in the abovementioned literatures are integer-order systems. In recent years, several fractional-order chaotic complex systems are investigated [20–22]. Nevertheless, in [20–22], the systems' parameters are taken as known in the synchronization of two identical fractional-order chaotic complex systems. As a matter of fact, many systems' parameters cannot be exactly determined in advance. The synchronization will not be achieved under the effects of unknown uncertainties. Therefore, it is urgent to consider the impacts of unknown parameters in synchronizing two chaotic complex systems.

On the other hand, in many practical systems, the master (drive) system and slave (response) system may evolve in different directions with a constant intersection angle. Thus,

the complex modified projective synchronization (CMPS) should be taken into consideration. This kind of CMPS is viewed as the generalization of several types of synchronization, such as complete synchronization (CS), antisynchronization (AS), modified projective synchronization (MPS), and projective synchronization (PS). However, to the best of our knowledge, up until now, there is no information available for the CMPS between fractional-order chaotic real and complex systems with unknown parameters.

Motivated by the above discussions, in this paper, the CMPS between two different fractional-order chaotic systems with unknown parameters and external disturbances is researched for the first time. Since the stability theory of integer-order system cannot be directly applied to fractional-order systems, so, inspired by contributions from [23, 24], we use frequency distributed model of fractional integrator and Lyapunov stability theory to prove the stability of closed-loop system.

The rest of this paper is organized as follows. In Section 2, the relevant definitions and lemma are introduced. In Section 3, the adaptive controller and update laws of unknown parameters are designed in detail. Simulation results are presented in Section 4. Finally, conclusions are included in Section 5.

2. Preliminaries

The Grunwald-Letnikov definition, Riemann-Liouville definition, and Caputo definition are the most frequently used definitions for fractional calculus.

Definition 1. The α order Riemann-Liouville fractional integration is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function, determined by

$$ \Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt. \tag{2} $$

Definition 2. For $n-1 < \alpha \le n$, the Riemann-Liouville fractional derivative of order α is defined as

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} f(t) &= \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\ &= \frac{d^n}{dt^n} I^{n-\alpha} f(t). \end{aligned} \tag{3} $$

Definition 3. The Grunwald-Letnikov fractional derivative of order α is written as

$$ {}_{t_0}D_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{[(t-t_0)/h]} (-1)^j \binom{\alpha}{j} f(t-jh). \tag{4} $$

Definition 4. The Caputo fractional derivative of order α is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m, \end{cases} \tag{5} $$

where m is the smallest integer number, larger than α.

In the rest of this paper, we will use the Caputo definition to describe the fractional chaotic systems and use the Grunwald-Letnikov approach to perform numerical simulations. For simplicity, in the rest of this paper, we use D^{α} instead of ${}_0D_t^{\alpha}$.

Lemma 5 (see [25–27]). *Consider a nonlinear fractional-order system*

$$ D^{\alpha} x(t) = f(x(t)), \tag{6} $$

where $\alpha \in (0,1)$. Then the system is exactly equivalent to the following continuous frequency distributed model:

$$ \begin{aligned} \frac{\partial z(\omega,t)}{\partial t} &= -\omega z(\omega,t) + f(x(t)) \\ x(t) &= \int_0^{\infty} \mu(\omega) z(\omega,t) d\omega, \end{aligned} \tag{7} $$

where $\mu(\omega) = ((\sin(\alpha\pi))/\pi)\omega^{-\alpha}$. In the above equations, $z(\omega,t)$ is the true state variable, and $x(t)$ is the pseudostate variable.

3. Main Results

Take the following n-dimensional chaotic real system as master system:

$$ D^{\alpha} x = F(x)\theta + f(x), \tag{8} $$

where $x = (x_1, x_2, \ldots, x_n)^T$ is a real state vector. $F(x) \in R^{n\times m_1}$ is a real matrix and its elements are the functions of state variables. $\theta \in R^{m_1}$ is a vector of unknown parameters. $f(x) \in R^n$ is a vector of continuous nonlinear functions.

Consider the following n-dimensional chaotic complex system with external disturbances as slave system:

$$ D^{\alpha} y = G(y)\delta + g(y) + d(t) + W(t), \tag{9} $$

where $y = (y_1, y_2, \ldots, y_n)^T$ is a complex state vector, and $y_k = y_k^r + jy_k^i$, $j = \sqrt{-1}$, and superscripts r and i are real and imaginary parts of state variables, respectively. $G(y) \in C^{n\times m_2}$ is a complex matrix and its elements are the functions of complex state variables. $\delta \in R^{m_2}$ (or C^{m_2}) is a vector of unknown parameters. $g(y) \in C^n$ is a vector of complex nonlinear functions. $d(t) = (d_1(t), d_2(t), \ldots, d_n(t))^T \in C^n$ is a vector of external disturbances. $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))^T$ is a vector of controllers to be designed later.

Remark 6. Most of fractional-order chaotic real and complex systems can be described by (8) and (9), such as fractional-order real and complex Chen systems, fractional-order real and complex Lorenz systems, fractional-order Lu system, and fractional-order complex T system.

Before introducing our approach, we first give the definition of CMPS between systems (8) and (9).

Definition 7. Consider the real master system (8) and complex slave system (9), and the error of CMPS is defined as

$$e = e^r + je^i = y - Hx = y^r - H^r x + j\left(y^i - H^i x\right), \quad (10)$$

where $e^r = (e_1, e_3, \ldots, e_{2n-1})^T$, $e^i = (e_2, e_4, \ldots, e_{2n})^T$, $H = \mathrm{diag}(h_1, h_2, \ldots, h_n)$, and $h_k = h_k^r + jh_k^i$. If $e \to 0$ as $t \to \infty$, then the CMPS between systems (8) and (9) is achieved.

Remark 8. Obviously, the problem of CMPS between (8) and (9) is equivalently transformed to the stabilization problem of error system.

Our goal in this paper is to design a robust adaptive controller to achieve the CMPS between fractional-order chaotic real and complex systems with unknown parameters.

In order to make the proposed method more reasonable and effective, an assumption is provided.

Assumption 9. In general, it is assumed that the external disturbances $d_k(t) = d_k^r(t) + jd_k^i(t)$ are bounded by

$$\left|d_k(t)\right| \le \phi_k, \quad (11)$$

where $k = 1, 2, \ldots, n$. $|\cdot|$ denotes the modulus of disturbance term and ϕ_k are known positive constants.

Theorem 10. *Consider the real master system* (8) *and the complex slave system* (9), *if the controller is designed as*

$$\begin{aligned} W_k(t) &= W_k^r(t) + jW_k^i(t) \\ W_k^r(t) &= -G_k^r(y)\hat{\delta} + h_k^r F_k(x)\hat{\theta} - g_k^r(y) + h_k^r f_k(x) \\ &\quad - L_k e_{2k-1} - \xi_k \operatorname{sgn}\left(e_{2k-1}\right) \\ W_k^i(t) &= -G_k^i(y)\hat{\delta} + h_k^i F_k(x)\hat{\theta} - g_k^i(y) + h_k^i f_k(x) \\ &\quad - L_k e_{2k} - \xi_k \operatorname{sgn}\left(e_{2k}\right), \end{aligned} \quad (12)$$

where $k = 1, 2, \ldots, n$. sgn *is a sign function, $G_k^r(y)$, $G_k^i(y)$, $g_k^r(y)$, $g_k^i(y)$, $F_k(x)$, and $f_k(x)$ are the kth row of $G^r(y)$, $G^i(y)$, $g^r(y)$, $g^i(y)$, $F(x)$, and $f(x)$, respectively, h_k^r and h_k^i are given constants, and L_k and ξ_k are control gains, updated by*

$$\begin{aligned} D^\alpha L_k &= \beta\left(e_{2k-1}^2 + e_{2k}^2\right) \\ D^\alpha \xi_k &= \sigma\left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right) \end{aligned} \quad (13)$$

in which β, σ are positive constants.

The adaptive rules of unknown parameters are selected as

$$\begin{aligned} D^\alpha \hat{\theta} &= -\left(H^r F(x)\right)^T e^r - \left(H^i F(x)\right)^T e^i \\ D^\alpha \hat{\delta} &= \left(G^r(y)\right)^T e^r + \left(G^i(y)\right)^T e^i, \end{aligned} \quad (14)$$

where $\hat{\theta}$ and $\hat{\delta}$ are the estimate values of θ and δ, respectively. Then the CMPS between systems (8) *and* (9) *can be achieved.*

Proof. According to the definition of CMPS, we obtain

$$\begin{aligned} D^\alpha e &= D^\alpha e^r + jD^\alpha e^i = D^\alpha y^r - H^r D^\alpha x \\ &\quad + j\left(D^\alpha y^i - H^i D^\alpha x\right) \\ &= G^r(y)\delta + g^r(y) + d^r(t) + W^r(t) \\ &\quad - H^r\left(F(x)\theta + f(x)\right) \\ &\quad + j\left[G^i(y)\delta + g^i(y) + d^i(t) + W^i(t)\right. \\ &\quad \left. - H^i\left(F(x)\theta + f(x)\right)\right]. \end{aligned} \quad (15)$$

Namely, the real and imaginary parts of complex error system (15) and the adaptation laws (13) and (14) constitute the following closed-loop adaptive system:

$$\begin{aligned} D^\alpha e_{2k-1} &= G_k^r(y)\delta + g_k^r(y) + d_k^r(t) + W_k^r(t) \\ &\quad - h_k^r\left(F_k(x)\theta + f_k(x)\right) \\ D^\alpha e_{2k} &= G_k^i(y)\delta + g_k^i(y) + d_k^i(t) + W_k^i(t) \\ &\quad - h_k^i\left(F_k(x)\theta + f_k(x)\right) \\ D^\alpha \hat{\theta} &= -\left(H^r F(x)\right)^T e^r - \left(H^i F(x)\right)^T e^i \\ D^\alpha \hat{\delta} &= \left(G^r(y)\right)^T e^r + \left(G^i(y)\right)^T e^i \\ D^\alpha L_k &= \beta\left(e_{2k-1}^2 + e_{2k}^2\right) \\ D^\alpha \xi_k &= \sigma\left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right). \end{aligned} \quad (16)$$

According to the continuous frequency distributed model of fractional integrator [28–30], system (16) is exactly equivalent to the following infinite-dimensional ordinary differential equations:

$$\begin{aligned} \frac{\partial z_{2k-1}(\omega,t)}{\partial t} &= -\omega z_{2k-1}(\omega,t) + G_k^r(y)\delta + g_k^r(y) + d_k^r(t) \\ &\quad + W_k^r(t) - h_k^r\left(F_k(x)\theta + f_k(x)\right) \\ e_{2k-1} &= \int_0^\infty \mu(\omega) z_{2k-1}(\omega,t)\, d\omega \\ \frac{\partial z_{2k}(\omega,t)}{\partial t} &= -\omega z_{2k}(\omega,t) + G_k^i(y)\delta + g_k^i(y) + d_k^i(t) \\ &\quad + W_k^i(t) - h_k^i\left(F_k(x)\theta + f_k(x)\right) \\ e_{2k} &= \int_0^\infty \mu(\omega) z_{2k}(\omega,t)\, d\omega \\ \frac{\partial z_{\tilde{\theta}}(\omega,t)}{\partial t} &= -\omega z_{\tilde{\theta}}(\omega,t) - \left(H^r F(x)\right)^T e^r - \left(H^i F(x)\right)^T e^i \\ \tilde{\theta} &= \int_0^\infty \mu(\omega) z_{\tilde{\theta}}(\omega,t)\, d\omega \end{aligned}$$

$$\begin{aligned}
\frac{\partial z_{\widetilde{\delta}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\delta}}(\omega,t) + \left(G^r(y)\right)^T e^r + \left(G^i(y)\right)^T e^i \\
\widetilde{\delta} &= \int_0^\infty \mu(\omega)\, z_{\widetilde{\delta}}(\omega,t)\, d\omega \\
\frac{\partial z_{\widetilde{L}_k}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{L}_k}(\omega,t) + \beta\left(e_{2k-1}^2 + e_{2k}^2\right) \\
\widetilde{L}_k &= \int_0^\infty \mu(\omega)\, z_{\widetilde{L}_k}(\omega,t)\, d\omega \\
\frac{\partial z_{\widetilde{\xi}_k}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\xi}_k}(\omega,t) + \sigma\left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right) \\
\widetilde{\xi}_k &= \int_0^\infty \mu(\omega)\, z_{\widetilde{\xi}_k}(\omega,t)\, d\omega,
\end{aligned} \tag{17}$$

where $k = 1,\ldots,n$. $\mu(\omega) = ((\sin(\alpha\pi))/\pi)\omega^{-\alpha}$, $\widetilde{\theta} = \widehat{\theta} - \theta$, $\widetilde{\delta} = \widehat{\delta} - \delta$, $\widetilde{L}_k = L_k - L_k^*$, and $\widetilde{\xi}_k = \xi_k - \xi_k^*$ are estimation errors. L_k^* and ξ_k^* are positive constants to be determined later. In the above model, $z_{2k-1}(\omega,t)$, $z_{2k}(\omega,t)$, $z_{\widetilde{\theta}}(\omega,t)$, $z_{\widetilde{\delta}}(\omega,t)$, $z_{\widetilde{L}_k}(\omega,t)$, and $z_{\widetilde{\xi}_k}(\omega,t)$ are true state variables (vectors), while e_{2k-1}, e_{2k}, $\widetilde{\theta}$, $\widetilde{\delta}$, $\widetilde{L}_k$, and $\widetilde{\xi}_k$ are pseudostate variables (vectors).

To prove the stability of (17), let us choose a positive definite Lyapunov function in the form of

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t), \tag{18}$$

where

$$\begin{aligned}
V_1(t) &= \int_0^\infty \mu(\omega)\, v_1(\omega,t)\, d\omega, \\
v_1(\omega,t) &= \frac{1}{2}\sum_{k=1}^n z_{2k-1}^2(\omega,t), \\
V_2(t) &= \int_0^\infty \mu(\omega)\, v_2(\omega,t)\, d\omega, \\
v_2(\omega,t) &= \frac{1}{2}\sum_{k=1}^n z_{2k}^2(\omega,t), \\
V_3(t) &= \int_0^\infty \mu(\omega)\, v_3(\omega,t)\, d\omega, \\
v_3(\omega,t) &= \frac{1}{2} z_{\widetilde{\theta}}^T(\omega,t)\, z_{\widetilde{\theta}}(\omega,t), \\
V_4(t) &= \int_0^\infty \mu(\omega)\, v_4(\omega,t)\, d\omega, \\
v_4(\omega,t) &= \frac{1}{2} z_{\widetilde{\delta}}^T(\omega,t)\, z_{\widetilde{\delta}}(\omega,t), \\
V_5(t) &= \int_0^\infty \mu(\omega)\, v_5(\omega,t)\, d\omega, \\
v_5(\omega,t) &= \frac{1}{2\beta}\sum_{k=1}^n z_{\widetilde{L}_k}^2(\omega,t), \\
V_6(t) &= \int_0^\infty \mu(\omega)\, v_6(\omega,t)\, d\omega, \\
v_6(\omega,t) &= \frac{1}{2\sigma}\sum_{k=1}^n z_{\widetilde{\xi}_k}^2(\omega,t).
\end{aligned} \tag{19}$$

Taking the derivative of $V(t)$ with respect to time, it yields

$$\begin{aligned}
\frac{dV(t)}{dt} = {} & \int_0^\infty \mu(\omega) \sum_{k=1}^n z_{2k-1} \frac{\partial z_{2k-1}}{\partial t} d\omega \\
& + \int_0^\infty \mu(\omega) \sum_{k=1}^n z_{2k} \frac{\partial z_{2k}}{\partial t} d\omega + \int_0^\infty \mu(\omega)\, z_{\widetilde{\theta}}^T \frac{\partial z_{\widetilde{\theta}}}{\partial t} d\omega \\
& + \int_0^\infty \mu(\omega)\, z_{\widetilde{\delta}}^T \frac{\partial z_{\widetilde{\delta}}}{\partial t} d\omega \\
& + \frac{1}{\beta}\int_0^\infty \mu(\omega) \sum_{k=1}^n z_{\widetilde{L}_k} \frac{\partial z_{\widetilde{L}_k}}{\partial t} d\omega \\
& + \frac{1}{\sigma}\int_0^\infty \mu(\omega) \sum_{k=1}^n z_{\widetilde{\xi}_k} \frac{\partial z_{\widetilde{\xi}_k}}{\partial t} d\omega.
\end{aligned} \tag{20}$$

Substituting (17) into (20), we have

$$\begin{aligned}
\frac{dV(t)}{dt} = {} & \int_0^\infty \mu(\omega) \sum_{k=1}^n z_{2k-1} \left[-\omega z_{2k-1} + G_k^r(y)\,\delta \right. \\
& \qquad + g_k^r(y) + d_k^r(t) + W_k^r(t) \\
& \qquad \left. - h_k^r\left(F_k(x)\,\theta + f_k(x)\right)\right] d\omega \\
& + \int_0^\infty \mu(\omega) \sum_{k=1}^n z_{2k} \left[-\omega z_{2k} + G_k^i(y)\,\delta + g_k^i(y) \right. \\
& \qquad + d_k^i(t) + W_k^i(t) \\
& \qquad \left. - h_k^i\left(F_k(x)\,\theta + f_k(x)\right)\right] d\omega \\
& + \int_0^\infty \mu(\omega)\, z_{\widetilde{\theta}}^T \left[-\omega z_{\widetilde{\theta}} - \left(H^r F(x)\right)^T e^r \right. \\
& \qquad \left. - \left(H^i F(x)\right)^T e^i\right] d\omega \\
& + \int_0^\infty \mu(\omega)\, z_{\widetilde{\delta}}^T \left[-\omega z_{\widetilde{\delta}} + \left(G^r(y)\right)^T e^r \right. \\
& \qquad \left. + \left(G^i(y)\right)^T e^i\right] d\omega \\
& + \frac{1}{\beta}\int_0^\infty \mu(\omega) \sum_{i=1}^n z_{\widetilde{L}_k} \left[-\omega z_{\widetilde{L}_k} + \beta\left(e_{2k-1}^2 + e_{2k}^2\right)\right] d\omega \\
& + \frac{1}{\sigma}\int_0^\infty \mu(\omega) \sum_{k=1}^n z_{\widetilde{\xi}_k} \left[-\omega z_{\widetilde{\xi}_k} \right. \\
& \qquad \left. + \sigma\left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right)\right] d\omega.
\end{aligned} \tag{21}$$

Inserting the control law (12) into (21), we obtain

$$\begin{aligned}
\frac{dV(t)}{dt} = &-\int_0^\infty \mu(\omega)\,\omega\sum_{k=1}^{n} z_{2k-1}^2 d\omega \\
&+\sum_{k=1}^{n}\Big[G_k^r(y)\left(\delta-\hat{\delta}\right)+d_k^r(t)-h_k^r F_k(x)\left(\theta-\hat{\theta}\right) \\
&\qquad -L_k e_{2k-1}-\xi_k \operatorname{sgn}\left(e_{2k-1}\right)\Big] \\
&\times\int_0^\infty \mu(\omega)\, z_{2k-1} d\omega-\int_0^\infty \mu(\omega)\,\omega\sum_{k=1}^{n} z_{2k}^2 d\omega \\
&+\sum_{k=1}^{n}\Big[G_k^i(y)\left(\delta-\hat{\delta}\right)+d_k^i(t)-h_k^i F_k(x)\left(\theta-\hat{\theta}\right) \\
&\qquad -L_k e_{2k}-\xi_k \operatorname{sgn}\left(e_{2k}\right)\Big] \\
&\times\int_0^\infty \mu(\omega)\, z_{2k} d\omega-\int_0^\infty \mu(\omega)\,\omega z_{\tilde{\theta}}^T z_{\tilde{\theta}} d\omega \\
&+\int_0^\infty \mu(\omega)\, z_{\tilde{\theta}}^T d\omega \\
&\times\left[-\left(H^r F(x)\right)^T e^r-\left(H^i F(x)\right)^T e^i\right] \\
&-\int_0^\infty \mu(\omega)\,\omega z_{\tilde{\delta}}^T z_{\tilde{\delta}} d\omega \\
&+\int_0^\infty \mu(\omega)\, z_{\tilde{\delta}}^T d\omega\left[\left(G^r(y)\right)^T e^r+\left(G^i(y)\right)^T e^i\right] \\
&-\frac{1}{\beta}\int_0^\infty \mu(\omega)\,\omega\sum_{k=1}^{n} z_{\tilde{L}_k}^2 d\omega \\
&+\frac{1}{\beta}\sum_{k=1}^{n}\beta\left(e_{2k-1}^2+e_{2k}^2\right)\int_0^\infty \mu(\omega)\, z_{\tilde{L}_k} d\omega \\
&-\frac{1}{\sigma}\int_0^\infty \mu(\omega)\,\omega\sum_{k=1}^{n} z_{\tilde{\xi}_k}^2 d\omega \\
&+\frac{1}{\sigma}\sum_{k=1}^{n}\sigma\left(\left|e_{2k-1}\right|+\left|e_{2k}\right|\right)\int_0^\infty \mu(\omega)\, z_{\tilde{\xi}_k} d\omega.
\end{aligned} \tag{22}$$

That is

$$\begin{aligned}
\frac{dV(t)}{dt} = &-J+\sum_{k=1}^{n}\Big[G_k^r(y)\left(\delta-\hat{\delta}\right)+d_k^r(t) \\
&\qquad -h_k^r F_k(x)\left(\theta-\hat{\theta}\right) \\
&\qquad -L_k e_{2k-1}-\xi_k \operatorname{sgn}\left(e_{2k-1}\right)\Big] e_{2k-1} \\
&+\sum_{k=1}^{n}\Big[G_k^i(y)\left(\delta-\hat{\delta}\right)+d_k^i(t)-h_k^i F_k(x)\left(\theta-\hat{\theta}\right) \\
&\qquad -L_k e_{2k}-\xi_k \operatorname{sgn}\left(e_{2k}\right)\Big] e_{2k} \\
&+\tilde{\theta}^T\left[-\left(H^r F(x)\right)^T e^r-\left(H^i F(x)\right)^T e^i\right] \\
&+\tilde{\delta}^T\left[\left(G^r(y)\right)^T e^r+\left(G^i(y)\right)^T e^i\right] \\
&+\sum_{k=1}^{n}\left(e_{2k-1}^2+e_{2k}^2\right)\tilde{L}_k+\sum_{k=1}^{n}\left(\left|e_{2k-1}\right|+\left|e_{2k}\right|\right)\tilde{\xi}_k,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
J = &\sum_{k=1}^{n}\int_0^\infty \mu(\omega)\,\omega z_{2k-1}^2 d\omega+\sum_{k=1}^{n}\int_0^\infty \mu(\omega)\,\omega z_{2k}^2 d\omega \\
&+\int_0^\infty \mu(\omega)\,\omega z_{\tilde{\theta}}^T z_{\tilde{\theta}} d\omega+\int_0^\infty \mu(\omega)\,\omega z_{\tilde{\delta}}^T z_{\tilde{\delta}} d\omega \\
&+\sum_{k=1}^{n}\frac{1}{\beta}\int_0^\infty \mu(\omega)\,\omega z_{\tilde{L}_k}^2 d\omega+\sum_{k=1}^{n}\frac{1}{\sigma}\int_0^\infty \mu(\omega)\,\omega z_{\tilde{\xi}_k}^2 d\omega > 0.
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{n}\left[G_k^r(y)\left(\delta-\hat{\delta}\right)\right] e_{2k-1}+\sum_{k=1}^{n}\left[G_k^i(y)\left(\delta-\hat{\delta}\right)\right] e_{2k} \\
&\quad=\left(e^r\right)^T\left[G^r(y)\left(\delta-\hat{\delta}\right)\right]+\left(e^i\right)^T\left[G^i(y)\left(\delta-\hat{\delta}\right)\right] \\
&\quad=\left(\delta-\hat{\delta}\right)^T\left[\left(G^r(y)\right)^T e^r+\left(G^i(y)\right)^T e^i\right] \\
&\quad=-\tilde{\delta}^T\left[\left(G^r(y)\right)^T e^r+\left(G^i(y)\right)^T e^i\right] \\
&\sum_{k=1}^{n}\left[-h_k^r F_k(x)\left(\theta-\hat{\theta}\right)\right] e_{2k-1}+\sum_{k=1}^{n}\left[-h_k^i F_k(x)\left(\theta-\hat{\theta}\right)\right] e_{2k} \\
&\quad=\left(e^r\right)^T\left[-H^r F(x)\left(\theta-\hat{\theta}\right)\right]+\left(e^i\right)^T\left[-H^i F(x)\left(\theta-\hat{\theta}\right)\right] \\
&\quad=\left(\theta-\hat{\theta}\right)^T\left[-\left(H^r F(x)\right)^T e^r-\left(H^i F(x)\right)^T e^i\right] \\
&\quad=-\tilde{\theta}^T\left[-\left(H^r F(x)\right)^T e^r-\left(H^i F(x)\right)^T e^i\right].
\end{aligned} \tag{25}$$

On the basis of (25), then (23) can be rewritten as

$$\begin{aligned}\frac{dV(t)}{dt} &= -J + \sum_{k=1}^{n} d_k^r(t)\, e_{2k-1} + \sum_{k=1}^{n} d_k^i(t)\, e_{2k} \\ &\quad - \sum_{k=1}^{n} L_k^* \left(e_{2k-1}^2 + e_{2k}^2\right) - \sum_{k=1}^{n} \xi_k^* \left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right) \\ &\le -J + \sum_{k=1}^{n} \left|d_k^r(t)\right| \left|e_{2k-1}\right| \\ &\quad + \sum_{k=1}^{n} \left|d_k^i(t)\right| \left|e_{2k}\right| - L^* E^T E - \xi^* \|E\| \\ &\le -J + \sum_{k=1}^{n} \phi_k \left|e_{2k-1}\right| \\ &\quad + \sum_{k=1}^{n} \phi_k \left|e_{2k}\right| - L^* E^T E - \xi^* \|E\| \\ &\le -J - \left(\xi^* - \phi\right) \|E\| - L^* E^T E,\end{aligned} \tag{26}$$

where $E = (e_1, e_3, \ldots, e_{2n-1}, e_2, e_4, \ldots, e_{2n})^T$, $\xi^* = \min\{\xi_k^*\}$, $\phi = \max\{\phi_k\}$, $L^* = \min\{L_k^*\}$, and $k = 1, 2, \ldots, n$. It is obvious that when $\xi^* \ge \phi$, $L^* > 0$, we have

$$\frac{dV(t)}{dt} \le -J - L^* E^T E < 0. \tag{27}$$

According to the conclusions of [23], the fractional-order closed-loop adaptive system (16) is asymptotically stable. This proves that the CMPS between systems (8) and (9) can be achieved by using the control law (12) and the fractional adaptation laws (13) and (14). Therefore, the proof is completed. □

Remark 11. Apparently, the theoretical results in [15, 31] are the special cases of our scheme. It should be noted that the feedback control gain in practical applications is desired as small as possible; however, the theoretical feedback control gains in [15, 31] are fixed values no matter where the initial values start; thus, the gains must be larger than the values needed, which means a kind of waste in practice. In our method, we use an adaptive controller to overcome the above drawbacks. The control gains L_k, ξ_k can be automatically adapted to the suitable values, which can simplify the design process and reduce the cost of control.

If δ is a complex unknown vector and can be written as $\delta = \delta^r + j\delta^i$, then slave system can be rewritten as

$$\begin{aligned}D^\alpha y &= G(y)\left(\delta^r + j\delta^i\right) + g(y) + d(t) + W(t) \\ &= G^r(y)\,\delta^r - G^i(y)\,\delta^i + g^r(y) + d^r(t) + W^r(t) \\ &\quad + j\left(G^r(y)\,\delta^i + G^i(y)\,\delta^r + g^i(y) + d^i(t) + W^i(t)\right).\end{aligned} \tag{28}$$

According to Definition 7, we can obtain

$$\begin{aligned}D^\alpha e &= G^r(y)\,\delta^r - G^i(y)\,\delta^i + g^r(y) + d^r(t) \\ &\quad + W^r(t) - H^r\left(F(x)\,\theta + f(x)\right) \\ &\quad + j\left[G^r(y)\,\delta^i + G^i(y)\,\delta^r + g^i(y) + d^i(t)\right. \\ &\quad \left. + W^i(t) - H^i\left(F(x)\,\theta + f(x)\right)\right].\end{aligned} \tag{29}$$

That is, the real and imaginary parts of (29) can be represented as

$$\begin{aligned}D^\alpha e_{2k-1} &= G_k^r(y)\,\delta^r - G_k^i(y)\,\delta^i + g_k^r(y) + d_k^r(t) \\ &\quad + W_k^r(t) - h_k^r\left(F_k(x)\,\theta + f_k(x)\right) \\ D^\alpha e_{2k} &= G_k^r(y)\,\delta^i + G_k^i(y)\,\delta^r + g_k^i(y) + d_k^i(t) \\ &\quad + W_k^i(t) - h_k^i\left(F_k(x)\,\theta + f_k(x)\right).\end{aligned} \tag{30}$$

Similarly, for stabilizing the error system (30), we give the following theorem.

Theorem 12. *Consider the real master system* (8) *and the complex slave system* (28)*, if the controller is designed as*

$$\begin{aligned}W_k(t) &= W_k^r(t) + W_k^i(t) \\ W_k^r(t) &= -G_k^r(y)\,\widehat{\delta}^r + G_k^i(y)\,\widehat{\delta}^i + h_k^r F_k(x)\,\widehat{\theta} - g_k^r(y) \\ &\quad + h_k^r f_k(x) - L_k e_{2k-1} - \xi_k \operatorname{sgn}\left(e_{2k-1}\right) \\ W_k^i(t) &= -G_k^r(y)\,\widehat{\delta}^i - G_k^i(y)\,\widehat{\delta}^r + h_k^i F_k(x)\,\widehat{\theta} - g_k^i(y) \\ &\quad + h_k^i f_k(x) - L_k e_{2k} - \xi_k \operatorname{sgn}\left(e_{2k}\right),\end{aligned} \tag{31}$$

where $k = 1, 2, \ldots, n$. L_k and ξ_k are control gains, which are updated by

$$\begin{aligned}D^\alpha L_k &= \beta\left(e_{2k-1}^2 + e_{2k}^2\right) \\ D^\alpha \xi_k &= \sigma\left(\left|e_{2k-1}\right| + \left|e_{2k}\right|\right),\end{aligned} \tag{32}$$

where β and σ are positive constants.

The parametric update laws are designed as

$$\begin{aligned}D^\alpha \widehat{\theta} &= -\left(H^r F(x)\right)^T e^r - \left(H^i F(x)\right)^T e^i \\ D^\alpha \widehat{\delta}^r &= \left(G^r(y)\right)^T e^r + \left(G^i(y)\right)^T e^i \\ D^\alpha \widehat{\delta}^i &= \left(-G^i(y)\right)^T e^r + \left(G^r(y)\right)^T e^i.\end{aligned} \tag{33}$$

Then the CMPS between systems (8) *and* (28) *can be achieved.*

Proof. It is similar to that of Theorem 10. Limited by the length of this paper, the proof is omitted here. □

In this paper, for simplicity, we merely consider the case of real unknown parameters.

4. Simulation Example

In this example, we take fractional-order real Chen system as master system, described by

$$\begin{aligned} D^{\alpha}x_1 &= a_1(x_2 - x_1) \\ D^{\alpha}x_2 &= (a_2 - a_1)x_1 + a_2x_2 - x_1x_3 \\ D^{\alpha}x_3 &= x_1x_2 - a_3x_3, \end{aligned} \tag{34}$$

where $x = (x_1, x_2, x_3)^T$ is a real state vector, and

$$\begin{aligned} F(x) &= \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ -x_1 & x_1 + x_2 & 0 \\ 0 & 0 & -x_3 \end{pmatrix}, \\ f(x) &= \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix} \qquad \theta = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \end{aligned} \tag{35}$$

Take the fractional-order complex Lorenz system with external disturbances as slave system, written as

$$\begin{aligned} D^{\alpha}y_1 &= b_1(y_2 - y_1) + d_1(t) + W_1(t) \\ D^{\alpha}y_2 &= b_2y_1 - y_2 - y_1y_3 + d_2(t) + W_2(t) \\ D^{\alpha}y_3 &= \frac{1}{2}(\overline{y}_1y_2 + y_1\overline{y}_2) - b_3y_3 + d_3(t) + W_3(t), \end{aligned} \tag{36}$$

where $y_1 = u_1 + ju_2$ and $y_2 = u_3 + ju_4$ are two complex state variables, $y_3 = u_5$ is a real state variable, and

$$\begin{aligned} G(y) &= \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & -y_3 \end{pmatrix} = \begin{pmatrix} u_3 - u_1 & 0 & 0 \\ 0 & u_1 & 0 \\ 0 & 0 & -u_5 \end{pmatrix} \\ &\quad + j\begin{pmatrix} u_4 - u_2 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ g(y) &= \begin{pmatrix} 0 \\ -y_2 - y_1y_3 \\ \frac{1}{2}(\overline{y}_1y_2 + y_1\overline{y}_2) \end{pmatrix} = \begin{pmatrix} 0 \\ -u_3 - u_1u_5 \\ u_1u_3 + u_2u_4 \end{pmatrix} \\ &\quad + j\begin{pmatrix} 0 \\ -u_4 - u_2u_5 \\ 0 \end{pmatrix} \\ d(t) &= \begin{pmatrix} 0.2\sin(\pi t) \\ 0.2\sin(0.5\pi t) \\ 0.2\sin(0.5\pi t) \end{pmatrix} + j\begin{pmatrix} 0.2\cos(\pi t) \\ 0.2\cos(\pi t) \\ 0 \end{pmatrix}, \\ \delta &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \end{aligned} \tag{37}$$

In this simulation, the complex scaling matrix $H = \mathrm{diag}(h_1, h_2, h_3)$ and $h_1 = h_1^r + jh_1^i$, $h_2 = h_2^r + jh_2^i$ are two complex constants, and h_3 is a real constant. According to Theorem 10, the controller can be designed as

$$\begin{aligned} W_1^r(t) &= -\widehat{b}_1(u_3 - u_1) + \widehat{a}_1h_1^r(x_2 - x_1) - L_1e_1 - \xi_1\,\mathrm{sgn}(e_1) \\ W_1^i(t) &= -\widehat{b}_1(u_4 - u_2) + \widehat{a}_1h_1^i(x_2 - x_1) - L_1e_2 - \xi_1\,\mathrm{sgn}(e_2) \\ W_2^r(t) &= -\widehat{b}_2u_1 + h_2^r(-\widehat{a}_1x_1 + \widehat{a}_2(x_1 + x_2)) - (-u_3 - u_1u_5) \\ &\quad + h_2^r(-x_1x_3) - L_2e_3 - \xi_2\,\mathrm{sgn}(e_3) \\ W_2^i(t) &= -\widehat{b}_2u_2 + h_2^i(-\widehat{a}_1x_1 + \widehat{a}_2(x_1 + x_2)) - (-u_4 - u_2u_5) \\ &\quad + h_2^i(-x_1x_3) - L_2e_4 - \xi_2\,\mathrm{sgn}(e_4) \\ W_3(t) &= \widehat{b}_3u_5 - h_3\widehat{a}_3x_3 - (u_1u_3 + u_2u_4) + h_3x_1x_2 \\ &\quad - L_3e_5 - \xi_3\,\mathrm{sgn}(e_5). \end{aligned} \tag{38}$$

The adaptation laws (13) and (14) are obtained in the form of

$$\begin{aligned} &D^{\alpha}L_1 = \beta(e_1^2 + e_2^2), \qquad D^{\alpha}L_2 = \beta(e_3^2 + e_4^2), \\ &D^{\alpha}L_3 = \beta e_5^2 \\ &D^{\alpha}\xi_1 = \sigma(|e_1| + |e_2|), \qquad D^{\alpha}\xi_2 = \sigma(|e_3| + |e_4|), \\ &D^{\alpha}\xi_3 = \sigma|e_5| \\ &D^{\alpha}\widehat{a}_1 = -h_1^r(x_2 - x_1)e_1 + h_2^rx_1e_3 - h_1^i(x_2 - x_1)e_2 + h_2^ix_1e_4 \\ &D^{\alpha}\widehat{a}_2 = -h_2^r(x_1 + x_2)e_3 - h_2^i(x_1 + x_2)e_4 \\ &D^{\alpha}\widehat{a}_3 = h_3x_3e_5 \\ &D^{\alpha}\widehat{b}_1 = (u_3 - u_1)e_1 + (u_4 - u_2)e_2 \\ &D^{\alpha}\widehat{b}_2 = u_1e_3 + u_2e_4 \\ &D^{\alpha}\widehat{b}_3 = -u_5e_5. \end{aligned} \tag{39}$$

Letting $\alpha = 0.998$, $H = \mathrm{diag}(1 + j, 1 + j, 1)$, the unknown parameters are $\theta = (35, 28, 3)^T$ and $\delta = (10, 28, 8/3)^T$, and the initial values can be randomly chosen as $x(t) = x(0^+) = (2, 2, 2)^T$, $y(t) = y(0^+) = (1 + j, 1 + j, 1)^T$, $\widehat{\theta}(t) = \widehat{\theta}(0^+) = (0, 0, 0)^T$, $\widehat{\delta}(t) = \widehat{\delta}(0^+) = (0, 0, 0)^T$, $L(t) = L(0^+) = (0, 0, 0)^T$, and $\xi(t) = \xi(0^+) = (0, 0, 0)^T$ (where $L(t) = (L_1, L_2, L_3)^T$ and $\xi(t) = (\xi_1, \xi_2, \xi_3)^T$) for $-\infty \le t \le 0$. The positive constants are as follows: $\beta = 10$ and $\sigma = 0.1$. The chaotic attractors of systems (34) and (36) without control are displayed in Figures 1 and 2.

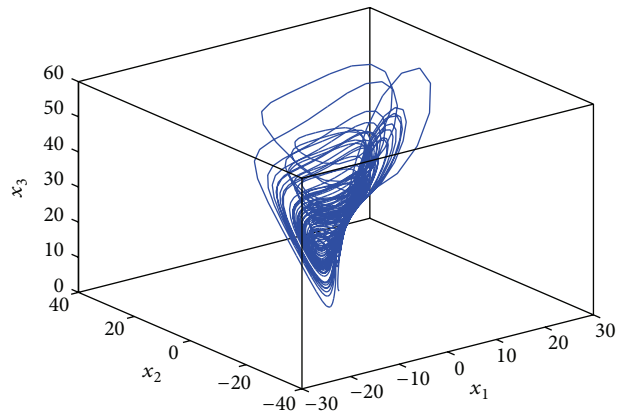

Figure 1: The 3D projection of chaotic attractor of (34).

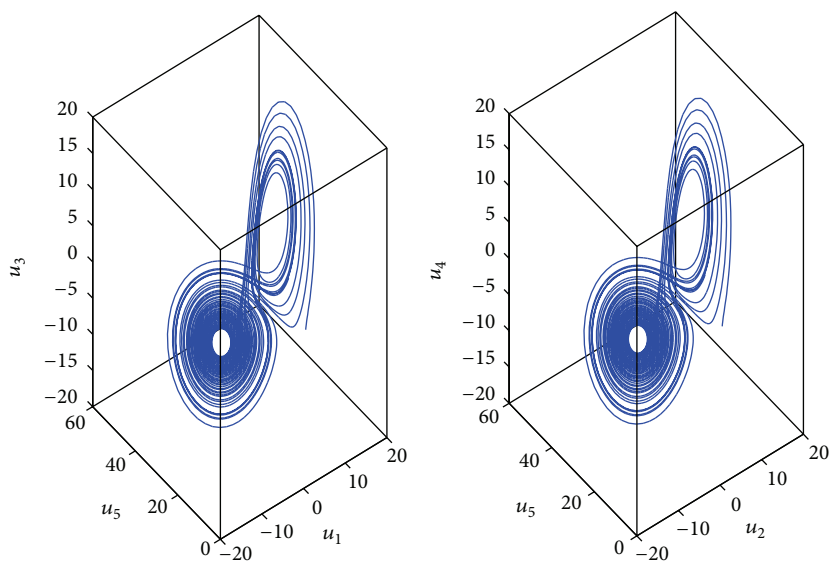

Figure 2: The 3D projections of chaotic attractors of (36) without control.

When the controller is activated, the time evolutions of CMPS errors between systems (34) and (36) are shown in Figure 3. It is clear that all error states asymptotically converge to zero.

The time responses of estimated parameter vectors $\widehat{\theta}$ and $\widehat{\delta}$ in master and slave systems are illustrated in Figure 4. It is obvious that all unknown parameters gradually converge to their actual values.

All simulation results imply that the proposed control scheme is effective in synchronizing fractional-order chaotic real system and fractional-order chaotic complex system in the sense of CMPS.

Remark 13. In above simulation example, the random choice of complex matrix H will not affect the theoretical results.

5. Conclusions

In this paper, the problem of complex modified projective synchronization (CMPS) between fractional-order real and complex systems is investigated. The parameters of both master and slave systems are assumed to be unknown in advance. Since the effects of external disturbances are fully taken into consideration, the proposed approach is more practical and meaningful than that of the existing methods. In order to prove the stability of the closed-loop system, the frequency distributed model of fractional integrator and Lyapunov stability theory are used. The simulation results have verified the effectiveness and applicability of the proposed synchronization scheme. Because our results contain and extend most existing works, we believe that there are high potentials in the proposed method.

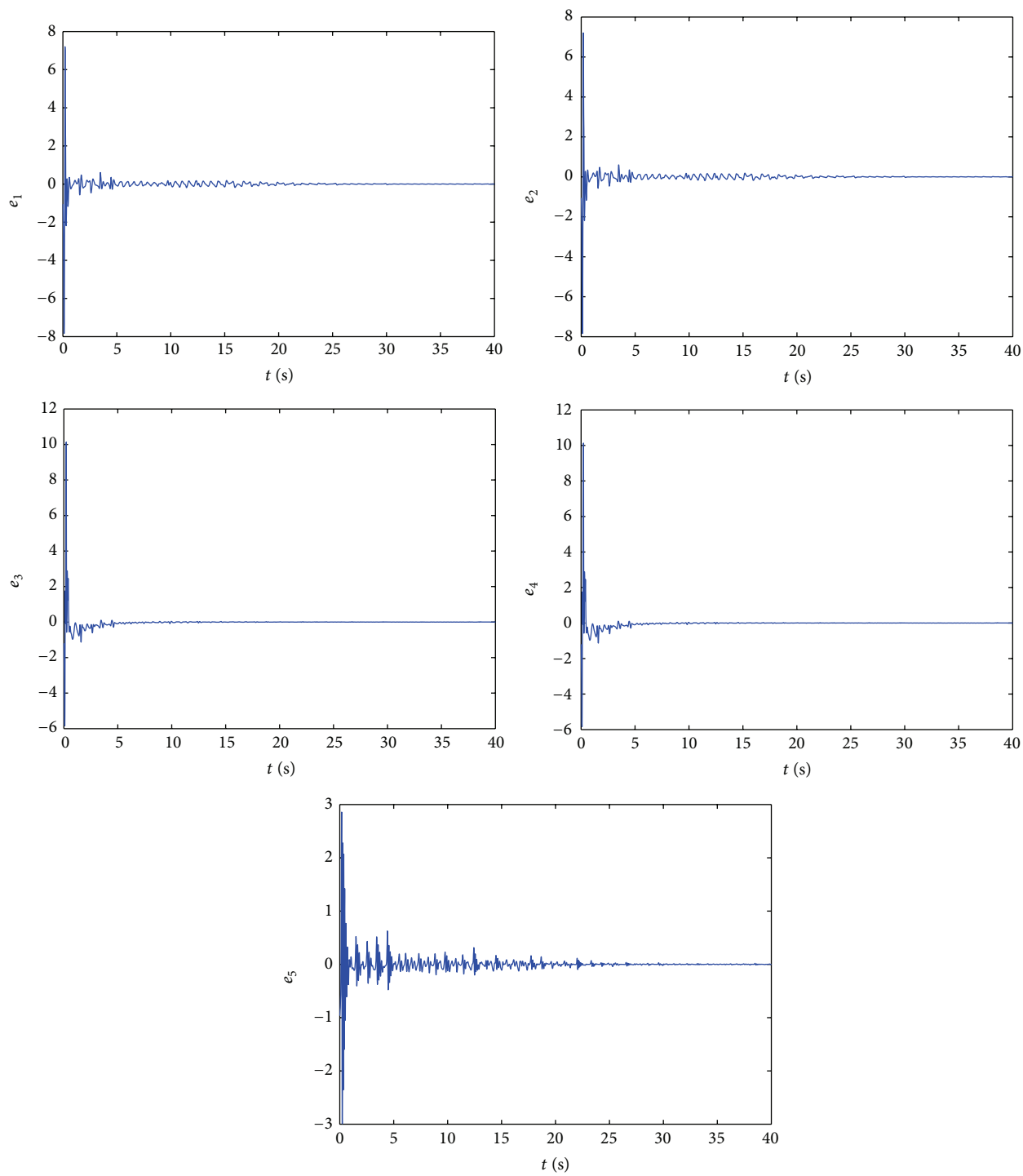

Figure 3: Time evolutions of CMPS errors between systems (34) and (36).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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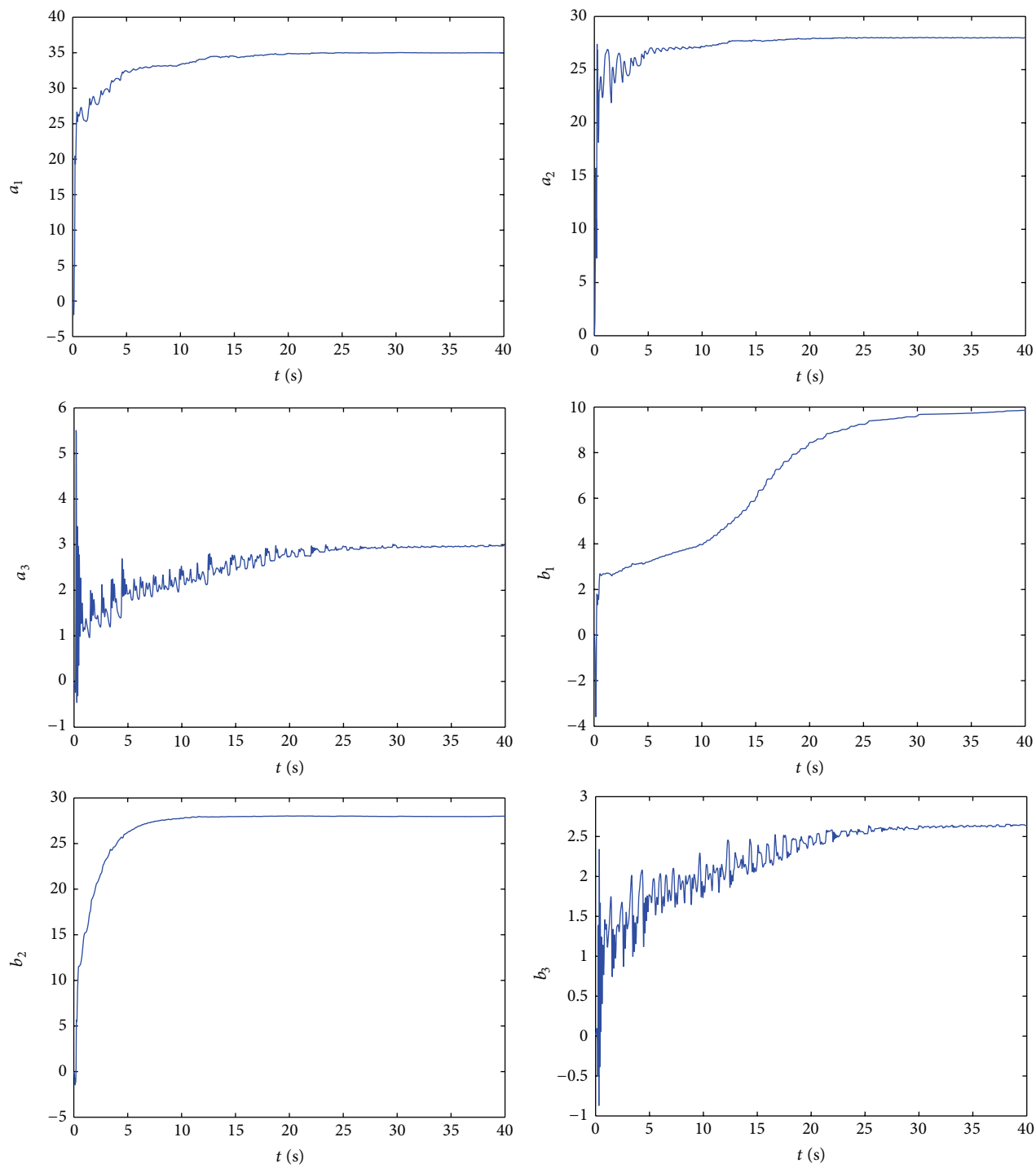

Figure 4: Time responses of estimate parameters in systems (34) and (36).

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