Research Article

Global Stability of a Stage-Structured Predator-Prey Model with Stochastic Perturbation

Liu Yang and Shouming Zhong

School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu 611731, China

Correspondence should be addressed to Liu Yang; ylazx@126.com

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This paper is concerned with a new predator-prey model with stage structure on prey, in which the immature prey and the mature prey are preyed on by predator. We think that the model is more realistic and interesting than the one in which only the immature prey or the mature prey is consumed by predator. Our work shows that the stochastic model and its corresponding deterministic system have a unique global positive solution and the positive solution is global asymptotic stability for each model. If the positive equilibrium point of the deterministic system is globally stable, then the stochastic model will preserve the nice property provided that the noise is sufficiently small. Results are analyzed with the help of graphical illustrations.

1. Introduction

Within the past decades, the dynamic behavior between predator and their prey has received considerable interest due to their wide applications in ecology and mathematical ecology. There is a great deal of attention for predator-prey models from many scholars [1–9]. In most of the cases, the study is based on interactions between homogeneous populations. However, in the nature, most of the species must go through two life stages from birth to death. In [6–9], some stage-structured models of population growth consisting of immature and mature individuals were discussed. In particular, [9] considered a predator-prey model with two populations, that is, predator and their food prey. In their model, only prey species is divided into two life stages, the immature prey and the mature prey. And the predators only consume the immature prey species. In [10], Chinese fire-bellied newt is described as an example, which is unable to prey on the mature *Rana chensinensis* and can only prey on the immature one. So, they consider the following nonlinear ordinary differential equations:

$$\frac{dx_1}{dt} = \alpha x_2 - r_1 x_1 - \eta x_1^2 - \beta x_1 - \beta_1 x_1 x_3;$$

$$\frac{dx_2}{dt} = \beta x_1 - r_2 x_2;$$

$$\frac{dx_3}{dt} = -r x_3 + k \beta_1 x_1 x_3 - \eta_1 x_3^2.$$

(1)

In [11], the authors have studied the dynamical properties of deterministic model (1) and the stochastic behavior of the corresponding stochastic model (19), which was first introduced by Beretta et al. [12] and Shaikhet [13]. Then, they obtained stochastic stability condition in mean square sense by utilizing Lyapunov function.

On one hand, the predators functional response, that is, the rate of prey consumption by an average predator, is one of the important components which can impact the relationship between predator and prey in population dynamics. There are many functional responses such as Holling type [5], Beddington-DeAngelis type [7], and Watt type [14]. On the other hand, population is inevitably affected by environmental noise in nature [15, 16]. May [17] also showed that, due to environmental fluctuation, the birth rate, the death rate, competition coefficients, and other parameters usually show random fluctuation to a certain extent

that should be stochastic. Therefore, many authors have taken stochastic perturbation into deterministic models and shown the effect of environmental variability on population dynamics in mathematical ecology [18–26]. For example, [26] considered the following stochastic stage-structured predator-prey model:

$$\begin{aligned} dx_1 &= [rx_2 - d_1x_1 - ax_1^2 - bx_1]dt - \sigma_1x_1(x_1 - x_1^*)dB_1(t); \\ dx_2 &= \left[bx_1 - d_2x_2 - a_{11}x_2^2 - \frac{a_{12}x_2x_3}{1 + mx_2 + nx_3} \right]dt \\ &\quad - \sigma_2x_2(x_2 - x_2^*)dB_2(t); \\ dx_3 &= \left[-d_3x_3 + \frac{a_{21}x_1x_3}{1 + mx_2 + nx_3} - a_{22}x_3^2 \right]dt \\ &\quad - \sigma_3x_3(x_3 - x_3^*)dB_3(t). \end{aligned} \quad (2)$$

In this paper, the authors mainly utilize Itô's formula, the theory of stochastic differential equations, and Lyapunov functions to investigate the global stability of the positive equilibrium of model (2).

Motivated by the above works, in this paper, we will consider the following stochastic stage-structured predator-prey model:

$$\begin{aligned} dx_1 &= \left[rx_2 - d_1x_1 - a_{11}x_1^2 - bx_1 - \frac{a_{13}x_1x_3}{1 + mx_1 + nx_3} \right]dt \\ &\quad - \sigma_1x_1(x_1 - x_1^*)dB_1(t); \\ dx_2 &= \left[bx_1 - d_2x_2 - a_{22}x_2^2 - \frac{a_{23}x_2x_3}{\alpha + \beta x_2} \right]dt \\ &\quad - \sigma_2x_2(x_2 - x_2^*)dB_2(t); \\ dx_3 &= \left[-d_3x_3 + \frac{k_1a_{13}x_1x_3}{1 + mx_1 + nx_3} + \frac{k_2a_{23}x_2x_3}{\alpha + \beta x_2} - a_{33}x_3^2 \right]dt \\ &\quad - \sigma_3x_3(x_3 - x_3^*)dB_3(t), \end{aligned} \quad (3)$$

where $x_1(t)$ denotes the population density of the immature prey, $x_2(t)$ represents the population density of the mature prey, and $x_3(t)$ stands for the population size of predator. $B_i(t)$ ($i = 1, 2, 3$) is a standard Brownian motion, which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ satisfying the usual conditions (i.e., right continuous and increasing while $\mathcal{F}_0$ contains all $\mathcal{P}$ -null sets). All parameters involved with the model are positive constants and can be interpreted in more detail: r is the birth rate of immature prey population, d_1 , d_2 , and d_3 represent the death rate of immature prey population, mature prey population, and predator population, respectively, a_{11} , a_{22} , and a_{33} denote intraspecies competition rate of immature prey population, mature prey population, and predator population, respectively, b represents transformation rate from immature prey population to mature prey population, a_{i3} ($i = 1, 2$) is the rate of predation, k_i ($i = 1, 2$) is fraction of prey

biomass converted into predator biomass, $x_1/(1 + mx_1 + nx_3)$ represents Beddington-DeAngelis functional response and $x_2/(\alpha + \beta x_2)$ denotes Holling-II functional response, and σ_i^2 ($i = 1, 2, 3$) is the intensity of the noise.

The initial condition of model (3) is any point in the biological meaning region $R_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > 0, x_2 > 0, x_3 > 0\}$. (x_1^*, x_2^*, x_3^*) is a positive equilibrium of model (3) which is the solution of the algebraic equations

$$\begin{aligned} rx_2 - d_1x_1 - a_{11}x_1^2 - bx_1 - \frac{a_{13}x_1x_3}{1 + mx_1 + nx_3} &= 0; \\ bx_1 - d_2x_2 - a_{22}x_2^2 - \frac{a_{23}x_2x_3}{\alpha + \beta x_2} &= 0; \\ -d_3x_3 + \frac{k_1a_{13}x_1x_3}{1 + mx_1 + nx_3} + \frac{k_2a_{23}x_2x_3}{\alpha + \beta x_2} - a_{33}x_3^2 &= 0, \end{aligned} \quad (4)$$

with initial conditions $x_i(0) > 0$, $i = 1, 2, 3$. Noting that if $\sigma_1 = \sigma_2 = \sigma_3 = 0$, then model (3) becomes the following corresponding deterministic stage-structured predator-prey systems:

$$\begin{aligned} \frac{dx_1}{dt} &= rx_2 - d_1x_1 - a_{11}x_1^2 - bx_1 - \frac{a_{13}x_1x_3}{1 + mx_1 + nx_3}; \\ \frac{dx_2}{dt} &= bx_1 - d_2x_2 - a_{22}x_2^2 - \frac{a_{23}x_2x_3}{\alpha + \beta x_2}; \\ \frac{dx_3}{dt} &= -d_3x_3 + \frac{k_1a_{13}x_1x_3}{1 + mx_1 + nx_3} + \frac{k_2a_{23}x_2x_3}{\alpha + \beta x_2} - a_{33}x_3^2. \end{aligned} \quad (5)$$

Therefore, in this paper, we only need to establish the sufficient conditions for global asymptotic stability of system (3). And, in this paper, we will also use Itô's formula, the theory of stochastic differential equations, and Lyapunov functions to study the global stability of the positive equilibrium point of stochastic system (3).

The paper is organized as follows. In Section 2, we study the existence and uniqueness of global positive solution of system (3). In Section 3, sufficient conditions for global asymptotic stability of system (3) are established. Then, we introduce some simulation figures to illustrate the main result in Section 4. In the last section, we give the conclusions.

2. Existence and Uniqueness of Solution

In this section, we will show that the solution of system (3) is positive and global. We give the following theorem.

Theorem 1. *For any given initial value $(x_1(0), x_2(0), x_3(0)) \in R_+^3$, there is a unique solution $(x_1(t), x_2(t), x_3(t))$ to (3) on $t \geq 0$. Furthermore, with probability one, R_+^3 are positive invariant for (3); that is, for all $t \geq 0$, $(x_1(t), x_2(t), x_3(t)) \in R_+^3$ a.s., if $(x_1(0), x_2(0), x_3(0)) \in R_+^3$.*

Proof. Since the coefficients of (3) are locally Lipschitz continuous, there is a unique local solution $(x_1(t), x_2(t), x_3(t))$ to (3) on $t \in [0, \tau_e)$, where τ_e is the explosion time [27, 28]. Therefore, to show that the solution is global, we only need

to show that $\tau_e = \infty$ a.s. We use the technique of localization [29, 30]. Let $k_0 > 0$ be sufficiently large for $x_i(0)$ ($i = 1, 2, 3$) lying within the interval $[1/k_0, k_0]$. Let us define a sequence of stopping time [29] for each integer $k \geq k_0$ by

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : x_i(t) \notin \left(\frac{1}{k}, k \right), i = 1, 2, 3 \right\}. \quad (6)$$

The convention here is that the infimum of the empty set is ∞ . Since τ_k is nondecreasing as $k \rightarrow \infty$, we set $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$. Then, $\tau_\infty \leq \tau_e$ a.s. Now, we will show that $\tau_\infty = \infty$ a.s. If the statement is not right, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Thus, by denoting $\Omega_k = \{\tau_k \leq T\}$, there exists $k_1 \geq k_0$ such that

$$P(\Omega_k) \geq \varepsilon \quad \forall k \geq k_1. \quad (7)$$

Consider the following function: $V(x_1, x_2, x_3) = 2 \sum_{i=1}^3 (\sqrt{x_i} - 1 - 0.5 \ln x_i)$. It is clear to see that $V \in C^2(R_+^3, R_+)$. If $(x_1(t), x_2(t), x_3(t)) \in R_+^3$, by using Itô formula, we get

$$dV(x_1, x_2, x_3) = f(x_1, x_2, x_3) dt - \sum_{i=1}^3 g_i(x_1, x_2, x_3) dB_i(t), \quad (8)$$

where

$$\begin{aligned} f(x_1, x_2, x_3) &= \frac{(\sqrt{x_1} - 1)}{x_1} \\ &\times \left[rx_2 - d_1 x_1 - a_{11} x_1^2 - bx_1 - \frac{a_{13} x_1 x_3}{1 + mx_1 + nx_3} \right] \\ &+ \frac{\sigma_1^2}{4} (-\sqrt{x_1} + 2) (x_1 - x_1^*)^2 \\ &+ \frac{(\sqrt{x_2} - 2)}{x_2} \left[bx_1 - d_2 x_2 - a_{22} x_2^2 - \frac{a_{23} x_2 x_3}{\alpha + \beta x_2} \right] \\ &+ \frac{\sigma_2^2}{4} (-\sqrt{x_2} + 2) (x_2 - x_2^*)^2, \\ &+ \frac{(\sqrt{x_3} - 1)}{x_3} \\ &\times \left[-d_3 x_3 + \frac{k_1 a_{13} x_1 x_3}{1 + mx_1 + nx_3} + \frac{k_2 a_{23} x_2 x_3}{\alpha + \beta x_2} - a_{33} x_3^2 \right] \\ &+ \frac{\sigma_3^2}{4} (-\sqrt{x_3} + 2) (x_3 - x_3^*)^2 \\ g_i(x_1, x_2, x_3) &= \sigma_i (\sqrt{x_i} - 1) (x_i - x_i^*). \end{aligned} \quad (9)$$

Since all coefficients of system (2) are positive constants, it is easy to see from (9) that the function $f(x_1, x_2, x_3)$ is bounded,

say by M . Thus, since $(x_1(t \wedge \tau_k), x_2(t \wedge \tau_k), x_3(t \wedge \tau_k)) \in R_+^3$ and we consider (9), we have

$$\begin{aligned} &\int_0^{T \wedge \tau_k} dV(x_1(t), x_2(t), x_3(t)) \\ &\leq \int_0^{T \wedge \tau_k} M dt - \sum_{i=1}^3 \int_0^{T \wedge \tau_k} g_i(x_1, x_2, x_3) dB_i(t). \end{aligned} \quad (10)$$

Taking expectations yields

$$\begin{aligned} &EV(x_1(T \wedge \tau_k), x_2(T \wedge \tau_k), x_3(T \wedge \tau_k)) \\ &\leq V(x_1(0), x_2(0), x_3(0)) + \int_0^{T \wedge \tau_k} M dt \\ &\leq V(x_1(0), x_2(0), x_3(0)) + MT. \end{aligned} \quad (11)$$

On the other hand, for every $\omega \in \Omega_k$, either $x_1(\tau_k, \omega)$ or $x_2(\tau_k, \omega)$ or $x_3(\tau_k, \omega)$ equals either k or $1/k$. Then,

$$\begin{aligned} &V(x_1(T \wedge \tau_k, \omega), x_2(T \wedge \tau_k, \omega), x_3(T \wedge \tau_k, \omega)) \\ &\geq \min \left\{ \sqrt{k} - 1 - 0.5 \ln k, \sqrt{\frac{1}{k}} - 1 - 0.5 \ln \left(\frac{1}{k} \right) \right\}. \end{aligned} \quad (12)$$

We therefore get from (7) that

$$\begin{aligned} &EV(x_1(t \wedge \tau_k), x_2(t \wedge \tau_k), x_3(t \wedge \tau_k)) \\ &\geq E[1_{\Omega_k} V(x_1(t \wedge \tau_k), x_2(t \wedge \tau_k), x_3(t \wedge \tau_k))] \\ &\geq \varepsilon \min \left\{ \sqrt{k} - 1 - 0.5 \ln k, \sqrt{\frac{1}{k}} - 1 - 0.5 \ln \left(\frac{1}{k} \right) \right\}, \end{aligned} \quad (13)$$

where 1_{Ω_k} is the indicator function of Ω_k . Then, it follows from (11) that

$$\begin{aligned} &V(x_1(0), x_2(0), x_3(0)) + MT \\ &\geq \varepsilon \min \left\{ \sqrt{k} - 1 - 0.5 \ln k, \sqrt{\frac{1}{k}} - 1 - 0.5 \ln \left(\frac{1}{k} \right) \right\}. \end{aligned} \quad (14)$$

Letting $k \rightarrow \infty$ leads to the contradiction $\infty > V(x_1(0), x_2(0), x_3(0)) + MT = \infty$. Therefore, $\tau_\infty = \infty$ a.s. Then, $\tau_e = \infty$ a.s. and $(x_1(t), x_2(t), x_3(t)) \in R_+^3$ a.s. This completes the proof of Theorem 1. $\square$

3. Globally Asymptotically Stable

For the sake of convenience, denote

$$\begin{aligned} A &= -\frac{x_1^*}{r} \left[a_{11} - \frac{ma_{13}x_3^*}{1 + mx_1^* + nx_3^*} - \frac{\sigma_1^2 x_1^*}{2} \right], \\ B &= -\frac{x_2^*}{b} \left[a_{22} - \frac{a_{23}\beta x_3^*}{\alpha(\alpha + \beta x_2^*)} - \frac{\sigma_2^2 x_2^*}{2} \right], \\ C &= -\frac{x_1^* (1 + mx_1^*)}{rk_1 (1 + nx_3^*)} \left[a_{33} - \frac{\sigma_3^2 x_3^*}{2} \right], \\ D &= \frac{a_{23}x_2^*}{b\alpha} + \frac{k_2 a_{23}x_1^* (1 + mx_1^*)}{rk_1 (1 + nx_3^*) (\alpha + \beta x_2^*)}. \end{aligned} \quad (15)$$

Theorem 2. *If*

$$\begin{aligned} A &< 0, \\ C &< 0 \end{aligned} \quad (16)$$

as well as

$$4BC - D^2 > 0. \quad (17)$$

Then, the positive equilibrium position (x_1^*, x_2^*, x_3^*) of model (3) is globally asymptotically stable with probability one; that is, for any positive initial data $(x_1(0), x_2(0), x_3(0))$, the solution of system (3) has the property that

$$\lim_{t \rightarrow \infty} x_i(t) = x_i^* \quad (i = 1, 2, 3) \quad (18)$$

almost surely.

Proof. From the stability theory of stochastic differential equations, we only need to find a suitable Lyapunov function $V(z)$ satisfying $LV(z) \leq 0$ and the identity holds if and only if $z = z^*$ [30], where $z = z(t)$ is the solution of the following stochastic differential equation:

$$dz(t) = f(t, z(t))dt + g(t, z(t))dB(t), \quad (19)$$

z^* is the positive equilibrium position of (19), and

$$LV(z) = V_z(z) f(t, z) + 0.5 \text{ trace} [g^T(t, z) V_{zz}(z) g(t, z)]. \quad (20)$$

For $x \in R_+^3$, define

$$V(x) = \sum_{i=1}^3 \frac{c_i}{2} \int_{x_i^*}^{x_i} \frac{x_i - x_i^*}{x_i} dx_i, \quad (21)$$

where

$$c_1 = \frac{x_1^*}{r}, \quad c_2 = \frac{x_2^*}{b}, \quad c_3 = \frac{c_1(1 + mx_1^*)}{k_1(1 + nx_3^*)}. \quad (22)$$

We can rewrite (3) as follows:

$$\begin{aligned} dx_1 &= \frac{r}{x_1^*} [x_1(x_2 - x_2^*) - x_2(x_1 - x_1^*)] dt \\ &\quad - a_{11}x_1(x_1 - x_1^*) dt \\ &\quad - a_{13}x_1 \frac{(1 + mx_1^*)(x_3 - x_3^*) - mx_3^*(x_1 - x_1^*)}{(1 + mx_1 + nx_3)(1 + mx_1^* + nx_3^*)} dt \\ &\quad - \sigma_1x_1(x_1 - x_1^*) dB_1(t); \\ dx_2 &= \frac{b}{x_2^*} [x_2(x_1 - x_1^*) - x_1(x_2 - x_2^*)] dt \\ &\quad - a_{22}x_2(x_2 - x_2^*) dt \\ &\quad - a_{23}x_2 \frac{(\alpha + \beta x_2^*)(x_3 - x_3^*) - \beta x_3^*(x_2 - x_2^*)}{(\alpha + \beta x_2)(\alpha + \beta x_2^*)} dt \\ &\quad - \sigma_2x_2(x_2 - x_2^*) dB_2(t); \end{aligned}$$

$$\begin{aligned} dx_3 &= -a_{33}x_3(x_3 - x_3^*) dt \\ &\quad + k_1a_{13}x_3 \frac{(1 + nx_3^*)(x_1 - x_1^*) - nx_1^*(x_3 - x_3^*)}{(1 + mx_1 + nx_3)(1 + mx_1^* + nx_3^*)} dt \\ &\quad + k_2a_{23}x_3 \frac{\alpha(x_2 - x_2^*)}{(\alpha + \beta x_2)(\alpha + \beta x_2^*)} dt \\ &\quad - \sigma_3x_3(x_3 - x_3^*) dB_3(t). \end{aligned} \quad (23)$$

Applying Itô formula to model (23), we can get that

$$\begin{aligned} LV(x) &= \frac{x_1^*}{r} \left\{ (x_1 - x_1^*) \frac{r}{x_1^*} \left[(x_2 - x_2^*) - \frac{x_2}{x_1} (x_1 - x_1^*) \right] \right. \\ &\quad - a_{11}(x_1 - x_1^*)^2 \\ &\quad - a_{13} \frac{(1 + mx_1^*)(x_3 - x_3^*) - mx_3^*(x_1 - x_1^*)}{(1 + mx_1 + nx_3)(1 + mx_1^* + nx_3^*)} \\ &\quad \times (x_1 - x_1^*) + \frac{\sigma_1^2 x_1^*}{2} (x_1 - x_1^*)^2 \Big\} \\ &\quad + \frac{x_2^*}{b} \left\{ (x_2 - x_2^*) \frac{b}{x_2^*} \left[(x_1 - x_1^*) - \frac{x_1}{x_2} (x_2 - x_2^*) \right] \right. \\ &\quad - a_{22}(x_2 - x_2^*)^2 \\ &\quad - a_{23} \frac{(\alpha + \beta x_2^*)(x_3 - x_3^*) - \beta x_3^*(x_2 - x_2^*)}{(\alpha + \beta x_2)(\alpha + \beta x_2^*)} \\ &\quad \times (x_2 - x_2^*) + \frac{\sigma_2^2 x_2^*}{2} (x_2 - x_2^*)^2 \Big\} \\ &\quad + \frac{x_1^*(1 + mx_1^*)}{rk_1(1 + nx_3^*)} \\ &\quad \times \left\{ -a_{33}(x_3 - x_3^*)^2 + k_1a_{13} \right. \\ &\quad \times \frac{(1 + nx_3^*)(x_1 - x_1^*) - nx_1^*(x_3 - x_3^*)}{(1 + mx_1 + nx_3)(1 + mx_1^* + nx_3^*)} \\ &\quad \times (x_3 - x_3^*) + k_2a_{23} \frac{\alpha(x_2 - x_2^*)(x_3 - x_3^*)}{(\alpha + \beta x_2)(\alpha + \beta x_2^*)} \\ &\quad \left. + \frac{\sigma_3^2 x_3^*}{2} (x_3 - x_3^*)^2 \right\} \\ &= (x_1 - x_1^*)(x_2 - x_2^*) \\ &\quad - \frac{x_2}{x_1} (x_1 - x_1^*)^2 - \frac{a_{11}x_1^*}{r} (x_1 - x_1^*)^2 \\ &\quad + \frac{\sigma_1^2 (x_1^*)^2}{2r} (x_1 - x_1^*)^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{ma_{13}x_1^*x_3^*}{r(1+mx_1+nx_3)(1+mx_1^*+nx_3^*)} \\
& \times (x_1 - x_1^*)^2 + (x_1 - x_1^*)(x_2 - x_2^*) - \frac{x_1}{x_2}(x_2 - x_2^*)^2 \\
& - \frac{a_{22}x_2^*}{b}(x_2 - x_2^*)^2 \\
& + \frac{\sigma_2^2(x_2^*)^2}{2b}(x_2 - x_2^*)^2 + \frac{a_{23}\beta x_2^*x_3^*}{b(\alpha + \beta x_2)(\alpha + \beta x_2^*)} \\
& \times (x_2 - x_2^*)^2 - \frac{x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)} \\
& \times \left(a_{33} - \frac{\sigma_3^2x_3^*}{2} \right) (x_3 - x_3^*)^2 \\
& - \frac{na_{13}(x_1^*)^2(1+mx_1^*)(x_3 - x_3^*)^2}{r(1+nx_3^*)(1+mx_1+nx_3)(1+mx_1^*+nx_3^*)} \\
& - \frac{a_{23}x_2^*(x_2 - x_2^*)(x_3 - x_3^*)}{b(\alpha + \beta x_2)} \\
& + \frac{\alpha k_2 a_{23} x_1^*(1+mx_1^*)(x_2 - x_2^*)(x_3 - x_3^*)}{rk_1(1+nx_3^*)(\alpha + \beta x_2)(\alpha + \beta x_2^*)} \\
& \leq - \left[\sqrt{\frac{x_2}{x_1}}(x_1 - x_1^*) - \sqrt{\frac{x_1}{x_2}}(x_2 - x_2^*) \right]^2 \\
& - \frac{x_1^*}{r} \left[a_{11} - \frac{ma_{13}x_3^*}{1+mx_1^*+nx_3^*} - \frac{\sigma_1^2x_1^*}{2} \right] \\
& \times (x_1 - x_1^*)^2 \\
& - \frac{x_2^*}{b} \left[a_{22} - \frac{a_{23}\beta x_3^*}{\alpha(\alpha + \beta x_2^*)} - \frac{\sigma_2^2x_2^*}{2} \right] \\
& \times (x_2 - x_2^*)^2 - \frac{x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)} \\
& \times \left[a_{33} - \frac{\sigma_3^2x_3^*}{2} \right] (x_3 - x_3^*)^2 \\
& + \frac{a_{23}x_2^*}{b\alpha} |x_2 - x_2^*| |x_3 - x_3^*| \\
& + \frac{k_2 a_{23} x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)(\alpha + \beta x_2^*)} \\
& \times |x_2 - x_2^*| |x_3 - x_3^*| \\
& \leq - \frac{x_1^*}{r} \left[a_{11} - \frac{ma_{13}x_3^*}{1+mx_1^*+nx_3^*} - \frac{\sigma_1^2x_1^*}{2} \right] \\
& \times (x_1 - x_1^*)^2 - \frac{x_2^*}{b} \left[a_{22} - \frac{a_{23}\beta x_3^*}{\alpha(\alpha + \beta x_2^*)} - \frac{\sigma_2^2x_2^*}{2} \right]
\end{aligned}$$

$$\begin{aligned}
& \times (x_2 - x_2^*)^2 - \frac{x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)} \left[a_{22} - \frac{\sigma_3^2x_3^*}{2} \right] \\
& \times (x_3 - x_3^*)^2 + \frac{a_{23}x_2^*}{b\alpha} |x_2 - x_2^*| |x_3 - x_3^*| \\
& + \frac{k_2 a_{23} x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)(\alpha + \beta x_2^*)} |x_2 - x_2^*| |x_3 - x_3^*|. \quad (24)
\end{aligned}$$

Let $|Z - Z^*| = (|x_1 - x_1^*|, |x_2 - x_2^*|, |x_3 - x_3^*|)^T$; then we have

$$LV(x) \leq \frac{1}{2} |Z - Z^*|^T \begin{pmatrix} 2A & 0 & 0 \\ 0 & 2B & D \\ 0 & D & 2C \end{pmatrix} |Z - Z^*|. \quad (25)$$

Clearly, the conditions of Theorem 2 and the above inequality denote $LV(x) < 0$ along all trajectories in R_+^3 except (x_1^*, x_2^*, x_3^*) . Then, we get the desired assertion immediately.

For deterministic system (5), we have that the following theorem holds. $\square$

Theorem 3. *If*

$$\begin{aligned}
& a_{11} - \frac{ma_{13}x_3^*}{1+mx_1^*+nx_3^*} > 0, \\
& a_{22} - \frac{a_{23}\beta x_3^*}{\alpha(\alpha + \beta x_2^*)} > 0, \\
& \frac{4a_{33}x_2^*x_1^*(1+mx_1^*)}{brk_1(1+nx_3^*)} \left[a_{22} - \frac{a_{23}\beta x_3^*}{\alpha(\alpha + \beta x_2^*)} \right] \\
& - \left[\frac{a_{23}x_2^*}{b\alpha} + \frac{k_2 a_{23} x_1^*(1+mx_1^*)}{rk_1(1+nx_3^*)(\alpha + \beta x_2^*)} \right]^2 > 0, \quad (26)
\end{aligned}$$

then the equilibrium position (x_1^*, x_2^*, x_3^*) of system (5) is globally asymptotically stable.

Remark 4. From Theorems 2 and 3, we can see that if the positive equilibrium of the deterministic system is globally stable and the noise perturbation is not very large, then the stochastic system will keep the nice property.

4. Numerical Simulations

In this section, we will utilize the Milstein method mentioned in Higham [31] to consolidate the analytical findings.

FIGURE 1: Solution of system (3) with initial conditions $x_1(0) = x_2(0) = x_3(0) = 0.2$ and $r = 0.6$, $d_1 = 0.2$, $a_{11} = 0.3$, $b = 0.8$, $a_{13} = 0.6$, $d_2 = 0.3$, $a_{22} = 0.2$, $a_{23} = 0.3$, $d_3 = 0.1$, $k_1 = 5/6$, $k_2 = 0.8$, $a_{33} = 0.2$, $\Delta t = 0.01$, $m = 0.3$, $n = 0.1$, $\alpha = 2$, and $\beta = 1$. (a) $\sigma_1 = \sigma_2 = \sigma_3 = 0$. (b) $\sigma_1^2 = 1.9$, $\sigma_2^2 = 0.6$, and $\sigma_3^2 = 1.9$. (c) $\sigma_1^2 = 20$, $\sigma_2^2 = 20$, and $\sigma_3^2 = 20$.

Here, we consider the discretization equations of model (3) as follows:

$$\begin{aligned}
 x_1(k+1) &= x_1(k) + \left[rx_2(k) - d_1x_1(k) - a_{11}x_1^2(k) \right. \\
 &\quad \left. - bx_1(k) - \frac{a_{13}x_3(k)}{1 + mx_1(k) + nx_3(k)} \right] \Delta t \\
 &\quad - \sigma_1x_1(k)(x_1(k) - x_1^*)\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}x_1(k)
 \end{aligned}$$

$$\begin{aligned}
 &\times (x_1(k) - x_1^*)^2 [\xi_k^2 - 1] \Delta t, \\
 x_2(k+1) &= x_2(k) + \left[bx_1(k) - d_2x_2(k) \right. \\
 &\quad \left. - a_{22}x_2^2(k) - \frac{a_{23}x_3(k)}{\alpha + \beta x_2(k)} \right] \Delta t \\
 &\quad - \sigma_2x_2(k)(x_2(k) - x_2^*)\sqrt{\Delta t}\eta_k \\
 &\quad + \frac{\sigma_2^2}{2}x_2(k)(x_2(k) - x_2^*)^2 [\eta_k^2 - 1] \Delta t,
 \end{aligned}$$

$$\begin{aligned}
& x_3(k+1) \\
&= x_3(k) + \left[-d_3 x_3(k) + \frac{k_1 a_{13} x_1(k)}{1 + m x_1(k) + n x_3(k)} \right. \\
&\quad \left. + \frac{k_2 a_{23} x_2(k)}{\alpha + \beta x_2(k)} - a_{33} x_3^2(k) \right] \Delta t \\
&\quad - \sigma_3 x_3(k) (x_3(k) - x_3^*) \sqrt{\Delta t} \zeta_k \\
&\quad + \frac{\sigma_3^2}{x_3} (k) (x_3(k) - x_3^*)^2 [\zeta_k^2 - 1] \Delta t,
\end{aligned} \tag{27}$$

where ξ_k , η_k , and ζ_k ($k = 1, 2, \dots$) are the Gaussian random variables which follow $N(0, 1)$.

Set $r = 0.6$, $d_1 = 0.2$, $a_{11} = 0.3$, $b = 0.8$, $a_{13} = 0.6$, $d_2 = 0.3$, $a_{32} = 0.2$, $a_{23} = 0.3$, $d_3 = 0.1$, $k_1 = 5/6$, $k_2 = 0.8$, $a_{33} = 0.2$, $\alpha = 2$, $\beta = 1$, $m = 0.3$, $n = 0.1$, and $\Delta t = 0.01$. From Figure 1, we can get that the equilibrium is $x_1 = 0.2137$, $x_2 = 0.4184$, and $x_3 = 0.2003$. The only difference between the conditions of Figures 1(a), 1(b), and 1(c) is the values of σ_1 , σ_2 , and σ_3 . In Figure 1(a), we suppose that $\sigma_1 = \sigma_2 = \sigma_3 = 0$. By Theorem 3, the equilibrium point (x_1^*, x_2^*, x_3^*) of deterministic system (5) is globally asymptotically stable. Figure 1(a) confirms this. In Figure 1(b), we choose $\sigma_1^2 = 1.9$, $\sigma_2^2 = 0.6$, and $\sigma_3^2 = 1.9$. From Theorem 2, we can easily get that the equilibrium position (x_1^*, x_2^*, x_3^*) of stochastic system (3) is globally asymptotically stable. By Figure 1(c), we can see that if we choose $\sigma_1^2 = 20$, $\sigma_2^2 = 20$, and $\sigma_3^2 = 20$, these values violate conditions (19); all the species will die out. That is to say, if the conditions of Theorem 2 are not satisfied, the positive equilibrium point (x_1^*, x_2^*, x_3^*) may be no longer globally asymptotically stable.

5. Conclusions

In this paper, we investigated two stage-structured predator-prey systems: deterministic one and stochastic one. In the models, we suppose that both immature prey and mature prey are consumed by predator. The model is more realistic and complicated than the one in which only the immature prey or mature prey is preyed on by predator. For each system, we established the sufficient conditions for global asymptotic stability. From the results and simulation figures, we can see that if the positive equilibrium position of the corresponding deterministic model is globally stable and the noise is sufficiently small, then the stochastic system will preserve the nice property. The result is useful and important for ecological balance. Up to our knowledge, the present work is the first attempt to study such stochastic model with stage structure on prey.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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