Research Article

Analytic Continuation of Euler Polynomials and the Euler Zeta Function

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Received 10 January 2014; Accepted 10 March 2014; Published 3 April 2014

Academic Editor: Binggen Zhang

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We study that the Euler numbers $E_n$ and Euler polynomials $E_n(z)$ are analytically continued to $E(s)$ and $E(s, w)$. We investigate the new concept of dynamics of the zeros of analytically continued polynomials. Finally, we observe an interesting phenomenon of “scattering” of the zeros of $E(s, w)$.

1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the ring of integers, the field of real numbers, and the complex numbers, respectively. Recently, many mathematicians have studied different kinds of the Euler, Bernoulli, and Genocchi numbers and polynomials (see [1–19]). The computing environment would make more and more rapid progress and there has been increasing interest in solving mathematical problems with the aid of computers. By using software, mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concepts. Numerical experiments of Bernoulli polynomials, Euler polynomials, and Genocchi polynomials have been the subject of extensive study in recent years and much progress has been made both mathematically and computationally. Using computer, a realistic study for Euler polynomials $E_n(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of “scattering” of the zeros of the Euler polynomials $E_n(x)$ in complex plane. First, we introduce the Euler numbers and Euler polynomials. As a well-known definition, the Euler numbers $E_n$ are defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (1)$$

Here is the list of the first Euler numbers:

$$E_0 = 1, \quad E_1 = -\frac{1}{2}, \quad E_2 = 0, \quad E_3 = \frac{1}{4},$$
$$E_4 = 0, \quad E_5 = -\frac{1}{2}, \quad E_6 = 0, \quad E_7 = \frac{17}{8},$$
$$E_8 = 0, \quad E_9 = -\frac{31}{2}, \quad E_{10} = 0, \quad E_{11} = \frac{691}{4},$$
$$E_{12} = 0, \quad E_{13} = -\frac{5461}{2}, \quad E_{14} = 0, \quad E_{15} = \frac{929569}{16}, \quad E_{16} = 0,$$
$$E_{17} = -\frac{3202291}{2}, \quad E_{18} = 0,$$
$$E_{19} = \frac{221930581}{4}, \quad E_{20} = 0,$$
$$E_{21} = -\frac{4722116521}{2}, \quad E_{22} = 0,$$
$$E_{23} = \frac{968383680827}{8},$$
$$E_{24} = 0,$$
$$E_{25} = -\frac{14717667114151}{2}.$$
The Euler polynomials $E_n(x)$ are defined by the generating function:

$$F(x, t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{e^{Etx} - e^t}{e^t - 1},$$

where we use the technique method notation by replacing $E(x)^n$ by $E_n(x)$ symbolically. Because

$$\frac{\partial F}{\partial x} (x, t) = tF(x, t) = \sum_{n=0}^{\infty} \frac{dE_n}{dx} (x) \frac{t^n}{n!},$$

an important relation follows:

$$\frac{dE_k}{dx} (x) = kE_{k-1} (x).$$

Then, it is easy to deduce that $E_k(x)$ are polynomials of degree $k$. Here is the list of the first Euler polynomials:

$$E_0 (x) = 1, \quad E_1 (x) = x - \frac{1}{2},$$
$$E_2 (x) = x^2 - x, \quad E_3 (x) = x^3 - \frac{3x^2}{2} + \frac{1}{4},$$
$$E_4 (x) = x^4 - 2x^3 + x, \quad E_5 (x) = x^5 - \frac{5x^4}{2} - \frac{1}{2},$$
$$E_6 (x) = x^6 - 3x^5 + 5x^3 - 3x,$$
$$E_7 (x) = x^7 - \frac{7x^6}{2} + \frac{35x^4}{4} - \frac{21x^2}{2} + 17x,$$
$$E_8 (x) = x^8 - 4x^7 + 14x^5 - 28x^3 + 17x,$$
$$E_9 (x) = x^9 - \frac{9x^8}{2} + 21x^6 - 63x^4 + \frac{153x^2}{2} - \frac{31}{2},$$
$$E_{10} (x) = x^{10} - 5x^9 + 30x^7 - 126x^5 + 255x^3 - 155x.$$

### 2. Generating Euler Polynomials and Numbers

Since

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{Etx} = e^{Etx} = e^{(E+x)t},$$

we have the following theorem.

**Theorem 1.** For $n$, one has

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}.$$

**Definition 2.** For $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, define the Euler zeta function by

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s};$$

see [7–10]. Notice that the Euler zeta function can be analytically continued to the whole complex plane, and these zeta functions have the values of the Euler numbers at negative integers. That is, Euler numbers are related to the Euler zeta function as

$$\zeta_E(-n) = E_n.$$  

**Definition 3.** We define the Hurwitz zeta function $\zeta_E(s, x)$ for $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ and $x \in \mathbb{R}$ with $0 \leq x < 1$ by

$$\zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s};$$

see [3, 7–10, 12]. Euler polynomials are related to the Hurwitz zeta function as

$$\zeta_E(-n, x) = E_n(x).$$

We now consider the function $E(s)$ as the analytic continuation of Euler numbers. From the above analytic continuation of Euler numbers, we consider

$$E_n \mapsto E(s), \quad \zeta_E(-n) = E_n \mapsto \zeta_E(-s) = E(s).$$

All the Euler numbers $E_n$ agree with $E(n)$, the analytic continuation of Euler numbers evaluated at $n$ (see Figure 1),

$$E_n = E(n) \quad \text{for } n \geq 1,$$

except $E(0) = -1$, but $E_0 = 1$.

In fact, we can express $E'(s)$ in terms of $\zeta'_E(s)$, the derivative of $\zeta_E(s)$,

$$E(s) = \zeta_E(-s), \quad E'(s) = -\zeta'_E(-s),$$

for $n \in \mathbb{N} \cup \{0\}$.

From relation (16), we can define the other analytic continued half of Euler numbers as

$$E(s) = \zeta_E(-s),$$

for $n \in \mathbb{N}$. By (17), we have

$$\lim_{n \to \infty} E_{-n} = \zeta_E(n) = -2.$$
which only changes the sign in the conventional definition of the only nonzero even Euler numbers, $E_0$, from $E_0 = 1$ to $E_0 = E(0) = -1$.

By using Cauchy product, we have

$$
\sum_{n=0}^{\infty} (-1)^n E_n(t) \frac{t^n}{n!} = \left( \sum_{k=0}^{\infty} (-1)^k E_k \frac{t^k}{k!} \right) \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} \right)
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k E_k \frac{t^{n-k}}{n!}.
$$

(20)

For consistency with the redefinition of $E_n = E(n)$ in (19), Euler polynomials should be analogously redefined as

$$
E_n(x) = \sum_{n=0}^{\infty} (-1)^{n+k} \binom{n}{k} E_k x^{n-k}.
$$

(21)

The analytic continuation can then be obtained as

$$
E_k \mapsto E(k+s-[s]) = \zeta_E((-k+(s-[s]))) ,
$$

$$
\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s])) \Gamma(1+[s]-k)}
= \sum_{k=1}^{\lfloor s \rfloor} (-1)^{k+[s]} \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s])) \Gamma(1+[s]-k)}
= \sum_{k=0}^{\lfloor s \rfloor+1} (-1)^{k+[s]} \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s])) \Gamma(2+[s]-k)}
$$

(22)

where $[s]$ gives the integer part of $s$, and so $s-[s]$ gives the fractional part.

By (22), we obtain analytic continuation of Euler polynomials:

$$
E_0(w) = -1, \quad E_1(w) = E(1,w) = -0.5 + w, \quad E_2(w) = E(2,w) = w - w^2,
$$

3. **Analytic Continuation of Euler Polynomials**

Looking back at (1) and (3), we can see that the sign convention of $E_0$ was actually arbitrary. Equation (15) suggests that consistent definition of Euler numbers should really have been

$$
\sum_{n=0}^{\infty} (-1)^n E_n \frac{t^n}{n!} = \frac{2}{e^t + 1}, \quad |t| < \pi, \ n = 0, 1, 2, \ldots \in \mathbb{N} \cup \{0\}
$$

(19)
\[ E(2.2, w) \approx 0.0799175 + 0.866594w - 1.1976514w^2 + 0.116134w^3, \]
\[ E(2.4, w) \approx 0.147412 + 0.693349w - 1.36116w^2 + 0.279503w^3, \]
\[ E(2.6, w) \approx 0.199873 + 0.485705w - 1.47547w^2 + 0.486915w^3, \]
\[ E(2.8, w) \approx 0.234806 + 0.251271w - 1.52609w^2 + 0.731015w^3, \]
\[ E_3(w) = E(3, w) = 0.25 - 1.5w^2 + w^3. \]

By using (23), we plot the deformation of the curve \( E(2, w) \) into the curve of \( E(3, w) \) via the real analytic continuation \( E(s, w) \), \( 2 \leq s \leq 3 \), \( w \in \mathbb{R} \) (see Figure 3).

Next, we investigate the beautiful zeros of the \( E(s, w) \) by using a computer. We plot the zeros of \( E(s, w) \) for \( s = 9, 9.6, 9.8, 10 \) and \( w \in \mathbb{C} \) (Figure 4).

In Figure 4(a), we choose \( s = 9 \). In Figure 4(b), we choose \( s = 9.6 \). In Figure 4(c), we choose \( s = 9.8 \). In Figure 4(d), we choose \( s = 10 \).

Since
\[
\sum_{n=0}^{\infty} E_n (1-x) \frac{(-1)^n t^n}{n!} = F(1-x,-t)
\]
\[ = \frac{2}{\text{e}^{x}+1} \text{e}^{(1-x)(-t)} = \frac{2}{\text{e}^{x}+1} \text{e}^{xt} \]  
\[ = F(x,t) = \sum_{n=0}^{\infty} E_n (x) \frac{t^n}{n!}, \]  
we obtain
\[ E_n (x) = (-1)^n E_n (1-x). \]

Hence, we have the following theorem.

**Theorem 4.** If \( n \equiv 1 \pmod{2} \), then \( E_n(1/2) = 0 \), for \( n \in \mathbb{N} \).

The question is, what happens with the reflexive symmetry (25) when one considers Euler polynomials? Prove that \( E_n(x), x \in \mathbb{C} \), has \( \text{Re}(x) = 1/2 \) reflection symmetry in addition to the usual \( \text{Im}(x) = 0 \) reflection symmetry analytic complex functions. However, we observe that \( E(s, w), w \in \mathbb{C} \), does not have \( \text{Re}(w) = 1/2 \) reflection symmetry analytic complex functions (Figure 4).

Our numerical results for approximate solutions of real zeros of \( E(s, w) \) are displayed. We observe a remarkably regular structure of the complex roots of Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of Euler polynomials (Table 1). Next, we calculated an approximate solution satisfying \( E(s, w), w \in \mathbb{R} \). The results are given in Table 2.

<table>
<thead>
<tr>
<th>( s )</th>
<th>Real zeros</th>
<th>Complex zeros</th>
</tr>
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<tbody>
<tr>
<td>1.5</td>
<td>2</td>
<td>0</td>
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<tr>
<td>2.5</td>
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<td>4</td>
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</tr>
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<td>5</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>9.6</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>9.8</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( s )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
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<tr>
<td>6.5</td>
<td>-0.24986, 0.749547, 6.94675</td>
</tr>
<tr>
<td>7</td>
<td>-0.49773, 0.49999, 1.49773</td>
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<tr>
<td>7.5</td>
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<td>8</td>
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<tr>
<td>8.5</td>
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</tr>
<tr>
<td>9</td>
<td>-1.21928, -0.501115, 0.500956, 1.49919, 2.22008</td>
</tr>
<tr>
<td>9.6</td>
<td>-1.32375, -0.810502, 0.206935, 1.19507, 1.90363, 7.44777</td>
</tr>
<tr>
<td>9.8</td>
<td>-1.34314, -0.922069, 0.112954, 1.08884, 1.90735, 4.56809</td>
</tr>
<tr>
<td>10</td>
<td>-1.34708, -1.0482, 0.0238982, 0.97898, 2.03661, 2.35464</td>
</tr>
</tbody>
</table>
Euler polynomials $E_n(w)$ are polynomials of degree $n$. Thus, $E_n(w)$ has $n$ zeros and $E_{n+1}(w)$ has $n + 1$ zeros. When discrete $n$ is analytically continued to continuous parameter $s$, it naturally leads to the following question: how does $E(s, w)$, the analytic continuation of $E_n(w)$, pick up an additional zero as $s$ increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials, the idea of looking at how the zeros move about in the $w$ complex plane as we vary the parameter $s$. For more studies and results in this subject you may see [11, 14–16].

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.
References


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