Research Article

Monotone Iterative Technique and Symmetric Positive Solutions to Fourth-Order Boundary Value Problem with Integral Boundary Conditions

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The purpose of this paper is to investigate the existence of symmetric positive solutions for a class of fourth-order boundary value problem: $u^{(4)}(t) + \beta u''(t) = f(t, u(t), u''(t))$, $0 < t < 1$, $u(0) = u(1) = \int_0^1 p(s)u(s)ds$, $u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds$, where $p, q \in L^1[0, 1]$, $f \in C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty))$. By using a monotone iterative technique, we prove that the above boundary value problem has symmetric positive solutions under certain conditions. In particular, these solutions are obtained via the iteration procedures.

1. Introduction

The deformation of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by a fourth-order ordinary differential equation BVP (short for boundary value problem). At present, two-point situation of fourth-order BVP has been studied by many authors, generally using the nonlinear alternatives of Leray-Schauder, the fixed point index theory, and the method of upper and lower solutions, monotone iteration; see [1–6].

Recently, problems with integral boundary value conditions arise naturally in thermal conduction problems [7], semiconductor problems [8], and hydrodynamic problems [9]. Hence, the existence results of positive solutions to this kind of problems have received a great deal of attentions. We refer the readers to [10–15].

In [13], Ma studied the following problem:

$$u^{(4)}(t) = h(t)f(t, u), \quad 0 < t < 1, \quad (1)$$

$$u(0) = u(1) = \int_0^1 p(s)u(s)ds, \quad (2)$$

$$u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds,$$

where $p, q \in L^1[0, 1]$ and h and f are continuous. The existence of at least one symmetric positive solution is obtained by the application of the fixed point index in cones.

In [14], authors study the existence and nonexistence of symmetric positive solutions of the following fourth-order BVP:

$$\begin{aligned} (\phi_p(x''(t)))'' &= w(t)f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x(1) &= \int_0^1 g(s)x(s)ds, \end{aligned} \quad (3)$$

$$\phi_p(x''(0)) = \phi_p(x''(1)) = \int_0^1 h(s)\phi_p(x''(s))ds.$$

The argument was based on the fixed point theory in cones. For fourth-order differential equation

$$u^{(4)}(t) + \beta u''(t) = \lambda f(t, u(t), u''(t)), \quad 0 < t < 1, \quad (4)$$

subject to boundary value conditions (2), author in [15] established the existence of positive solutions by the use of the Krasnosel'skiĭ fixed point theorem in cone.

The existing literature indicates that researches of fourth-order two point BVPs are excellent and methods are developed to be various. However, as to fourth-order BVPs with

integral boundary value conditions, methods applied are relatively limited. Most of results are obtained by the use of fixed point theory in the cone or the fix point index theorem.

In this paper, we will apply the monotone iterative technique to the following fourth-order BVP with integral boundary conditions:

$$u^{(4)}(t) + \beta u''(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \quad (5)$$

$$u(0) = u(1) = \int_0^1 p(s) u(s) ds, \quad (6)$$

$$u''(0) = u''(1) = \int_0^1 q(s) u''(s) ds.$$

We do not assume that the upper and lower solutions to the boundary value problem should exist but construct the specific form of the symmetric upper and lower solutions. And we will construct successive iterative schemes for approximating solutions. In addition, it is worth stating that the first term of our iterative scheme is a simple function or a constant function. Therefore, the iterative scheme is feasible. Under the appropriate assumptions on nonlinear term, a new and general result to the existence of symmetric positive solution of BVP (5) and (6) is obtained.

We assume that the following conditions hold throughout the paper:

$$(S_1) \quad \pi^2/4 \leq \beta < \pi^2;$$

$$(S_2) \quad \int_0^1 p(s) ds < 1, \int_0^1 q(s) \sin \sqrt{\beta} s ds + \int_0^1 q(s) \sin \sqrt{\beta} (1-s) ds < \sin \sqrt{\beta};$$

$$(S_3) \quad f \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty)), \quad p, q \in L^1[0, 1], \quad p(s) \geq 0, \quad q(s) \geq 0.$$

2. Preliminaries

Given $m(x) \in L^1[0, 1]$, let

$$\begin{aligned} \delta_1 &= 1 - \int_0^1 m(x) dx; \\ \delta_2 &= \sin \sqrt{\beta} - \int_0^1 m(x) \sin \sqrt{\beta} x dx \\ &\quad - \int_0^1 m(x) \sin \sqrt{\beta} (1-x) dx \end{aligned} \quad (7)$$

and $\delta_i \neq 0$, $i = 1, 2$. Denoted by $H_i(t, s)$, $i = 1, 2$, the Green's function of the following problem:

$$\begin{aligned} -u''(t) + \lambda_i u(t) &= 0, \quad 0 < t < 1, \\ u(0) = u(1) &= \int_0^1 m(s) u(s) ds. \end{aligned} \quad (8)$$

Then, careful calculation yields

$$\begin{aligned} H_1(t, s) &= G_1(t, s) + \frac{1}{\delta_1} \int_0^1 G_1(s, x) m(x) dx, \\ G_1(t, s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \\ H_2(t, s) &= G_2(t, s) \\ &\quad + \frac{\sin \sqrt{\beta} t + \sin \sqrt{\beta} (1-t)}{\delta_2} \int_0^1 G_2(s, x) m(x) dx, \end{aligned} \quad (9)$$

$$G_2(t, s) = \begin{cases} \frac{\sin \sqrt{\beta} t \sin \sqrt{\beta} (1-s)}{\sqrt{\beta} \sin \sqrt{\beta}}, & 0 \leq t \leq s \leq 1, \\ \frac{\sin \sqrt{\beta} s \sin \sqrt{\beta} (1-t)}{\sqrt{\beta} \sin \sqrt{\beta}}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Denote

$$\begin{aligned} \rho_1 &= \frac{1}{1 - \int_0^1 p(x) dx}; \\ \rho_2 &= \sin \sqrt{\beta} - \int_0^1 q(x) \sin \sqrt{\beta} x dx \\ &\quad - \int_0^1 q(x) \sin \sqrt{\beta} (1-x) dx; \\ \rho_2(t) &= \frac{1}{\rho_2} (\sin \sqrt{\beta} t + \sin \sqrt{\beta} (1-t)). \end{aligned} \quad (10)$$

Lemma 1 (see [15]). Suppose that (S_1) – (S_3) hold. Then, for any $g \in C[0, 1]$, u solves the problem

$$\begin{aligned} u^{(4)}(t) + \beta u''(t) &= g(t), \quad 0 < t < 1, \\ u(0) = u(1) &= \int_0^1 p(s) u(s) ds, \\ u''(0) = u''(1) &= \int_0^1 q(s) u''(s) ds \end{aligned} \quad (11)$$

if and only if $u(t) = \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) g(\tau) d\tau ds$, where

$$\begin{aligned} K_1(t, s) &= G_1(t, s) + \rho_1 \int_0^1 G_1(s, x) p(x) dx, \\ K_2(s, \tau) &= G_2(s, \tau) + \rho_2(s) \int_0^1 G_2(\tau, x) q(x) dx. \end{aligned} \quad (12)$$

During the process of getting the above solution, we can also know

$$u''(t) = - \int_0^1 K_2(t, s) g(s) ds. \quad (13)$$

Lemma 2. *If (S_2) is satisfied, the following results are true:*

- (1) $K_i(t, s) \geq 0$, for $t, s \in [0, 1]$, $i = 1, 2$;
- (2) $G_i(1-t, 1-s) = G_i(t, s)$, $G_i(t, s) \leq G_i(t, t)$, for $t, s \in [0, 1]$, $i = 1, 2$.

Denote

$$\begin{aligned} l_1 &= \int_0^1 G_1(x, x) p(x) dx; \\ l_2 &= \int_0^1 G_2(x, x) q(x) dx; \\ h(t) &= G_2(t, t) + \rho_2(t) l_2; \\ l_3 &= \int_0^1 h(t) dt. \end{aligned} \quad (14)$$

As $\sqrt{\beta} \in [\pi/2, \pi)$, it is easy to check that $h(t) = h(1-t)$ and $h''(t) \leq 0$, for $t \in [0, 1]$. Hence, from the symmetry and concavity of $h(t)$, we have

$$h(t) \leq h\left(\frac{1}{2}\right) \leq 2 \int_0^1 h(t) dt = 2l_3. \quad (15)$$

In addition, for $t, s \in [0, 1]$, the following results hold:

$$\begin{aligned} K_1(t, s) &= G_1(t, s) + \rho_1 \int_0^1 G_1(s, x) p(x) dx \\ &\leq G_1(t, t) + \rho_1 \int_0^1 G_1(x, x) p(x) dx \\ &= t(1-t) + \rho_1 l_1, \\ K_1\left(\frac{1}{2}, s\right) &\leq \frac{1}{4} + \rho_1 l_1. \end{aligned} \quad (16)$$

Further,

$$\begin{aligned} K_2(s, \tau) &= G_2(s, \tau) + \rho_2(s) \int_0^1 G_2(\tau, x) q(x) dx \\ &\leq G_2(s, s) + \rho_2(s) \int_0^1 G_2(x, x) q(x) dx \\ &= G_2(s, s) + \rho_2(s) l_2 \\ &= h(s) \end{aligned} \quad (18)$$

and therefore

$$K_2\left(\frac{1}{2}, \tau\right) \leq h\left(\frac{1}{2}\right). \quad (19)$$

We consider Banach space $E = C^2[0, 1]$ equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u''\|_\infty\}$, where $\|u\|_\infty = \sup_{t \in [0, 1]} |u(t)|$. In this paper, a symmetric positive solution u^* of (5) means a function which is symmetric and positive on $(0, 1)$ and satisfies (5) as well as the boundary conditions (6).

In this paper, we always suppose that the following assumptions hold:

- (H₁) $f(t, u, v) = f(1-t, u, v)$ for $t \in [0, 1]$, $u \in [0, \infty)$, $v \in (-\infty, 0]$;
- (H₂) $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, for $0 \leq u_1 \leq u_2 < +\infty$, $0 \leq |v_1| \leq |v_2| < +\infty$;
- (H₃) $f(t, 0, 0) \neq 0$, for $t \in [0, 1]$.

Denote

$$\begin{aligned} P &= \{u \in E : u(t) \geq 0, u''(t) \leq 0, \\ &\quad u(t) = u(1-t), t \in [0, 1]\}. \end{aligned} \quad (20)$$

It is easy to see that P is a cone in E .

We define the operator T as follows:

$$\begin{aligned} (Tu)(t) &= \iint_0^1 K_1(t, s) K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds, \\ &\quad t \in [0, 1]. \end{aligned} \quad (21)$$

By the above argument, we know that, for any $u \in E$, $Tu \in E$ and

$$\begin{aligned} (Tu)''(t) &= - \int_0^1 K_2(t, \tau) f(\tau, u(\tau), u''(\tau)) d\tau, \\ &\quad t \in [0, 1]. \end{aligned} \quad (22)$$

Lemma 3. *If (S_1) – (S_3) are satisfied, $T : P \rightarrow P$ is completely continuous; that is, T is continuous and compact.*

Proof. For any $u \in P$, from (21) and (22), combining Lemma 2 and (S_3) , we know that $(Tu)(t) \geq 0$ and $(Tu)''(t) \leq 0$ for $t \in [0, 1]$. We now prove that Tu is symmetric about $1/2$.

For $t \in [0, 1]$,

$$\begin{aligned} (Tu)(1-t) &= \iint_0^1 K_1(1-t, s) K_2(s, \tau) \\ &\quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &= \iint_1^0 G_1(1-t, 1-s) G_2(1-s, 1-\tau) \\ &\quad \cdot f(1-\tau, u(1-\tau), u''(1-\tau)) \\ &\quad \cdot d(1-\tau) d(1-s) \\ &\quad + \int_1^0 \int_0^1 G_1(1-t, 1-s) \\ &\quad \cdot \left(\rho_2(1-s) \int_0^1 G_2(\tau, x) q(x) dx \right) \\ &\quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau d(1-s) \end{aligned}$$

$$\begin{aligned}
& + \iint_0^1 \left(\rho_1 \int_0^1 G_1(s, x) p(x) dx \right) \\
& \quad \cdot G_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& + \iint_0^1 \left(\rho_1 \int_0^1 G_1(s, x) p(x) dx \right) \\
& \quad \cdot G_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& = \iint_0^1 G_1(t, s) G_2(s, \tau) \\
& \quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& + \iint_0^1 G_1(t, s) \left(\rho_2(s) \int_0^1 G_2(\tau, x) q(x) dx \right) \\
& \quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& + \iint_0^1 \left(\rho_1 \int_0^1 G_1(s, x) p(x) dx \right) \\
& \quad \cdot G_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& + \iint_0^1 \left(\rho_1 \int_0^1 G_1(s, x) p(x) dx \right) \\
& \quad \cdot G_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& = (Tu)(t).
\end{aligned} \tag{23}$$

So, $TP \subset P$. The continuity of T is obvious. We now prove that T is compact. Let $\Omega \subset P$ be a bounded set. Then, there exists R such that

$$\Omega = \{u \in P : \|u\| \leq R\}. \tag{24}$$

For any $u \in \Omega$, we have

$$\begin{aligned}
0 & \leq f(t, u(t), u''(t)) \\
& \leq \max \{f(t, u, u'') \mid t \in [0, 1], \\
& \quad u \in [0, R], u'' \in [-R, 0]\} \\
& =: M.
\end{aligned} \tag{25}$$

Therefore, from (17) and (18), we have

$$\begin{aligned}
\|Tu\|_\infty & = (Tu)\left(\frac{1}{2}\right) \\
& = \iint_0^1 K_1\left(\frac{1}{2}, s\right) K_2(s, \tau) \\
& \quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau ds
\end{aligned}$$

$$\begin{aligned}
& \leq M \iint_0^1 \left(\frac{1}{4} + \rho_1 l_1 \right) h(s) d\tau ds \\
& = \left(\frac{1}{4} + \rho_1 l_1 \right) l_3 M,
\end{aligned} \tag{26}$$

and from (19), we have

$$\begin{aligned}
\|(Tu)''\|_\infty & = -(Tu)''\left(\frac{1}{2}\right) \\
& = \int_0^1 K_2\left(\frac{1}{2}, \tau\right) f(\tau, u(\tau), u''(\tau)) d\tau \\
& \leq M \int_0^1 h\left(\frac{1}{2}\right) d\tau \\
& = Mh\left(\frac{1}{2}\right).
\end{aligned} \tag{27}$$

So, $\|Tu\|$ is uniformly bounded. Next we prove that Tu is equicontinuous.

For $0 \leq t_1 \leq t_2 \leq s \leq 1$, we have

$$\begin{aligned}
& \|(Tu)(t_2) - (Tu)(t_1)\|_\infty \\
& = \iint_0^1 |K_1(t_2, s) - K_1(t_1, s)| \\
& \quad \cdot K_2(s, \tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\
& \leq M \iint_0^1 |K_1(t_2, s) - K_1(t_1, s)| \\
& \quad \cdot K_2(s, \tau) d\tau ds \\
& = M \iint_0^1 |G_1(t_2, s) - G_1(t_1, s)| \\
& \quad \cdot K_2(s, \tau) d\tau ds \\
& \leq M \iint_0^1 |t_2 - t_1| K_2(s, \tau) d\tau ds \\
& = N_1 |t_2 - t_1|,
\end{aligned} \tag{28}$$

where $N_1 = M \iint_0^1 K_2(s, \tau) d\tau ds$ and

$$\begin{aligned}
& \|(Tu)''(t_2) - (Tu)''(t_1)\|_\infty \\
& \leq M \int_0^1 |K_2(t_2, \tau) - K_2(t_1, \tau)| d\tau \\
& \leq M \int_0^1 |G_2(t_2, \tau) - G_2(t_1, \tau)| d\tau \\
& \quad + M |\rho_2(t_2) - \rho_2(t_1)| \\
& \quad \cdot \iint_0^1 G_2(\tau, x) q(x) dx d\tau.
\end{aligned} \tag{29}$$

According to the Lagrange mean value theorem, we obtain that

$$\begin{aligned} & |G_2(t_2, \tau) - G_2(t_1, \tau)| \\ &= \frac{\sin \sqrt{\beta} (1 - \tau)}{\sqrt{\beta} \sin \sqrt{\beta}} \left| \sin \sqrt{\beta} t_2 - \sin \sqrt{\beta} t_1 \right| \\ &\leq \frac{1}{\sqrt{\beta} \sin \sqrt{\beta}} \left| \sqrt{\beta} t_2 - \sqrt{\beta} t_1 \right| \\ &= \frac{1}{\sin \sqrt{\beta}} |t_2 - t_1|. \end{aligned} \quad (30)$$

Similarly, we have

$$|\rho_2(t_2) - \rho_2(t_1)| \leq \frac{2\sqrt{\beta}}{\rho_2} |t_2 - t_1|. \quad (31)$$

Hence, there exists a positive constant N_2 such that

$$\|(Tu)''(t_2) - (Tu)''(t_1)\|_{\infty} \leq N_2 |t_2 - t_1|. \quad (32)$$

And the similar results can be obtained for $0 \leq s \leq t_1 \leq t_2 \leq 1$ and $0 \leq t_1 \leq s \leq t_2 \leq 1$.

The Arzelà-Ascoli theorem guarantees that $T\Omega$ is relatively compact which means that T is compact. $\square$

3. Existence and Iterative of Solutions for BVP (5) and (6)

Theorem 4. Assume that (H_1) – (H_3) hold. If there exists two positive numbers $a_1 < a$ such that

$$\sup_{t \in [0,1]} f(t, a, a) \leq a_1, \quad (33)$$

where a and a_1 satisfy

$$a \geq \max \left\{ \left(\frac{1}{4} + \rho_1 l_1 \right) l_3, 2l_3, h\left(\frac{1}{2}\right) \right\} a_1. \quad (34)$$

Then, problem (5) and (6) has concave symmetric positive solution $w^*, v^* \in P$ with

$$\begin{aligned} & \|w^*\|_{\infty} \leq a, \\ & \lim_{n \rightarrow \infty} T^n w_0 = w^*, \end{aligned} \quad (35)$$

where

$$\begin{aligned} & w_0(t) = a_1 l_3 (t(1-t) + \rho_1 l_1), \\ & \|v^*\|_{\infty} \leq a, \\ & \lim_{n \rightarrow \infty} T^n v_0 = v^*, \end{aligned} \quad (36)$$

where

$$v_0(t) = 0. \quad (37)$$

Proof. We denote $P_a = \{w \in P : \|w\| \leq a\}$. In what follows, we first prove that $TP_a \subset P_a$.

Let $w \in P_a$; then $0 \leq w(t) \leq \sup_{t \in [0,1]} w(t) = \|w\|_{\infty} \leq a$, $|w''(t)| \leq \sup_{t \in [0,1]} |w''(t)| = \|w''\|_{\infty} \leq a$.

By assumption (H_2) and (33), for $t \in [0, 1]$, we have

$$0 \leq f(t, w(t), w''(t)) \leq a_1. \quad (38)$$

For any $w(t) \in P_a$, by Lemma 3, we know that $Tw \in P$. According to (17), (18), and (33), we get

$$\begin{aligned} \|Tw\|_{\infty} &= (Tw) \left(\frac{1}{2} \right) \\ &= \int_0^1 K_1 \left(\frac{1}{2}, s \right) K_2(s, \tau) \\ &\quad \cdot f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &\leq a_1 \int_0^1 \left(\frac{1}{4} + \rho_1 l_1 \right) h(s) d\tau ds \\ &= a_1 \left(\frac{1}{4} + \rho_1 l_1 \right) \int_0^1 h(s) ds \\ &= a_1 l_3 \left(\frac{1}{4} + \rho_1 l_1 \right) \leq a, \end{aligned} \quad (39)$$

and from (19) and (33), we get

$$\begin{aligned} \|(Tw)''\|_{\infty} &= -(Tw)'' \left(\frac{1}{2} \right) \\ &= \int_0^1 K_2 \left(\frac{1}{2}, \tau \right) f(\tau, u(\tau), u''(\tau)) d\tau \\ &\leq a_1 \int_0^1 K_2 \left(\frac{1}{2}, \tau \right) d\tau \\ &\leq a_1 \int_0^1 h \left(\frac{1}{2} \right) d\tau \\ &= a_1 h \left(\frac{1}{2} \right) \leq a. \end{aligned} \quad (40)$$

Hence, $\|Tw\| \leq a$. Thus, we get $TP_a \subset P_a$. Let $w_0(t) = a_1 l_3 (t(1-t) + \rho_1 l_1)$, for $t \in [0, 1]$; then $\|w_0\| \leq a$ and $w_0(t) \in P_a$. Let $w_1(t) = Tw_0$; then $w_1 \in P_a$. We denote

$$w_{n+1} = Tw_n = T^{n+1} w_0 \quad (n = 0, 1, 2, \dots). \quad (41)$$

From the definition of T , (16), (18), and (38), it follows that

$$\begin{aligned} w_1(t) &= (Tw_0)(t) \\ &= \int_0^1 K_1(t, s) K_2(s, \tau) \\ &\quad \cdot f(\tau, w_0(\tau), w_0''(\tau)) d\tau ds \end{aligned}$$

$$\begin{aligned}
&\leq a_1 \int_0^1 \int_0^1 K_1(t, s) K_2(s, \tau) d\tau ds \\
&\leq a_1 \int_0^1 (t(1-t) + \rho_1 l_1) h(s) d\tau ds \\
&= a_1 (t(1-t) + \rho_1 l_1) \int_0^1 h(s) ds \\
&= a_1 l_3 (t(1-t) + \rho_1 l_1) \\
&= w_0(t).
\end{aligned} \tag{42}$$

On the other hand, from (15), (18), and (38), we have

$$\begin{aligned}
|w_1''(t)| &= |(Tw_0)''(t)| \\
&= \int_0^1 K_2(t, \tau) f(\tau, w_0(\tau), w_0''(\tau)) d\tau \\
&\leq a_1 \int_0^1 K_2(t, \tau) d\tau \\
&\leq a_1 \int_0^1 h(t) d\tau \\
&= a_1 h(t) \\
&\leq 2a_1 l_3 \\
&= |w_0''(t)|.
\end{aligned} \tag{43}$$

From (H₂), it follows that

$$\begin{aligned}
w_2(t) &= (Tw_1)(t) \leq (Tw_0)(t) = w_1(t), \quad 0 \leq t \leq 1, \\
|w_2''(t)| &= |(Tw_1)''(t)| \leq |(Tw_0)''(t)| = |w_1''(t)|, \quad 0 \leq t \leq 1.
\end{aligned} \tag{44}$$

By induction,

$$\begin{aligned}
w_{n+1}(t) &\leq w_n(t), \\
|w_{n+1}''(t)| &\leq |w_n''(t)|, \\
0 \leq t \leq 1 \quad (n = 0, 1, 2, \dots).
\end{aligned} \tag{45}$$

Since $TP_a \subset P_a$, we have $w_n \in P_a$, $n = 0, 1, 2, \dots$. From Lemma 3, T is completely continuous. We assert that $\{w_n\}_{n=1}^\infty$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^\infty$ and there exists $w^* \in P_a$ such that $w_{n_k} \rightarrow w^*$.

Let $v_0 = 0$, $t \in [0, 1]$; then $v_0 \in P_a$. Let $v_1 = Tv_0$; then $v_1 \in P_a$; we denote

$$v_{n+1} = Tv_n = T^{n+1}v_0, \quad n = 0, 1, 2, \dots \tag{46}$$

Similarly to $\{w_n\}_{n=1}^\infty$, we assert that $\{v_n\}_{n=1}^\infty$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^\infty$ and there exists $v^* \in P_a$, such that $v_{n_k} \rightarrow v^*$.

Since $v_1(t) = (Tv_0)(t) = (T0)(t) \in P_a$, we have

$$\begin{aligned}
v_1(t) &= (Tv_0)(t) = (T0)(t) \geq 0, \quad 0 \leq t \leq 1, \\
|v_1''(t)| &= |(Tv_0)''(t)| = |(T0)''(t)| \geq 0, \\
0 &\leq t \leq 1.
\end{aligned} \tag{47}$$

Hence,

$$\begin{aligned}
v_2(t) &= (Tv_1)(t) \geq (T0)(t) = v_1(t), \quad 0 \leq t \leq 1, \\
|v_2''(t)| &= |(Tv_1)''(t)| \geq |(T0)''(t)| = |v_1''(t)|, \\
0 &\leq t \leq 1.
\end{aligned} \tag{48}$$

By induction, $v_{n+1} \geq v_n$, $|v_{n+1}''(t)| \geq |v_n''(t)|$, $0 \leq t \leq 1$, ($n = 0, 1, 2, \dots$). Hence, we assert that $v_n \rightarrow v^*$, $Tv^* = v^*$.

If $f(t, 0, 0) \neq 0$, $0 \leq t \leq 1$, then the zero function is not the solution of BVP (5) and (6). Thus, $\max_{0 \leq t \leq 1} |v^*(t)| > 0$; we have

$$v^* \geq \min\{t, 1-t\} \max_{0 \leq t \leq 1} |v^*(t)| > 0, \quad 0 < t < 1. \tag{49}$$

It is well known that the fixed point of operator T is the solution of BVP (5) and (6). Therefore, w^* and v^* are two positive, concave, and symmetric solutions of BVP (5) and (6). $\square$

Example 5. Consider the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{aligned}
u^{(4)}(t) + \frac{\pi^2}{4} u''(t) &= f(t, u(t), u''(t)), \quad 0 < t < 1, \\
u(0) = u(1) &= \int_0^1 su(s) ds, \\
u''(0) = u''(1) &= \int_0^1 su''(s) ds,
\end{aligned} \tag{50}$$

where

$$f(t, u, v) = \frac{1}{30} \left(\sqrt{t(1-t)} + \frac{1}{2} \right) [v^2 + \ln(u+2)]. \tag{51}$$

The calculation yields

$$\begin{aligned}
\rho_1 &= 2, \\
\rho_2 &= \frac{\pi-2}{\pi}, \\
l_1 &= \frac{1}{12}, \\
l_2 &= \frac{1}{\pi^2}, \\
l_3 &= \frac{2}{\pi(\pi-2)},
\end{aligned}$$

$$\begin{aligned}
\rho_2(t) &= \frac{\sin(\pi/2)t + \sin(\pi/2)(1-t)}{\rho_2}, \\
h(t) &= \frac{2}{\pi} \sin \frac{\pi}{2} t \sin \frac{\pi}{2} (1-t) \\
&\quad + \frac{\sin(\pi/2)t + \sin(\pi/2)(1-t)}{\pi(\pi-2)}, \\
h\left(\frac{1}{2}\right) &= \frac{\pi-2+\sqrt{2}}{\pi(\pi-2)}.
\end{aligned} \tag{52}$$

It is easy to check that assumptions (S_1) – (S_3) hold. Set $a = 12$, $a_1 = 10$. Then we can verify that conditions (H_1) and (H_2) and (33) are satisfied. Then applying Theorem 4, BVP (50) has two concave symmetric positive solutions $w^*, v^* \in P$ with

$$\begin{aligned}
\|w^*\|_\infty &\leq 10, \\
\lim_{n \rightarrow \infty} T^n w_0 &= w^*,
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
w_0(t) &= \frac{60t(1-t) + 10}{3\pi(\pi-2)}, \\
\|v^*\|_\infty &\leq 10, \\
\lim_{n \rightarrow \infty} T^n v_0 &= v^*,
\end{aligned} \tag{54}$$

where

$$v_0(t) = 0. \tag{55}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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