Global Structure of Positive Solutions for a Singular Fourth-Order Integral Boundary Value Problem

Wenguo Shen and Tao He

Department of Basic Courses, Lanzhou Institute of Technology, Lanzhou 730050, China

Correspondence should be addressed to Wenguo Shen; shenwg369@163.com

Received 11 November 2013; Revised 14 December 2013; Accepted 16 December 2013; Published 8 January 2014

Academic Editor: Gabriele Bonanno

Copyright © 2014 W. Shen and T. He. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider fourth-order boundary value problems
\[ u'''' + k x''' + lx = \lambda h(t) f(x), \quad 0 < t < 1, \]
\[ u(0) = \int_0^1 u(s) d\alpha(s), \quad u'(0) = u(1) = u''(0) = 0, \]
where \( \int_0^1 u(s) d\alpha(s) \) is a Stieltjes integral with \( \alpha(t) \) being nondecreasing and \( \alpha(t) \) being not a constant on \([0,1]\); \( h(t) \) may be singular at \( t = 0 \) and \( t = 1 \), \( h \in C((0,1],[0,\infty)) \) with \( h(t) \neq 0 \) on any subinterval of \((0,1)\); \( f \in C([0,\infty],[0,\infty)) \) and \( f(s) > 0 \) for all \( s > 0 \), and \( f_0 = \infty, \quad f_\infty = 0, \quad f_0 = \lim_{s \to 0^+} f(s)/s, \quad f_\infty = \lim_{s \to \infty} f(s)/s. \)

We investigate the global structure of positive solutions by using global bifurcation techniques.

1. Introduction

Recently, fourth-order boundary value problem
\[ x'''' + k x''' + lx = \lambda h(t) f(x), \quad 0 < t < 1, \]
\[ x(0) = x'(0) = x''(0) = x'''(0) = 0, \tag{1} \]
has been investigated by the fixed point theory in cones, see [1–4] \((k = l = 0)\). By applying bifurcation techniques, see Rynne [5] \((k = l = 0)\), Korman [6] \((k = l = 0)\), Xu and Han [7] \((k = 0, l \neq 0)\), Shen [8, 9] \((k \neq 0, l \neq 0)\), and references therein. However, these papers only studied the nonsingular boundary value problems.

In 2008, Webb et al. [10] studied the existence of multiple positive solutions of nonlinear nonlocal boundary value problems (BVPs) for equations of the form
\[ u''''(t) = g(t) \tilde{f}(t, u(t)), \quad \text{for almost every } t \in (0, 1), \]
\[ u(0) = \int_0^1 u(s) dA(s), \quad u'(0) = u(1) = u''(1) = 0, \tag{2} \]
where \( g, \tilde{f} \) are continuous and nonnegative functions and \( A \) is a function of bounded variation. They treat many boundary conditions appearing in the literature in a unified way. The main tool they used is the fixed point index theory in cones. In 2009, Ma and An [11] studied the global structure for second-order nonlocal boundary value problem involving Stieltjes integral conditions by applying bifurcation techniques.

Motivated by above papers, in this paper, we will use global bifurcation techniques to study the global structure of positive solutions of the singular problem
\[ u''''(t) = \lambda h(t) f(u(t)), \quad 0 < t < 1, \]
\[ u(0) = \int_0^1 u(s) d\alpha(s), \quad u'(0) = u(1) = u''(1) = 0, \tag{3} \]
where \( h(t) \) may be singular at \( t = 0 \) and \( t = 1 \), and \( \lambda \in (0, \infty) \) is a parameter.

In order to prove our main result, let us make the assumptions as follows:

(A1) \( \alpha : [0, 1] \to \mathbb{R} \) is nondecreasing and \( \alpha(t) \) is not a constant on \([0, 1]\), \( \int_0^1 k(t,s)d\alpha(t) \geq 0 \) for \( s \in [0, 1] \), and \( 0 \leq a < 1 \) with \( a := \int_0^1 \gamma(t)d\alpha(t), \quad \gamma(t) = (t - 1)^2(2t + 1); \)

(A2) \( h \in C((0,1],[0,\infty)) \) with \( h(t) \neq 0 \) on any subinterval of \((0,1), \) and \( 0 < \int_0^1 h(s)ds < \infty; \)
Discrete Dynamics in Nature and Society

(A3) \( f \in C([0, \infty), [0, \infty]) \) satisfies \( f(s) > 0 \) for all \( s > 0 \);
(A4) \( f_0 = \lim_{s \to 0^+} (f(s)/s) = \infty \);
(A5) \( f_\infty = \lim_{s \to \infty} (f(s)/s) = 0 \).

Remark 1. For other results on the existence and multiplicity of positive solutions and nodal solutions for the boundary value problems of fourth-order ordinary differential equations based on bifurcation techniques, see Ma et al. [12–15] and Bai and Wang [16] and their references.

The rest of the paper is arranged as follows: In Section 2, we state some properties of superior limit of certain infinity collection of connected sets. In Section 3, we give some preliminary results. In Section 4, we state and prove our main results.

2. Superior Limit and Component

In order to treat the case \( f_0 = \infty \), \( f_\infty = 0 \), we will need the following definition and lemmas.

Definition 2 (see [17]). Let \( X \) be a Banach space and let \( \{ C_n : n = 1, 2, \ldots \} \) be a family of subsets of \( X \). Then the superior limit \( D \) of \( \{ C_n \} \) is defined by

\[
D := \limsup_{n \to \infty} C_n = \{ x \in X | \exists \{ n_i \} \subset \mathbb{N}, x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x \}.
\]

Lemma 3 (see [17]). Each connected subset of metric space \( X \) is contained in a component, and each connected component of \( X \) is closed.

Lemma 4 (see [11]). Let \( X \) be a Banach space and let \( \{ C_n : n = 1, 2, \ldots \} \) be a family of closed connected subsets of \( X \). Assume that

(i) there exist \( z_n \in C_n, n = 1, 2, \ldots \), and \( z^* \in X \), such that \( z_n \to z^* \);
(ii) \( r_n = \sup \| x \| \in C_n \) = \( \infty \);
(iii) for all \( R > 0 \), \( \bigcup_{n=1}^{\infty} C_n \cap B_R \) is a relative compact set of \( X \), where

\[
B_R = \{ x \in X | \| x \| \leq R \}.
\]

Then there exists an unbounded connected component \( C \) in \( D \) and \( z^* \in C \).

3. Preliminaries

We consider the problem as follows:

\[
u'''(t) = y(t), \quad 0 < t < 1,
\]

\[
u(0) = \int_0^1 u(s) \, d\alpha(s), \quad u'(0) = u(1) = u'(1) = 0.
\]

Lemma 5. For any \( y \in C[0,1] \), the problem (6) has a unique solution

\[
u(t) = \int_0^1 K(t, s) \, y(s) \, ds,
\]

where

\[
K(t, s) = \frac{y(t)}{1 - a} \int_0^1 k(t, s) \, d\alpha(t) + k(t, s),
\]

\[
k(t, s) = \frac{1}{6} \left( s^2 (1 - s)^2 (s - t) + 2 (1 - t) s \right), \quad 0 \leq s \leq t \leq 1,
\]

\[
K(t, s) = \frac{1}{6} \left( (1 - s)^2 (1 - t)^2 (t - s) + 2 (1 - s) t \right), \quad 0 \leq t \leq s \leq 1,
\]

\[
a = \int_0^1 y(t) \, d\alpha(t), \quad y(t) = (t - 1)^2 (2t + 1).
\]

Proof. By [10], the problem (6) can be equivalently written as

\[
u(t) = \int_0^1 u(s) \, d\alpha(s) + \int_0^1 k(t, s) \, y(s) \, ds.
\]

Applying \( \alpha \) to both sides of (10), we obtain

\[
\int_0^1 u(t) \, d\alpha(t) = \int_0^1 \left[ y(t) \int_0^1 u(s) \, d\alpha(s) \right] \, d\alpha(t)
\]

\[
+ \int_0^1 \left[ \int_0^1 k(t, s) \, y(s) \, ds \right] \, d\alpha(t).
\]

Thus, we have

\[
\int_0^1 u(t) \, d\alpha(t) = \int_0^1 y(t) \, d\alpha(t) \cdot \int_0^1 u(s) \, d\alpha(s)
\]

\[
+ \int_0^1 \left[ \int_0^1 k(t, s) \, d\alpha(t) \right] y(s) \, ds.
\]

Furthermore, it follows that

\[
\int_0^1 u(s) \, d\alpha(s)
\]

\[
= \frac{1}{1 - \int_0^1 y(t) \, d\alpha(t)} \int_0^1 \left[ \int_0^1 k(t, s) \, d\alpha(t) \right] y(s) \, ds.
\]

So, we obtain

\[
u(t) = \int_0^1 \left[ \frac{y(t)}{1 - a} \int_0^1 k(t, s) \, d\alpha(t) + k(t, s) \right] y(s) \, ds.
\]

Lemma 6 (see [2–4]). Green’s function \( k(t, s) \) defined by (9) satisfies the following:

(i) \( k(t, s) \geq 0 \) is continuous for all \( t, s \in [0, 1] \);
(ii) \( c(t) k(\tau(s), s) \leq k(t, s) \leq c(\tau(s), s), \) for all \( t, s \in [0, 1] \), and for any \( \delta \in (0, 1/2) \) and \( t \in [\delta, 1 - \delta] \), such that

\[
k(t, s) \geq \frac{2 \sigma^2}{3} k(\tau(s), s), \forall s \in [0, 1],
\]
where
\[\tau(s) = \begin{cases} 
\frac{1}{3 - 2s}, & 0 \leq s \leq \frac{1}{2}, \\
\frac{2s}{1 + 2s}, & \frac{1}{2} \leq s \leq 1,
\end{cases}\]

\[k(\tau(s), s) = \max_{t \in [0,1]} k(t, s) = \begin{cases} 
\frac{2^2(1 - s)^3}{3(3 - 2s)^2}, & 0 \leq s \leq \frac{1}{2}, \\
\frac{2s^3(1 - s)^2}{3(1 + 2s)^2}, & \frac{1}{2} \leq s \leq 1,
\end{cases}\]

\[c(t) = \frac{2}{3} \min\{t^2, (1 - t)^2\}, \quad t \in [0,1],\]

\[\min_{t \in [0,1]} c(t) = \frac{2}{3}\delta^2.\]

**Lemma 7.** Green's function \(K(t, s)\) defined by (8) satisfies the following:

(i) \(K(t, s) \geq 0\) is continuous for all \(t, s \in [0,1]\);

(ii) \(K(t, s) \leq K(s), \) for all \(t, s \in [0,1], \) and for any \(\delta \in (0, 1/2), \) there exists a constant \(\gamma_\delta > 0, \) for any \(t \in [\delta, 1 - \delta], \) such that

\[K(t, s) \geq \gamma_\delta K(s), \quad \forall s \in [0,1],\]

\[K(s) = \frac{1 - a + \alpha(1) - \alpha(0)}{1 - a} \cdot k(\tau(s), s),\]

\[\gamma_\delta = \frac{2}{3}\delta^2 \cdot \frac{1 - a + \delta^2(3 - 2\delta)(\alpha(1) - \alpha(0))}{1 - a + \alpha(1) - \alpha(0)},\]

where \(k(t, s)\) is defined by (9), \(\max_{t \in [0,1]} \gamma(t) = 1, \)

\[\min_{t \in [\delta, 1 - \delta]} \gamma(t) = \gamma_\delta.\]

**Proof.** (i) From Lemma 6 (i), we get the proof of Lemma 7 (i) immediately.

(ii) By Lemma 6 (ii), we get

\[K(t, s) \leq \frac{1}{1 - a} \int_0^1 k(\tau(s), s) \alpha(t) + k(\tau(s), s)\]

\[\leq \frac{1 - a + \alpha(1) - \alpha(0)}{1 - a} \cdot k(\tau(s), s) = K(s),\]

\[\forall t, s \in [0,1].\]

By Lemma 6 (ii), for any \(\delta \in (0, 1/2)\) and \(t \in [\delta, 1 - \delta], s \in [0,1],\) we obtain

\[K(t, s) \geq \frac{\delta^2(3 - 2\delta)}{1 - a} \int_0^1 \frac{2}{3}\delta^2 k(\tau(s), s) \alpha(t)\]

\[+ \frac{2}{3}\delta^2 k(\tau(s), s)\]

\[\geq \frac{2}{3}\delta^2 \cdot \frac{1 - a + \delta^2(3 - 2\delta)(\alpha(1) - \alpha(0))}{1 - a} \cdot k(\tau(s), s) = \gamma_\delta K(s), \quad \forall s \in [0,1].\]

**Lemma 8.** For \(y \in C[0,1]\) and \(y \geq 0, \) the unique solution of the problem (6) satisfies the following:

(i) \(u(t) \geq 0, \) for all \(t \in [0,1];\)

(ii) \(\min_{t \in [\delta, 1 - \delta]} u(t) \geq \gamma_\delta \|u\|_\infty,\]

where \(\gamma_\delta\) is defined by Lemma 7 (ii), \(\|u\|_\infty = \max_{t \in [0,1]} |u|.

**Proof.** (i) From Lemma 7 (i), we get the proof of Lemma 8 (i) immediately.

(ii) From (7) and Lemma 7, we have

\[\min_{t \in [\delta, 1 - \delta]} u(t) = \min_{t \in [\delta, 1 - \delta]} \int_0^1 K(t, s) y(s) ds\]

\[\geq \gamma_\delta \int_0^1 K(t, s) y(s) ds\]

\[\geq \gamma_\delta \int_0^1 \max_{t \in [0,1]} K(t, s) y(s) ds\]

\[\geq \gamma_\delta \max_{t \in [0,1]} \int_0^1 K(t, s) y(s) ds = \gamma_\delta \|u\|_\infty.\]

Therefore, the proof of Lemma 8 is complete. □

Let \(Y = C[0,1]\) be the Banach space with the norm \(\|u\|_\infty = \max_{t \in [0,1]} |u|.

Let \(E = \{u \in C^2[0,1] | u(0) = \int_0^1 u(s) d\alpha(s), u'(0) = u(1) = u'(1) = 0\}\) with the norm

\[\|u\|_E = \max \{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}.\]

(21)

Let

\[P = \left\{ u \in C[0,1] | u(t) \geq 0, t \in [0,1], \right\}\]

\[\min_{t \in [\delta, 1 - \delta]} u(t) \geq \gamma_\delta \|u\|_\infty \right\},\]

and for \(r > 0, \) let \(\Omega_r = \{u \in P \|u\|_E < r\}.

In order to use bifurcation technique to study the problem (3), we consider the linear eigenvalue problem

\[u''(t) = \lambda h(t) u(t), \quad 0 < t < 1,\]

\[u(0) = \int_0^1 u(s) d\alpha(s), \quad u'(0) = u(1) = u'(1) = 0.\]

(23)

Let

\[L_\lambda u(t) = \lambda \int_0^1 K(t, s) h(s) u(s) ds, \quad t \in [0,1],\]

(24)

\[T_\lambda u(t) = \lambda \int_0^1 K(t, s) f(u(s)) ds, \quad t \in [0,1].\]

(25)

By [18], it is easy to show the following lemma.
Lemma 9. Assume that (A1)–(A3) hold the following.

$L_{\lambda} : P \to P$ is a completely continuous linear operator and $L_{\lambda}(P) \subset P$, and the fixed points of the operator $L_{\lambda}$ in $P$ are the positive solutions of the BVP (23).

$T_{\lambda} : P \to P$ is a completely continuous operator and $T_{\lambda}(P) \subset P$, and the fixed points of the operator $T_{\lambda}$ in $P$ are the positive solutions of the BVP (3).

By virtue of Krein-Rutman theorem (Theorem 2.5 in [19]), one has (see [18] or [20]) the following lemma.

Lemma 10. Suppose that $L_{\lambda} : C[0,1] \to C[0,1]$ is a completely continuous linear operator and $L_{\lambda}(P) \subset P$. If there exist $\psi \in C[0,1] \setminus (-P)$ and a constant $c > 0$ such that $c L_{\lambda} \psi \geq \psi$, then the spectral radius $r(L_{\lambda}) \neq 0$ and $L_{\lambda}$ has a positive eigenfunction $\phi_{1}$ corresponding to its first eigenvalue \( \lambda_{1} \), that is, $\phi_{1} = \lambda_{1} L_{\lambda} \phi_{1}$.

Lemma 11. Suppose (A1) and (A2) are satisfied, then for the operator $L_{\lambda}$ defined by (24), the spectral radius $r(L_{\lambda}) \neq 0$ and $L_{\lambda}$ has a positive eigenfunction $\phi_{1} \in \text{int } P$ corresponding to its first eigenvalue $\lambda_{1} = 1/r(L_{\lambda})$.

Proof. It is easy to see that there is $t_{1} \in (0, 1)$ such that $K(t_{1}, t_{1})h(t_{1}) > 0$. Thus there exists $[\alpha, \beta] \subset (0, 1)$ such that $t_{1} \in (\alpha, \beta)$ and $K(t, s)h(s) > 0$, for all $t, s \in [\alpha, \beta]$. Take $\psi \in C[0,1]$ such that $\psi(t) \geq 0$, for all $t \in [0, 1]$, $\psi(t_{1}) > 0$, and $\psi(t) = 0$, for all $t \notin [\alpha, \beta]$. Then for $t \in [\alpha, \beta]$, we have

$$
(L_{\lambda} \psi) (t) = \lambda \int_{0}^{1} K(t, s) h(s) \psi(s) ds 
$$

$$
\geq \lambda \int_{\alpha}^{\beta} K(t, s) h(s) \psi(s) ds > 0.
$$

So there exists a constant $c > 0$ such that $c \lambda L_{\lambda} \psi \geq \psi$, for all $t \in [0, 1]$. From Lemma 10, we know that the spectral radius $r(L_{\lambda}) \neq 0$ and $L_{\lambda}$ has a positive eigenfunction corresponding to its first eigenvalue $\lambda_{1} = 1/r(L_{\lambda})$.

Lemma 12. Let (A1)–(A3) hold. The solution $u(t)$ of the problem (3) satisfies

$$
\|u\|_{\infty} \leq \|u'\|_{\infty} \leq \|u''\|_{\infty} \leq \|u'''\|_{\infty}.
$$

Proof. From $u'(0) = u'(1) = 0$, there exists $\xi \in (0, 1)$, such that $u''(\xi) = 0$. Using a similar proof of (10) in [21, page 212], it is easy to show that

$$
\|u''(t)\| = \left| \int_{0}^{t} u''(s) ds - u'(1) \right| = \left| \int_{\xi}^{1} u''(s) ds \right| 
$$

$$
\leq \int_{\xi}^{1} |u''(s)| ds \leq \int_{0}^{1} |u''(s)| ds,
$$

$$
\|u'(t)\| = \|u'(0) + \int_{0}^{t} u''(s) ds\| = \left| \int_{0}^{t} u''(s) ds \right| \leq \int_{0}^{1} |u''(s)| ds,
$$

$$
\|u(t)\| = \left| \int_{0}^{t} u'(s) ds - u(1) \right| = \left| \int_{\xi}^{1} u'(s) ds \right| 
$$

$$
= \left| \int_{0}^{1} u'(s) ds \right| \leq \int_{0}^{1} |u'(s)| ds.
$$

Furthermore, we obtain

$$
\|u'''\|_{\infty} \leq \|u''\|_{\infty} \leq \|u''\|_{\infty} \leq \|u'''\|_{\infty}.
$$

Lemma 13. Let (A1)–(A3) hold. Assume that $\{(u_{k}, \mu_{k})\} \subset (0, \infty) \times P$ is a sequence of positive solutions of (3). Assume that $\|\mu_{k}\| \leq \epsilon_{0}$ for some constant $\epsilon_{0} > 0$, and

$$
\lim_{k \to \infty} \|u_{k}\|_{E} = \infty.
$$

Then

$$
\lim_{k \to \infty} \|u_{k}\|_{\infty} = \infty.
$$

Proof. Assume on the contrary that

$$
\|u_{k}\|_{\infty} \leq M_{0}
$$

for some constant $M_{0} > 0$.

Since $(u_{k}, \mu_{k})$ is a solution of the problem (3), we have

$$
u_{k}(t) = \mu_{k} \int_{0}^{1} K(t, s) h(s) f(u_{k}(s)) ds, \quad t \in [0, 1].
$$

Thus,

$$
u_{k}'''(t) = \mu_{k} \int_{0}^{1} \frac{\partial^{3}}{\partial t^{3}} K(t, s) h(s) f(u_{k}(s)) ds,
$$

$$
\left| \frac{\partial^{3}}{\partial t^{3}} C(t, s) \right| = \left| \frac{12}{1-a} \int_{0}^{1} k(t, s) d(c(t) + \frac{\partial^{3}}{\partial t^{3}} k(t, s)) \right|
$$

$$
\leq \frac{12(1-a)}{1-a} \int_{0}^{1} k(t, s) d(c(t) + \frac{\partial^{3}}{\partial t^{3}} k(t, s))
$$

$$
\leq \frac{\alpha(1-a)}{16(1-a)} + 5 := M_{1},
$$

where $0 \leq |k(t, s)| \leq 1/192, \sup_{0 \leq t \leq 1} (\partial^{3}/\partial t^{3}) k(t, s) \leq 5$ (see [3]).

Furthermore, it follows that

$$
\|u_{k}'''\|_{\infty} \leq \epsilon_{0} M_{1} B_{0} \int_{0}^{1} h(s) ds,
$$

where $B_{0} = \max_{s \in [0, M_{0}]} \{ |f(s)| \}$, together with (A2), which implies that $\|u_{k}'''\|_{\infty}$ is bounded whenever $\|u_{k}\|_{\infty}$ is bounded. Together with Lemma 12, we obtain

$$
\|u_{k}\|_{E} \leq M_{2}
$$

for some constant $M_{2} > 0$. This is a contradiction.
4. Main Results

Let $\Sigma$ be the closure of the set of positive solutions of (3) in $[0, \infty) \times E$. The main results of this paper are the following.

**Theorem 14.** Let (A1)–(A5) hold, then (3) has at least one solution for any $\lambda \in (0, \infty)$.

Let $L : D(L) \subset E \to E$ be an operator defined by

$$Lu = u'''(t) = \lambda h(t) g[n] u(t) + \lambda h(t) \zeta[n] (u(t)),$$

for each $u \in D(L)$, with

$$D(L) = \{ u \in C^k [0,1] \mid u(0) = \int_0^1 u(s) \alpha(s) \mathrm{d}s, \quad u'(0) = u(1) = u'(1) = 0 \}.$$

Then $L$ is a closed operator and $L^{-1} : Y \to E$ is completely continuous.

For each $n \in \mathbb{N}$, define $f[n](s) : [0, \infty) \to [0, \infty)$ by

$$f[n](s) = \begin{cases} f(s), & s \in \left[0, \frac{1}{n}, \infty) \\ nf \left( \frac{1}{n} \right), & s \in \left[0, \frac{1}{n} \right] \end{cases}.$$

Then $f[n] \in C([0, \infty), [0, \infty))$ with

$$f[n](s) > 0, \quad \forall s \in (0, \infty), \quad \left( f[n] \right)_0 = nf \left( \frac{1}{n} \right).$$

By (A4), it follows that

$$\lim_{n \to \infty} f[n]_0 = \infty. \quad (41)$$

To apply the global bifurcation theorem, one extends $f$ to an odd function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s), & s \geq 0 \\ -f(-s), & s < 0 \end{cases} \quad (42)$$

Similarly one may extend $f[n]$ to an odd function $g[n] : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$.

Now let one consider the auxiliary family of the equations

$$u'''(t) = \lambda h(t) g[n](u(t)), \quad 0 < t < 1,$$

$$u(0) = \int_0^1 u(s) \alpha(s) \mathrm{d}s, \quad u'(0) = u(1) = u'(1) = 0.$$ \quad (43)

Let $\xi[n] \in C(\mathbb{R})$ be such that

$$g[n](u) = \left( g[n] \right)_0 u + \xi[n](u) = nf \left( \frac{1}{n} \right) u + \xi[n](u). \quad (44)$$

Then

$$\lim_{n \to 0} \xi[n](s) = 0. \quad (45)$$

Let one consider

$$Lu = \lambda h(t) \left( g[n] \right)_0 u + \lambda h(t) \zeta[n] (u),$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

From Lemma 5, (46) can be converted to the equivalent equation

$$u(t) = \int_0^1 K(t,s) \left[ \lambda h(s) \left( g[n] \right)_0 u(s) + \lambda h(s) \zeta[n] (u(s)) \right] \mathrm{d}s$$

$$:= \lambda L^{-1} \left[ h(\cdot) \left( g[n] \right)_0 u(\cdot) + \lambda L^{-1} \left[ h(\cdot) \zeta[n] (u(\cdot)) \right] (t) \right]. \quad (47)$$

Further one has that

$$\| L^{-1} [h(\cdot) \zeta[n] (u(\cdot))] \|_E = o \left( \| u \|_E \right), \quad \text{as} \quad \| u \|_E \to 0. \quad (48)$$

Indeed, (8) implies that, for all $(t, s) \in [0, 1] \times [0, 1],$

$$\left| \frac{\partial}{\partial t} K(t,s) \right| = \frac{6(t-1)}{1-a} \int_0^1 k(t,s) \alpha(t) + \frac{\partial}{\partial t} k(t,s) \right| \leq \frac{\alpha(1) - \alpha(0)}{128 (1-a)} + 3,$$

$$\left| \frac{\partial^2}{\partial t^2} K(t,s) \right| \leq \frac{6(2t-1)}{1-a} \int_0^1 k(t,s) \alpha(t) + \frac{\partial^2}{\partial t^2} k(t,s) \right| \leq \frac{\alpha(1) - \alpha(0)}{32 (1-a)} + 8, \quad (49)$$

where $0 \leq |k(r(s),s)| \leq 1/192, \max_{0 \leq t,s \leq 1} |\partial/\partial t k(t,s)| \leq 3,$

$$\max_{0 \leq t,s \leq 1} |\partial^2/\partial t^2 k(t,s)| \leq 3 \text{ (see [3])}.$$
(51) in Lemma 4 is satisfied with $z^* = (0,0)$. Obviously

$$ r_\ast = \sup \{ \lambda + \|u\| : (\lambda, u) \in C^\ast \} = \infty, $$

(53)

and accordingly (ii) in Lemma 4 holds. (iii) in Lemma 4 can be deduced directly from the Arzela-Ascoli theorem and the definition of $g[n]$. Therefore, the superior limit of $\{C[n]+\}$, $\mathcal{D}$, contained an unbounded connected component $C$ with $(0,0) \in C$. Since $C^\ast \subset \Phi^+$, one concludes $C \subset \Phi^+$. Moreover, $C \subset \Sigma$ by (3).

Proof of Theorem 14. We firstly prove

$$ \text{Proj}_R C = (0,\infty). $$

(54)

Assume on the contrary that $\sup \{ \lambda : (\lambda, u) \in C \} < \infty$. (55)

Then there exists a sequence $(\mu_k, u_k) \in C$ such that

$$ \lim_{k \to \infty} \|u_k\|_E = \infty, \quad \mu_k \leq c_0, $$

(56)

for some positive constant $c_0$ with doing not depend on $k$. From Lemma 13, we have

$$ \lim_{k \to \infty} \|u_k\|_E = \infty. $$

(57)

This together with the fact

$$ \min_{t \in [0,1]} u_k(t) \geq \gamma_0 \|u_k\|_E \quad \forall \delta \in \left(0, \frac{1}{2}\right) $$

(58)

imply that for arbitrary $\delta \in (0,1/2)$

$$ \lim_{k \to \infty} u_k(t) = \infty, \quad \text{uniformly for } t \in [\delta, 1-\delta]. $$

(59)

Since $(\mu_k, u_k) \in C$, we have that

$$ u_k'' = \mu_k h(t) f(u_k), \quad 0 < t < 1, $$

$$ u_k(0) = 1, \quad u_k'(0) = u_k(1) = u_k'(1) = 0. $$

(60)

Set $v_k(t) = u_k(t)/\|u_k\|_E$. Then

$$ \|v_k(t)\|_E = 1, $$

(61)

$$ v_k''(t) = \mu_k h(t) \frac{f(u_k(t))}{u_k(t)} v_k(t), \quad 0 < t < 1, $$

(62)

$$ v_k(0) = 1, \quad v_k'(0) = v_k(1) = v_k'(1) = 0. $$

(63)

Now, choosing a subsequence and relabeling if necessary, it follows that there exists $(\mu_\ast, v_\ast) \in [0,c_0] \times E$ with

$$ \|v_\ast\|_E = 1. $$

(64)

such that

$$ \lim_{k \to \infty} (\mu_k, v_k) = (\mu_\ast, v_\ast), \quad \text{in } \mathbb{R} \times Y. $$

(65)

By (A3), let

$$ f(u) = \max_{0 \leq s \leq u} f(s). $$

(66)

Then $f$ is nondecreasing and

$$ \lim_{u \to \infty} \frac{f(u)}{u} = 0. $$

(67)

Further it follows from (66) that

$$ \frac{f(u)}{\|u\|_E} \leq \frac{\bar{f}(u)}{\|u\|_E} \to 0, \quad \|u\|_E \to +\infty. $$

Thus,

$$ \lim_{k \to \infty} \frac{f(u_k)}{u_k} = 0. $$

(68)

Notice that (62) is equivalent to

$$ v_\ast(t) = \mu_\ast \int_0^1 K(t,s) h(s) f(u_k(s)) \cdot v_k(s) \, ds, \quad t \in [0,1]. $$

(69)

Furthermore, by (59), (68), and (69), together with the Lebesgue dominated convergence theorem, it follows that

$$ v_\ast(t) = \mu_\ast \int_0^1 K(t,s) h(s) \cdot v_\ast(s) \, ds, \quad t \in [0,1]. $$

It follows that

$$ v_\ast(t) = 0. $$

(70)

(71)

This contradicts (63). Therefore

$$ \sup \{ \lambda : (\lambda, u) \in C^\ast \} = \infty. $$

(72)

Noticing that $\lambda = 0$ is the only solution of the problem (3), thus

$$ \text{Proj}_R C = (0,\infty). $$

(73)

Furthermore, it follows the proof of Theorem 14.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Thanks are given to the anonymous referee for his/her valuable suggestions. The authors were supported by the NSFC (no. 11261052) and the Scientific Research Foundation of the Education department of Gansu Province (no. 1114-04).
References


Submit your manuscripts at http://www.hindawi.com