Research Article

Output Tracking via Adaptive Backstepping Higher Order Integral Sliding Mode for Uncertain Nonlinear Systems

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The authors propose a new tracking control design strategy for uncertain non-linear systems which are convertible to Semi-Strict Feedback Form (SSFF). The system in SSFF is first converted into new variables via existing adaptive backstepping control techniques. The control law is obtained by combining adaptive backstepping procedure and higher order integral sliding mode. The component of control law designed via backstepping is continuous which shows robustness against parametric uncertainties where as the discontinuous control component provides robustness against unmodeled dynamics and external disturbances. Since, this strategy relies on an integral manifold of the adaptively developed variables, therefore, the reaching phase is eliminated in this approach, which is an advantage in term of robustness. Furthermore, the parameters update law correctly provides the estimation of parameters which is again results in enhanced robustness of the strategy. The stability of proposed method is analysed theoretically and validated through a numerical example.

1. Introduction

The design process of nonlinear feedback control system is quite complex and challenging. Major concern is to design a controller that guarantees some specific behaviour of the system and is robust against unmodeled dynamics, parametric uncertainties, and external disturbances. There are no specific methods or set of analysis and design tools that can fit universally for wide range of situations and applications. However several tools are available which are generally applied to control nonlinear feedback system to achieve desired regulation or output tracking.

Sliding mode control (SMC) reported by [1, 2] is well known for its ability to compensate the external disturbance, unmodeled dynamics, and parametric uncertainties satisfying matching conditions. The classical SMC suffers from high frequency control switching the chattering phenomenon that severely damages the actuators and the system. Another limitation of this method is that it can only be applied to the systems of relative degree one. Relative degree is the

relationship of output to the input of the system. The relative degree number is the count of derivatives of the output variable on which input variable appears. In the system which satisfies 01 relative degree condition, on taking first derivative of the output variable, input variable appears in the equation. Many physical systems, such as mechanical systems and satellite control system, do not satisfy 01 relative degree condition. The need of the relative degree one and chattering reduction/elimination was handled via different techniques (see, e.g., [3–5]).

A systematic design procedure that resolves the issue of relative degree is the backstepping algorithm [6] and it is suitable for feedback linearisable systems [7–9]. This is essentially a recursive design methodology in which a virtual control is designed at each step until an actual control law is realized. Conventional feedback linearisation method [3] has a drawback in the sense that this requires a precise model around some specific equilibrium points where some useful nonlinearities responsible for stability may disappear. However, the backstepping method avoids

wasteful cancellations to achieve the global stability and to improve the performance.

Sira-Ramirez et al. [10] have combined the fundamental adaptive backstepping algorithm [11] with dynamically input-output linearization technique [7]. The blended method was extended in combination with the SMC, for those systems which can be transformed into triangular, Parametric Pure Feedback Form (PPFF) or Parametric Strict Feedback Form (PSFF) (see, e.g., [12–14]). Furthermore, some interesting SMC based techniques were put forwarded by, for example, [15–17] for nonlinear systems which are transferable into semistrict feedback form (SSFF) suffering from unmodeled and unmatched uncertainties. In this way, the benefit of SMC and adaptive backstepping were realised. An additional benefit is that the requirement of transformation into PPFF or PSFF and the justification of extra conditions on parameters and sufficient condition for the existence of sliding mode [14] were not needed. Recently, Khan et al. [18] reported strategy of dynamic control for MIMO uncertain, square, and minimum phase nonlinear system based on integral sliding mode technique. This method has synthesized dynamic sliding mode control [19, 20] and integral sliding mode strategies [1] into dynamically integral sliding mode control (DISMC) which provided the establishment of sliding mode without reaching phase. In this way the robustness against uncertainties is enhanced and performance is improved with considerable attenuation in the chattering across the integral manifold.

Fascinated by the strategy of DISMC, the authors have combined the adaptive backstepping and Higher Order Integral Sliding Mode Control (HOISMC) (see, e.g., [21, 22]) into a new technique named Adaptive Backstepping Higher Order Integral Sliding Mode Control (ABHOISMC); this will provide robustness from the initial time instant against parametric uncertainty along with considerable reduction in chattering. This newly proposed technique is developed for Single Input Single Output (SISO) nonlinear systems which are transferable into SSFF in the presence of some unknown parameters. The design process involves the subdivision of the control law into two parts, continuous control law which can be realized by applying adaptive backstepping method and discontinuous control law which is developed via an integral manifold. The parameter update law is also a sum of two terms. The first term emerges from the adaptive backstepping method and the second term is the contribution of the HOISMC. The remainder of the paper is managed as follows: the problem description/formulation is presented in Section 2, the development of the control law is explained in Section 3, an illustrative example is presented in Section 4, and the conclusions are drawn in Section 5.

2. Preliminaries

2.1. System Description. Consider a class of nonlinear systems which can be represented/transformed into the following SSFF [15]:

$$\begin{aligned}\dot{\zeta}_i &= \zeta_{i+1} + \varphi_i^T(\zeta_1, \zeta_2, \dots, \zeta_i) \theta + \eta_i(\zeta, \omega, t), \\ \dot{\zeta}_n &= f(\zeta) + g(\zeta) u + \varphi_n^T(\zeta) \theta + \eta_n(\zeta, \omega, t), \\ y &= \zeta_1; \quad 1 \leq i \leq n-1,\end{aligned}\quad (1)$$

where $\zeta = [\zeta_1, \zeta_2, \dots, \zeta_n]^T$ is the state vector, u is the scalar control input, y is the output of interest, $\varphi_i(\zeta_1, \zeta_2, \dots, \zeta_i) \in \mathcal{R}^p$, $i = 1, \dots, n$ are known and sufficiently smooth functions, $f(\zeta)$ and $g(\zeta)$ are known multivariable functions, $\theta \in R^p$ is the vector of unknown parameters, and $\eta_i(\zeta, \omega, t)$, $i = 1, \dots, n$, are the unknown nonlinear scalar functions including all the disturbances, whereas ω is an uncertain time-varying parameter.

It is required that the output y of the system should follow a specific bounded desired continuous signal $y_d(t)$.

Assumption 1. It is assumed that the n th order time derivatives of y_d exist and are bounded in nature. In addition, it is assumed that the scalar nonlinear functions $\eta_i(\zeta, \omega, t)$, $i = 1, \dots, n$, are bounded by known positive functions; that is,

$$\begin{aligned}h_i(\zeta_1, \zeta_2, \dots, \zeta_i) &\in \mathcal{R}^p, \\ |\eta_i(\zeta, \omega, t)| &\leq h_i(\zeta_1, \zeta_2, \dots, \zeta_i).\end{aligned}\quad (2)$$

2.2. Adaptive Backstepping Design Approach Development. In this study, a generalised recursive approach analogous to that of [13] is presented for the tracking of a time varying signal $y_d(t)$. The step by step development is given as follows.

2.2.1. Step 1. Consider a new variable which represents the error between the actual output of the system and the desired output in the following form.

$z_1 = \zeta_1 - y_d(t)$ and its derivative $\dot{z}_1 = \dot{\zeta}_1 - \dot{y}_d(t)$ along (1) becomes

$$\dot{z}_1 = \zeta_2 + \varphi_1^T(\zeta_1) \theta + \eta_1(\zeta, \omega, t) - \dot{y}_d(t). \quad (3)$$

The expression in (3) may be written in an alternate form as follows:

$$\dot{z}_1 = \zeta_2 + \omega_1(t)^T \hat{\theta} + \eta_1(\zeta, \omega, t) - \dot{y}_d(t) + \omega_1(t)^T \tilde{\theta}, \quad (4)$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\omega_1(t) = \varphi_1(\zeta_1)$, and $\hat{\theta}$ is the estimated value of unknown parameters.

Now, a virtual control input for the stabilization of the first step is designed by considering the Lyapunov function as follows:

$$V_1(z_1, \hat{\theta}) = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad (5)$$

where Γ is a positive definite matrix. The time derivative of V_1 along (3) takes the form

$$\dot{V}_1(z_1, \hat{\theta}) = z_1 \dot{z}_1 - \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}, \quad (6)$$

$$\begin{aligned}\dot{V}_1(z_1, \hat{\theta}) &= z_1 (\zeta_2 + \omega_1(t)^T \hat{\theta} + \eta_1(\zeta, \omega, t) - \dot{y}_d(t)) \\ &\quad + \tilde{\theta}^T \Gamma^{-1} (\Gamma \omega_1 z_1 - \dot{\tilde{\theta}}).\end{aligned}\quad (7)$$

Define the parameter update law of the first stage $\dot{\hat{\theta}} = \tau_1 = \Gamma \omega_1 z_1$ and the stabilizing control

$$\alpha_1(\zeta_1, \hat{\theta}, t) = -\omega_1(t)^T \hat{\theta} - c_1 z_1 - \frac{h_1^2 z_1}{h_1 |z_1| + (\epsilon/n) e^{-at}}, \quad (8)$$

where c_1 , a , and ϵ are positive constants. The last term in α_1 is a saturation function which avoids the chattering in the evaluation of the control input. Now, define a second new variable of the form

$$z_2 = \zeta_2 - \alpha_1(\zeta_1, \hat{\theta}, t) - \dot{y}_d(t). \quad (9)$$

By using (9) and (8) in (7) and applying Assumption 1 in (2), one gets

$$\dot{V}_1(z_1, \hat{\theta}) \leq -c_1 z_1^2 + z_1 z_2 + \frac{\epsilon}{n} e^{-at} + \bar{\theta}^T \Gamma^{-1}(\tau_1 - \dot{\hat{\theta}}). \quad (10)$$

Consequently, one has

$$\dot{z}_1 = -c_1 z_1 + z_2 + \omega_1(t)^T \bar{\theta} + \eta_1(\zeta, \omega, t) - \frac{h_1^2 z_1}{h_1 |z_1| + (\epsilon/n) e^{-at}}. \quad (11)$$

The second step is developed as follows.

2.2.2. Step 2. The time derivative of the new variable defined in (9) along (1) is given by

$$\begin{aligned} \dot{z}_2 &= \dot{\zeta}_2 - \dot{\alpha}_1(\zeta_1, \hat{\theta}, t) - \ddot{y}_d(t) \\ \dot{z}_2 &= \zeta_3 + \varphi_2^T(\zeta_1, \zeta_2) \bar{\theta} + \eta_2(\zeta, \omega, t) - \frac{\partial \alpha_1}{\partial \zeta_1} \dot{\zeta}_1 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \\ &\quad - \frac{\partial \alpha_1}{\partial t} - \ddot{y}_d(t). \end{aligned} \quad (12)$$

Now, define a Lyapunov function as follows.

$V_2 = (1/2)z_1^2 + (1/2)z_2^2 + (1/2)\bar{\theta}^T \Gamma^{-1} \bar{\theta}$ and its time derivative along (11) and (12) becomes

$$\begin{aligned} \dot{V}_2(z_1, z_2, \hat{\theta}) &= -c_1 z_1^2 + \frac{\epsilon}{n} e^{-at} \\ &\quad + z_2 \left[z_1 + \zeta_3 + \omega_2^T \bar{\theta} + \left(\eta_2 - \frac{\partial \alpha_1}{\partial \zeta_1} \eta_1 \right) \right. \\ &\quad \left. - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_1}{\partial \zeta_1} \dot{\zeta}_2 - \frac{\partial \alpha_1}{\partial t} - \ddot{y}_d(t) \right] \\ &\quad + \bar{\theta}^T \Gamma^{-1}(\tau_2 - \dot{\hat{\theta}}), \end{aligned} \quad (13)$$

where $\tau_2 = \Gamma(\omega_1 z_1 + \omega_2 z_2)$.

Now, define a third new variable of the form $z_3 = \zeta_3 - \alpha_2(z_1, z_2, \hat{\theta}, t) - \dot{y}_d(t)$. The selection of $\alpha_2(z_1, z_2, \hat{\theta}, t)$ follows as

$$\begin{aligned} \alpha_2(z_1, z_2, \hat{\theta}, t) &= -z_1 - c_2 z_2 - \omega_2^T \bar{\theta} - \frac{h_2^2 z_2}{h_2 |z_2| + (\epsilon/n) e^{-at}} \\ &\quad - \frac{h_1^2 z_2}{h_1 |z_2| (\partial \alpha_1 / \partial \zeta_1) + (\epsilon/n) e^{-at}} \left(\frac{\partial \alpha_1}{\partial \zeta_1} \right)^2 \\ &\quad + \frac{\partial \alpha_1}{\partial \zeta_1} \zeta_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2 + \frac{\partial \alpha_1}{\partial t} + \ddot{y}_d(t) \end{aligned} \quad (14)$$

and reduces (13) to the following form:

$$\dot{V}_2(z_1, z_2, \hat{\theta}) \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \frac{3\epsilon}{n} e^{-at}, \quad (15)$$

where c_2 is a positive constant, $\tau_2 = \dot{\hat{\theta}}$, and

$$\begin{aligned} \dot{z}_2 &= -z_1 - c_2 z_2 + z_3 + \omega_2^T \bar{\theta} + (\eta_2(\zeta, \omega, t)) \\ &\quad - \frac{\partial \alpha_1}{\partial \zeta_1} - \left(\frac{h_2^2}{h_2 |z_2| + (\epsilon/n) e^{-at}} \right. \\ &\quad \left. - \frac{h_1^2}{h_1 |z_2| (\partial \alpha_1 / \partial \zeta_1) + (\epsilon/n) e^{-at}} \left(\frac{\partial \alpha_1}{\partial \zeta_1} \right)^2 \right) z_2 \\ &\quad + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\tau_2 - \dot{\hat{\theta}}). \end{aligned} \quad (16)$$

Similarly, the recursive development in the new variables is in the sequel.

2.2.3. Step k ($1 \leq k \leq n-1$). The variable z_k is defined as $z_k = \zeta_k - \alpha_{k-1} - y_d^{(k-1)}$.

The time derivative of z_k appears as follows:

$$\begin{aligned} \dot{z}_k &= \zeta_{k+1} + \varphi_k^T(\zeta_1 \cdots \zeta_k) \bar{\theta} + \eta_k(\zeta, \omega, t) + \varphi_k^T \bar{\theta} \\ &\quad - \frac{\partial \alpha_{k-1}}{\partial t} - \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{k-1}}{\partial \zeta_1} \dot{\zeta}_1 \\ &\quad - \frac{\partial \alpha_{k-1}}{\partial \zeta_2} \dot{\zeta}_2 \cdots - \frac{\partial \alpha_{k-1}}{\partial \zeta_{k-1}} \dot{\zeta}_{k-1} - y_d^{(k)}, \end{aligned} \quad (17)$$

where

$$\dot{\zeta}_{k-1} = \zeta_k + \varphi^T(\zeta_1, \zeta_2, \dots, \zeta_{k-1}) \bar{\theta} + \eta_{k-1}. \quad (18)$$

The uncertain time varying function in recursive procedure appears in the following form:

$$\begin{aligned} \omega_k &= \varphi_k(\zeta_1, \dots, \zeta_k) - \frac{\partial \alpha_{k-1}}{\partial \zeta_1} \varphi_1(\zeta_1) \\ &\quad - \frac{\partial \alpha_{k-1}}{\partial \zeta_2} \varphi_2(\zeta_1, \zeta_2) \cdots - \frac{\partial \alpha_{k-1}}{\partial \zeta_{k-1}} \varphi_{k-1}(\zeta_1, \dots, \zeta_{k-1}), \\ \xi_k &= \eta_k - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \zeta_i} \eta_i, \end{aligned} \quad (19)$$

where $1 \leq k \leq n$.

The time derivative of the z_k becomes

$$\begin{aligned} \dot{z}_k &= \zeta_{k+1} + \omega_k^T \bar{\theta} - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \zeta_i} \zeta_{i+1} - \frac{\partial \alpha_{k-1}}{\partial t} \\ &\quad - \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \omega_k^T \bar{\theta} + \xi_k - y_d^{(k)}. \end{aligned} \quad (20)$$

In addition, the next variable is defined according to the following formulation: $z_{k+1} = \zeta_{k+1} - \alpha_k - y_d^{(k)}$, where the virtual control input α_k appears in the subsequent form

$$\begin{aligned} \alpha_k(\zeta, \hat{\theta}, t) = & -z_{k-1} - c_k z_k - \omega_k^T \hat{\theta} + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \zeta_i} \zeta_{i+1} \\ & + \frac{\partial \alpha_{k-1}}{\partial t} - \xi_k z_k + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k \\ & - \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k, \end{aligned} \quad (21)$$

where c_k are positive constants. Now, substituting the previous available data, the $\dot{z}_k$ carry the following form:

$$\begin{aligned} \dot{z}_k = & -z_{k-1} - c_k z_k + z_{k+1} + \omega_k^T \tilde{\theta} \\ & + \xi_k - \delta_k z_k - \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} (\dot{\hat{\theta}} - \tau_k) \\ & + \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \omega_k, \end{aligned} \quad (22)$$

where δ_k is defined by

$$\begin{aligned} \delta_k = & \frac{h_k^2}{h_k |z_k| + (\epsilon/n) e^{-at}} \\ & + \sum_{i=1}^{k-1} \left(\frac{\partial \alpha_{k-1}}{\partial \zeta_i} \right)^2 \frac{h_1^2}{h_i |z_k| (\partial \alpha_{k-1} / \partial \zeta_i) + (\epsilon/n) e^{-at}}. \end{aligned} \quad (23)$$

Now, the time derivative of a Lyapunov candidate function

$$V_k(\zeta_1, \dots, \zeta_k, \hat{\theta}) = V_{k-1} + \frac{1}{2} z_k^2 \quad (24)$$

takes the form

$$\begin{aligned} \dot{V}_k \leq & - \sum_{i=1}^k c_i z_i^2 + z_k z_{k+1} + \frac{k(k+1)\epsilon}{2n} e^{at} \\ & + \tilde{\theta}^T \Gamma^{-1} (\tau_k - \dot{\hat{\theta}}) + \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \\ & \times (\tau_k - \dot{\hat{\theta}}), \end{aligned} \quad (25)$$

where $\tau_k = \tau_{k-1} + \Gamma \omega_k z_k = \Gamma \sum_{i=1}^k \omega_i z_i$.

2.2.4. Step n . Finally, define a variable z_n as follows: $z_n = \zeta_n - \alpha_{n-1} - y_d^{(n)}$, where the virtual control input α_{n-1} can be obtained by putting $k = n - 1$ in (21). Thus, the final expression of $\dot{z}_n$ becomes

$$\begin{aligned} \dot{z}_n = & f(\zeta) + g(\zeta) u + \omega_n^T(\zeta, t) \hat{\theta} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \zeta_i} \zeta_{i+1} \\ & - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial t} + \omega_n^T(\zeta, t) \tilde{\theta} + \xi_n - y_d^{(n)}, \end{aligned} \quad (26)$$

where the terms $\omega_n(\zeta, \hat{\theta})$ and ξ_n can be obtained from (19) by substituting $k = n$.

Note that, z_n is the variable whose time derivative bears explicitly the actual control input u . In the next section, the control input is developed.

3. Control Design via Higher Integral Sliding Mode

In this study, which is the main contribution, a control design for the nonlinear systems convertible into SSFF is considered. The design is carried out by making use of first order and second order adaptive backstepping integral sliding mode control which are discussed in detail in forthcoming subsections.

3.1. Control Design via First Integral Sliding Mode. The control design for the systems which can be converted in new variables $[z_1, z_2, \dots, z_n]^T$ is the study of this section. The authors have developed an integral sliding mode control with adaptive backstepping techniques. The control law appears as sum of a continuous and discontinuous component which may take the following mathematical form:

$$u = u_0 + u_1. \quad (27)$$

The main advantage of this technique is that sliding mode is enforced from the very beginning which enhances robustness against uncertainties. In addition, the system operates in sliding mode under the action of the continuous control component $u_0 \in \mathfrak{R}$ [18] which is quite robust in this development because it is designed via the adaptive backstepping technique. The discontinuous component $u_1 \in \mathfrak{R}$ comes into action, when the system reaches the vicinity of the sliding manifold. In addition, the parameter updates law is formulated as

$$\dot{\hat{\theta}} = \dot{\hat{\theta}}_0 + \dot{\hat{\theta}}_1, \quad (28)$$

where the first term comes from adaptive backstepping, while the second term comes from the sliding mode approach. In the subsequent study, the development is presented.

3.1.1. Design of the Adaptive Backstepping Controller u_0 . The development of the continuous control component is presented in the form of the following proposition.

Proposition 2. Consider that the nonlinear system with new variables $[z_1, z_2, \dots, z_n]$ exists, if a control law is chosen in the following form:

$$\begin{aligned} u = & \frac{1}{g(\zeta)} \left[-z_{n-1} - c_n z_n - f(\zeta) - \omega_n^T \hat{\theta} \right. \\ & + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \zeta_i} \zeta_{i+1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n + \frac{\partial \alpha_{n-1}}{\partial t} \\ & \left. - \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \omega_n + y_d^{(n)} - \delta_n z_n \right] \end{aligned} \quad (29)$$

with parameter update law

$$\dot{\hat{\theta}} = \tau_n = \tau_{n-1} + \Gamma \omega_n^T z_n; \quad (30)$$

then the energy in this newly transformed system decays to zero asymptotically.

Proof. Consider a Lyapunov candidate function of the form

$$V_n = V_{n-1} + \frac{1}{2} z_n^2. \quad (31)$$

Calculating the time derivative of V_n along (26) and then inserting the value of the control input u from (29) and parameter update law from (30), one has

$$\dot{V}_n \leq -\sum_{i=1}^n c_i z_i^2 + \frac{(n+1)\epsilon}{2} e^{-at}. \quad (32)$$

Expression in (32) indicates that $V \rightarrow 0$ asymptotically (analogous to that of [17]), which ensures that the continuous control component is responsible for steering the actual system output to the desired output asymptotically.

Note that the update law mentioned in (30) is the first components of (28). In the forthcoming subsection, the design of the discontinuous term via integral sliding mode is presented. $\square$

3.1.2. The Design of the Discontinuous Component $u_1 \in \mathfrak{R}$ of (27). In [17] a conventional sliding surface is taken for conventional adaptive sliding mode which is by definition a Hurwitz polynomial of the states. However, in the present development, an integral manifold is designed which results in reaching phase free sliding mode. The integral manifold under study is defined as follows:

$$\sigma(z) = \sigma_0(z) + \varrho, \quad (33)$$

where $\sigma_0(z)$ is the sliding manifold which usually appears as a linear combination of the states; that is, $\sigma_0(z) = \sum_{i=1}^n k_i z_i$, where $k_i > 0$, $i = 1, \dots, n-1$ with $k_n = 1$ are the designer parameters which are chosen according to the performance of the system. The second term on the right-hand side, that is, ϱ , is the integral term which always contains the nominal dynamics of the system. The design of ϱ is presented in the following theorem.

Theorem 3. Consider the transformed system with the state vector $z = [z_1, z_2, \dots, z_n]^T$. If the integral manifold is defined according to (33), the integral dynamics is chosen according to the following equation:

$$\begin{aligned} \dot{\varrho} = & (-k_1 z_2 + k_1 c_1 z_1 - k_2 z_3 + k_2 c_2 z_2 + \dots - k_{n-1} z_n \\ & + k_{n-1} c_{n-1} z_{n-1}) + (k_2 z_1 + \dots + k_{n-1} z_{n-2}) \\ & + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-i}}{\partial \zeta_i} \zeta_{i+1} + \frac{\partial \alpha_{n-1}}{\partial t} + y_d^{(n)} - g(\zeta) u_0 \\ & + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_{n-1} - \omega_n^T(\zeta, t) \hat{\theta}, \end{aligned} \quad (34)$$

and the discontinuous control component is selected as follows:

$$u_1 = -k \operatorname{sign}(\sigma), \quad (35)$$

where k is positive constant equal to another constant $Kg(\zeta)^{-1}$. Then sliding mode can be enforced along the integral manifold asymptotically.

Proof. We consider the Lyapunov function of the form

$$V_2 = \frac{1}{2} \sigma^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}. \quad (36)$$

The time derivative of the above Lyapunov function along (33) becomes

$$\dot{V}_2 = \sigma (k_1 \dot{z}_1 + k_2 \dot{z}_2 \dots k_{n-1} \dot{z}_{n-1} + \dot{z}_n + \dot{\varrho}) - \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}, \quad (37)$$

substituting the values of $\dot{z}_i$ from (22) where $i = 1, 2, \dots, n$; one has

$$\begin{aligned} \dot{V}_2 = & \sigma \left[k_1 \left(-c_1 z_1 + z_2 + \omega_1^T \tilde{\theta} + \eta_1(\zeta, \omega, t) \right. \right. \\ & \left. \left. - \frac{h_1^2 z_1}{h_1 |z_1| + (\epsilon/n) e^{-at}} \right) \right. \\ & + k_2 \left(-z_1 - c_2 z_2 + z_3 + \omega_2^T \tilde{\theta} \right. \\ & \left. + \left(\eta_2(\zeta, \omega, t) - \frac{\partial \alpha_1}{\partial \zeta_1} \eta_1(\zeta, \omega, t) \right) \right. \\ & \left. - \left(\frac{h_2^2}{h_2 |z_2| + (\epsilon/n) e^{-at}} \right. \right. \\ & \left. \left. - \frac{h_1^2}{h_1 |z_2| (\partial \alpha_1 / \partial \zeta_1)| + (\epsilon/n) e^{-at}} \right. \right. \\ & \left. \left. \times \left(\frac{\partial \alpha_1}{\partial \zeta_1} \right)^2 \right) \right. \\ & \left. \times \left(\frac{\partial \alpha_1}{\partial \zeta_1} \right)^2 \right) \times z_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\tau_2 - \hat{\theta}) \right) \\ & \vdots \\ & + k_{n-1} \left(-z_{n-2} - c_{n-1} z_{n-1} \right. \\ & \left. + \omega_{n-1}^T \tilde{\theta} + \delta_{n-1} z_{n-1} + z_n \right. \\ & \left. - \frac{\partial \alpha_{n-2}}{\partial \hat{\theta}} (\hat{\theta} - \tau_{n-1}) \right. \\ & \left. + \left(\sum_{i=1}^{k-3} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \omega_{n-1} \right) \\ & + f(\zeta) + g(\zeta) (u_0 + u_1) + \omega_n^T \hat{\theta} \\ & - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \zeta_i} \zeta_{i+1} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \hat{\theta} - \frac{\partial \alpha_{n-1}}{\partial t} \\ & \left. + \omega_n^T(\zeta, t) \tilde{\theta} + \xi_n - y_d^{(n)} + \dot{\varrho} \right] - \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}. \end{aligned} \quad (38)$$

Now, inserting (34) in the above expression, one has

$$\begin{aligned} \dot{V}_2 = & \sigma \left[k_1 \eta_1 + k_2 \left(\eta_2 - \frac{\partial \alpha_1}{\partial \zeta_1} \eta_1(\zeta, \omega, t) \right) \right. \\ & \left. + \dots + k_{n-1} \xi_{n-1} + g(\zeta) u_1 + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} (\tau_n - \hat{\theta}) \right] \\ & + \tilde{\theta}^T \Gamma^{-1} (\tau_n - \hat{\theta}). \end{aligned} \quad (39)$$

The terms $\tilde{\theta}$ in (39) will vanish by using parameter update law $\dot{\hat{\theta}} = \tau_n$, where $\tau_n = \Gamma \sigma (\sum_{i=1}^{n-1} \omega_i^T k_i + \omega_n)$.

Therefore, the above expression reduces to the following form:

$$\dot{V}_2 = \sigma \left[g(\zeta) u_1 + k_1 \eta_1 + k_2 \left(\eta_2 - \frac{\partial \alpha_1}{\partial \zeta_1 \eta_1} \right) + \cdots + \xi_{n-1} \right]. \quad (40)$$

We denote $k_1 \eta_1 + k_2 (\eta_2 - (\partial \alpha_1 / \partial \zeta_1 \eta_1)) + \cdots = A$ and $\xi_{n-1} = B$, and, using (35), the expression in (40) takes the form

$$\dot{V}_2 = -k |\sigma| + |\sigma| [A + B] \quad (41)$$

or

$$\dot{V}_2 \leq |\sigma| [-k + |A| + |B|]. \quad (42)$$

Let $k > |A| + |B|$ and $-k + |A| + |B| \leq -\lambda$, where $\lambda > 0$.

Thus,

$$\dot{V}_2 \leq -\lambda |\sigma|. \quad (43)$$

This proves that the sliding mode is enforced in finite time. However, the parameter behaves an asymptotic convergence. Thus, the overall system is asymptotically convergent. $\square$

Note that the parameter update law which appears before (40) is the second component of the parameter update law in (28). The algebraic sum of the two parameter update laws gives birth to the final expression of the parameters update law.

3.2. Second Order Adaptive Integral Sliding Mode Control Design (SOAISM). This subsection is dedicated to the study of second order adaptive integral sliding mode control design. Since higher order integral sliding modes (see, e.g., [21, 22]) are famous for their robustness from the very beginning with the use of an integral manifold, the real twisting and super twisting controller, reported in [5, 23], are utilized here to develop the discontinuous control component of the control law mentioned in (27). The mathematical expressions of the universal super twisting controller reported in [5] are given by

$$u_{\text{dis}} = u_1 + u_2, \quad (44)$$

where

$$u_1 = \begin{cases} -\kappa |\sigma_0|^l \text{sign}(\sigma), & |\sigma| > \sigma_0, \\ -\kappa |\sigma|^l \text{sign}(\sigma), & |\sigma| \leq \sigma_0, \end{cases} \quad (45)$$

where $0 < l \leq 0.5$ and

$$u_2 = \begin{cases} -u, & |u| > 1, \\ -W \text{sign}(\sigma), & |u| \leq 1. \end{cases} \quad (46)$$

The beauty of this controller is that it is of relative degree one and it does not need the derivatives of the sliding manifolds.

On the other hand, the real twisting controller needs the time derivative of the sliding manifold which will be recovered in this work via the uniform robust exact differentiator [24]. Assume that the error between the available signal σ_1 (which is the integral manifold in our case defined in (33)) and the estimated signal $\hat{\sigma}_1$ is defined by $\tilde{\sigma} = \hat{\sigma}_1 - \sigma_1$; then the first derivative can be estimated via the following algorithm:

$$\dot{\hat{\sigma}}_1 = \hat{\sigma}_2 - R_1 \omega_1(\tilde{\sigma}), \quad \dot{\hat{\sigma}}_2 = -R_2 \omega_2(\tilde{\sigma}), \quad (47)$$

where R_1 and R_2 are positive gains to be designed and

$$\begin{aligned} \omega_1(\tilde{\sigma}) &= |\tilde{\sigma}|^{0.5} \text{sign}(\tilde{\sigma}) + \mu |\tilde{\sigma}|^{3/2} \text{sign}(\tilde{\sigma}) \\ \omega_2(\tilde{\sigma}) &= \frac{1}{2} \text{sign}(\tilde{\sigma}) + 2\mu \tilde{\sigma} + \frac{3}{2} \mu^2 |\tilde{\sigma}|^2 \text{sign}(\tilde{\sigma}). \end{aligned} \quad (48)$$

The output of $\dot{\hat{\sigma}}_2$, by integrating once, is $\hat{\sigma}_2$ which is the first estimated derivative of σ_1 which is estimated very accurately. Since the real twisting controller, which appears as follows, will make use of the above estimated derivative:

$$u_{\text{dis}} = -r_1(\hat{\sigma}_1) - r_2 \text{sign}(\hat{\sigma}_2) \quad r_1 > r_2 > 0, \quad (49)$$

the use of u_{dis} refers to the second component of (27) which will enforce sliding mode against intersection of the manifolds $\sigma = \dot{\sigma} = 0$ from the very beginning of the process. Consequently, the reaching phase will be eliminated and the chattering reduction will also occur across the switching manifold.

Note that the parameter update law in this case bears the first component of (28) because the higher order sliding manifold $\sigma = \dot{\sigma} = 0$ contributes nothing to parameter update law. In the subsequent illustrative examples, the switching manifold and chattering reduction claim is verified. The parameter update law in this case becomes $\tau = \theta_0$ because $\sigma = 0$ is justified from $t = 0$.

4. Illustrative Example

This section is dedicated to verify the aforementioned claims. Therefore, consider a second order nonlinear system reported in [16] which is in the semistrict feedback form:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + \zeta_1 \theta + A \zeta_1^2 \cos(B \zeta_1 \zeta_2), \\ \dot{\zeta}_2 &= u, \end{aligned} \quad (50)$$

where A and B are considered unknown, but it is known that $|A| \leq 2$ and $|B| \leq 3$. We also assume that

$$\begin{aligned} h_1 &= 2\zeta_1^2, \\ z_1 &= \zeta_1 - y_d, \\ \alpha_1 &= -\zeta_1 \hat{\theta} - c_1 z_1 + \frac{4\zeta_1^4 z_1}{h_1 |z_1| + (\epsilon/2) e^{(-at)}}, \\ \omega_1 &= \zeta_1, \\ \omega_2 &= -\frac{\partial \alpha_1}{\partial \zeta_1} \zeta_1, \\ \dot{\hat{\theta}}_0 &= \Gamma (\omega_1 z_1 + \omega_2 z_2), \end{aligned} \quad (51)$$

$$\delta_2 = \frac{4\zeta_1^4}{2\zeta_1^2 |(\partial \alpha_1 / \partial \zeta_1) z_2| + (\epsilon/2) e^{(-at)}} \left(\frac{\partial \alpha_1}{\partial \zeta_1} \right)^2.$$

FIGURE 1: Output tracking of desired signal y_d .

Following the procedure of Section 2, the above system can be transformed into the new variables z_1 and z_2 as follows:

$$\begin{aligned} \dot{z}_1 &= z_2 - c_1 z_1 + 2\zeta_1^2 \cos(3\zeta_1 \zeta_2) \\ &\quad - \frac{4\zeta_1^4}{2\zeta_1^2 |(\partial\alpha_1/\partial\zeta_1) z_2| + (\epsilon/2) e^{(-at)}} + \zeta\tilde{\theta}, \\ \dot{z}_2 &= u - \frac{\partial\alpha_1}{\partial\zeta_1} \zeta_2 + \omega_2^T \hat{\theta} + \omega_2^T \tilde{\theta} - \frac{\partial\alpha_1}{\partial\zeta_1} 2\zeta_1^2 \cos(3\zeta_1 \zeta_2) \\ &\quad - \frac{\partial\alpha_1}{\partial\hat{\theta}} \dot{\hat{\theta}} - \frac{\partial\alpha_1}{\partial t} - y_d^{(2)}. \end{aligned} \quad (52)$$

Following the formulation of the continuous control law reported in (21), one gets

$$\begin{aligned} u_0 &= -z_1 - c_2 z_2 - \omega_2^T \hat{\theta} + \frac{\partial\alpha_1}{\partial\zeta_1} \zeta_2 + \frac{\partial\alpha_1}{\partial\zeta_1} \omega_1^T \hat{\theta} \\ &\quad + \frac{\partial\alpha_1}{\partial\hat{\theta}} \tau_2 + \frac{\partial\alpha_1}{\partial t} + y_d^{(2)} - \delta_2 z_2. \end{aligned} \quad (53)$$

The discontinuous control component can be obtained by considering an integral manifold of the form

$$\begin{aligned} \sigma &= k_1 z_1 + z_2 + \varrho, \\ \dot{\sigma} &= k_1 \dot{z}_1 + \dot{z}_2 + \dot{\varrho}. \end{aligned} \quad (54)$$

For analysis, a Lyapunov function of the following form is considered:

$$V_2 = \frac{1}{2} \sigma^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}, \quad (55)$$

with the discontinuous control component as follows:

$$u_1 = -k \operatorname{sign}(\sigma). \quad (56)$$

The expression of the integral compensator dynamics is given by

$$\begin{aligned} \dot{\varrho} &= -k_1 z_2 + k_1 c_1 z_1 + \frac{\partial\alpha_1}{\partial\zeta_1} \zeta_2 + \omega_2^T \hat{\theta} \\ &\quad + \frac{\partial\alpha_1}{\partial t} + y_d^{(2)} - u_0 + \frac{\partial\alpha_1}{\partial\hat{\theta}} \tau_2, \end{aligned} \quad (57)$$

where the parameters update law in case of adaptive first order integral sliding mode is given by

$$\dot{\hat{\theta}}_1 = \Gamma \sigma (\omega_1^T k_1 + \omega_2). \quad (58)$$

The final parameter update law can then become

$$\tau_2 = \dot{\hat{\theta}}_0 + \dot{\hat{\theta}}_1, \quad (59)$$

and the final expression of the control law becomes

$$u = u_0 + u_1. \quad (60)$$

Using the approach presented in the Theorem 3, one gets

$$\dot{V}_2 \leq -\lambda |\sigma|. \quad (61)$$

This expression shows that the sliding mode is enforced in finite time. However parameter convergence is asymptotic. Thus the overall system is asymptotically convergent.

The above example is simulated with the control law and parameter update laws defined above. The tracking of the $y_d = 0.05 \sin(2\pi t)$ is shown in Figure 1 which indicates that the time varying signals can be tracked very accurately in presence of uncertainties shown in Figure 7. The first order integral manifold displayed in Figure 3 ensures the sliding mode enforcement from the very beginning and, consequently, ensures robustness. The respective control input and parameter estimates are shown in Figures 4 and 2, respectively. Thus it is clear from these figures that the control input is appearing with reduced chattering.

The AHOISM results of tracking are somewhat similar to that reported in Figure 1. Therefore, it is not displayed here. The clear advantage comes in the chattering reduction which is clear from Figures 3 and 6 comparison. That higher order sliding mode with adaptation provides almost chatter free control input. The sliding manifold convergence of this strategy is displayed in Figure 5. In addition, the phase trajectory of $\sigma = \dot{\sigma} = 0$ is given in Figure 6 which, once again, shows the establishment of higher order sliding modes from the very beginning. This is a clear advantage of this proposed technique over the existing adaptive sliding modes

FIGURE 2: The estimated parameter.

FIGURE 3: Integral manifold σ convergence.

FIGURE 4: The applied control input.

FIGURE 5: The σ convergence via the SOAISM. FIGURE 6: The manifold $\sigma_1 = \sigma_2 = 0$ (phase portrait of $\sigma_1 = \sigma_2 = 0$) for SOAISM.

techniques. The parameters used in simulation results are $\epsilon = 1.997$, $a = 0.7$, $k = 30$, $k_1 = 1$, $c_1 = 200$, and $c_2 = 140$, $\lambda = 0.1$.

From the aforementioned discussions of figures, it is verified that our new proposed techniques provide robustness from the very beginning via the integral manifold approach and the robustness against parametric variations is provided via adaptive backstepping. It is therefore claimed that the above development outshines the existing adaptive sliding mode techniques.

Comment. Proposed design technique is recursive; therefore it is costly in computation and complex due to complex mathematical derivations. Thus its computational cost is relatively higher than adaptive sliding mode. However beauty of this method is that its control action is global and robust.

FIGURE 7: Output disturbance added.

5. Conclusion

The backstepping technique which is famous for its robust nature against parameter variation is utilized in combination with integral higher order sliding mode control strategy in order to enhance the robustness of the system from the very beginning of the process with considerable attenuation in chattering. The approach is relying on an adaptively developed new system. The integral manifold is designed in the new state variables. In other words, the manifold is adaptive in nature because of the adaptively developed states variable. The control law is designed via higher order integral sliding mode with adaptation which is capable of providing robustness against uncertainties caused by external/internal disturbances. The stability analysis is elaborated in terms of a proposition and a theorem. A numerical simulation result has verified the design approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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