Research Article

Uniform Attractor for the Fractional Nonautonomous Long-Short Wave Equations

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Received 14 March 2014; Accepted 3 September 2014; Published 20 October 2014

Academic Editor: Prasanta K. Panigrahi

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We firstly proved the existence and the uniqueness of the solution for the 2π -periodic fractional nonautonomous long-short wave equations with translation compact force by using Galerkin method and then obtained the compact uniform attractor of the system.

1. Introduction

In this paper, we consider the following fractional nonautonomous long-short wave equations with translation compact forces:

$$iu_t - (-\Delta)^\alpha u - nu + i\delta u = f(x, t), \quad (1)$$

$$n_t + \beta n + |u|_x^2 = g(x, t), \quad (2)$$

with initial and periodic boundary conditions:

$$u(x, t)|_{t=\tau} = u_\tau(x), \quad n(x, t)|_{t=\tau} = n_\tau(x), \quad x \in \mathbb{R}, \quad (3)$$

$$u(x + 2\pi, t) = u(x, t), \quad n(x + 2\pi, t) = n(x, t), \quad (4)$$

$$x \in \mathbb{R}, \quad t \geq \tau,$$

where $u(x, t)$ is an unknown and complex-valued function, $n(x, t)$ is an unknown real valued function, $x \in \Omega = (0, 2\pi) \subset \mathbb{R}$; $t \geq \tau \in \mathbb{R}^+$; $\alpha, \delta, \beta > 0$, and nonautonomous terms $f(x, t)$ and $g(x, t)$ are time-dependant external forces and translation compact (see Definition 1).

We all know that the long-short wave resonance equations play an important role in fluid mechanics and have rich physical and mathematical properties. There are more and more recent papers treating the long-short wave resonance equations. Guo studied the global solution for one class of the system of LS nonlinear wave interaction in [1] and the periodic initial value problems and initial value problems for one class of generalized long-short type equations in [2].

The papers [3–5] studied the existence of a global attractor of it. Cui et al. developed the weakly compact uniform attractor for the nonautonomous long-short wave equations with translation compact forces in [6].

The Schrödinger type equation has been of great importance describing nonrelativistic quantum mechanical behavior. It is well known that Feynman and Hibbs derive the standard (nonfractional) Schrödinger type equation by applying path integrals over Brownian paths in [7]. Recently Laskin generalized the Schrödinger equation to space fractional cases using path integrals over Lévy trajectories in [8, 9]. In [10], the authors discussed the models and numerical methods of the fractional calculus. The fractional Schrödinger type equation is used to describe better physical phenomenon and has attracted more and more attention of researchers. Guo and Xu studied some applications of the Schrödinger equation in physics (see [11]). In [12], the authors obtained the approximate analytical solutions of the fractional nonlinear Schrödinger equations by using the homotopy perturbation method. Eid et al. studied the α -dimensional fractional Schrödinger equation and obtained its exact solutions in [13]. Guo et al. investigated the fractional nonlinear Schrödinger equation and showed the existence and uniqueness of its global smooth solution by using energy method in [14]. Goubet [15] studied regularity of the attractor for a weakly damped nonlinear Schrödinger equation in $\mathbb{R}^2$.

The rest of this paper is arranged as follows. In Section 2, we recall some basic definitions, introduce preparation results, and analyse some fractional calculation laws which

depend heavily on 2π -period. In Section 3, we introduce some preparation lemma and give the uniform a priori estimates (uniform in initial data and symbol in the symbol space and large time). In Section 4, we show the existence and uniqueness of the solution of the system. In Section 5, we prove the existence of strong compact uniform attractor of the system.

Through the paper, we denote the norm of $L^2_{\text{per}}(\Omega)$ with the usual inner product $(\cdot, \cdot)$ by $\|\cdot\|$. We denote the norm of $L^p_{\text{per}}(\Omega)$ for all $1 < p \leq \infty$ by $\|\cdot\|_{L^p}$. For simplicity and convenience, the letter C represents a constant, which may be different from one to others. $C(\cdot, \cdot)$ represents the constant C expressed by the parameters appearing in the parentheses.

2. Preliminaries

In this section, we introduce notations definition and preliminary facts. We firstly recall the following known definitions (see [6, 16–18]) and some main lemmas (see [16, 19, 20]).

Definition 1. Suppose X is a Banach space, $f(t) : \mathbb{R} \rightarrow X$ is a function, and $T(\cdot)$ is the translation operator. The hull $\mathcal{H}(f)$ of f is defined by

$$\mathcal{H}(f) = \overline{\{T(s)f(t) = f(t+s) \mid s \in \mathbb{R}\}}. \quad (5)$$

- (i) f is said to be translation bounded in $L^2(\mathbb{R}; X)$ if $\mathcal{H}(f)$ is bounded in which

$$\|f\|_{L^2_b(\mathbb{R}; X)}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f\|_X^2 ds < \infty. \quad (6)$$

Then $L^2_b(\mathbb{R}; X)$ consists of all the translation bounded functions in $L^2(\mathbb{R}; X)$.

- (ii) f is called translation compact in $L^2(\mathbb{R}; X)$ if $\mathcal{H}(f)$ is compact in $L^2(\mathbb{R}; X)$, where the convergence is taken in the sense of local quadratic mean convergence topology of $L^2_{\text{loc}}(\mathbb{R}; H^\alpha)$. The collection of all the translation compact functions in $L^2(\mathbb{R}; X)$ is denoted by $L^2_c(\mathbb{R}; X)$.

Let $(X, \|\cdot\|_X)$ be a Banach space, and the following definitions are common.

Definition 2. Let Σ be a parameter set. $\{U_\sigma(t, \tau), t \geq \tau, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$ is said to be a family of processes in X , if, for each $\sigma \in \Sigma$, $U_\sigma(t, \tau)$ is a process; that is, the two-parameter mapping $U_\sigma(t, \tau)$ from X to X satisfies

- (i) $U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), \forall t \geq s \geq \tau, \tau \in \mathbb{R}$,
- (ii) $U_\sigma(\tau, \tau) = I$ (the identity operator on X), $\tau \in \mathbb{R}$,

where Σ is called the symbol space and $\sigma \in \Sigma$ is called the symbol.

A subset $B_0 \subset E$ is said to be uniformly absorbing set for the family of processes $\{U_{\sigma \in \Sigma}(t, \tau)\}$, if, for any $\tau \in \mathbb{R}$ and

subset $B \in \mathcal{B}(E)$ denoting the set of all bounded subsets of E , there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\bigcup_{\sigma \in \Sigma} U_\sigma(t, \tau)B \subseteq B_0$ for all $t \geq t_0$. A set $Y \subset E$ is called uniformly attracting for the family of process $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ if, for each fixed $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(E)$, it satisfies that

$$\lim_{t \rightarrow +\infty} \left(\sup_{\sigma \in \Sigma} \text{dist}_E(U_\sigma(t, \tau)B, Y) \right) = 0. \quad (7)$$

Definition 3. A closed set $\mathcal{A}_\Sigma \subset X$ is called the uniform attractor of the family of processes $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma}$ if it is uniformly attracting (attracting property) and it is contained in any closed uniformly attracting set $\mathcal{A}'$ of the family of processes $\{u_\sigma(t, \tau)\}_{\sigma \in \Sigma} : \mathcal{A}_\Sigma \subseteq \mathcal{A}'$ (minimality property).

Definition 4. $\{U_{\sigma \in \Sigma}(t, \tau)\}$, a family of processes in X , is said to be $(X \times \Sigma, X)$ -continuous, if, for any fixed T and $\tau, T \geq \tau$, projection $(u_\tau, \sigma) \rightarrow U_\sigma(T, \tau)u_\tau$ is continuous from $X \times \Sigma$ to X .

Definition 5. The space $L^p(0, T; X)$ denotes all measurable functions $f : [0, T] \rightarrow X$ with the norm

$$\|f\|_{L^p(0, T; X)} = \left(\int_0^T \|f\|_X^p dt \right)^{1/p} < \infty, \quad (8)$$

for $1 \leq p < \infty$, and

$$\|f\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 \leq t \leq T} \|f\|_X < \infty. \quad (9)$$

Lemma 6. Let Σ be a compact metric space and suppose $\{T(h) \mid h \geq 0\}$ is a family of operators defined on Σ , satisfying

$$(i) \quad T(h)\Sigma = \Sigma, \quad \forall h \in \mathbb{R}_+; \quad (10)$$

(ii) translation identity:

$$U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t, \tau), \quad \forall \sigma \in \Sigma, t \geq \tau, \quad (11)$$

$$\tau \in \mathbb{R}, \quad h \geq 0,$$

where $U_\sigma(T, \tau)$ is an arbitrary process in compact metric space E . Note that if the family of processes $\{U_{\sigma \in \Sigma}(T, \tau)\}$ is $(E \times \Sigma, E)$ continuous and it has a uniform compact attracting set, then the skew product flow corresponding to it has a global attractor $\mathcal{A}$ on $E \times \Sigma$. And the projection of $\mathcal{A}$ on Σ , $\mathcal{A}_\Sigma$, is the compact uniform attractor of $\{U_{\sigma \in \Sigma}(T, \tau)\}$.

Remark 7. Assumption (11) holds if the system has a unique solution.

Lemma 8. Let $(X, \|\cdot\|_X)$ be a uniform convex Banach space (particularly, a Hilbert space), and let $\{x_k\}_{k \geq 0}$ be a sequence in X . If $x_k \rightarrow x_0$ and $\|x_k\|_X \rightarrow \|x_0\|_X$, then $x_k \rightarrow x_0$.

Lemma 9. Let $\{x_k\}_{k \geq 0}$ be a sequence in B^* space X . If $x_k \rightarrow x_0$, then

$$\sup_{k \geq 1} \|x_k\|_X < \infty, \quad \|x_0\|_X \leq \liminf_{k \rightarrow \infty} \|x_k\|_X. \quad (12)$$

Since the solution $u(x, t)$, if it exists, is a 2π -periodic function, we have the Fourier expansion:

$$u(x, t) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx}, \quad (13)$$

where $a_k(t) : \mathbb{R}^+ \rightarrow \mathbb{R}, x \in \mathbb{R}$. Therefore,

$$\partial_x u = \sum_{k \in \mathbb{Z}} ika_k e^{ikx}, \quad (14)$$

and $(-\Delta)^\alpha u$ is defined by

$$(-\Delta)^\alpha u = \sum_{k \in \mathbb{Z}} |k|^{2\alpha} a_k e^{ikx}. \quad (15)$$

Since

$$|e^{irx}|^2 = 1, \quad \int_{\Omega} e^{irx} dx = 0, \quad \forall r \in \mathbb{R}, \quad (16)$$

the following definitions make sense. Let

$$A = \left\{ u \mid u = \sum_{k \in \mathbb{Z}} a_k e^{ikx}, \sum_{k \in \mathbb{Z}} |k|^{4\alpha} a_k^2 < \infty, \sum_{k \in \mathbb{Z}} |a_k|^2 < \infty \right\}, \quad (17)$$

and let $H^{2\alpha}$ be a complete space of the set A under the norm:

$$\|u\|_{H^{2\alpha}} = \|(-\Delta)^\alpha u\| + \|u\| \\ = \left(2\pi \sum_{k \in \mathbb{Z}} |k|^{4\alpha} a_k^2 \right)^{1/2} + \left(2\pi \sum_{k \in \mathbb{Z}} |a_k|^2 \right)^{1/2}. \quad (18)$$

Then we can easily check that $H^{2\alpha}$ is a Banach space and that, for $\forall u \in H^{2\alpha}$, u is space-periodic with the period 2π and its 2α order derivatives are in $L^2_{\text{per}}(\Omega)$. And for $\forall u, v \in H^\alpha$,

$$\frac{d}{dt} uv = u \frac{d}{dt} v + v \frac{d}{dt} u, \quad (19) \\ ((-\Delta)^\alpha u, v) = ((-\Delta)^{\alpha_1} u, (-\Delta)^{\alpha_2} v),$$

when $\alpha_1 + \alpha_2 = \alpha, 0 \leq \alpha_1, \alpha_2 \leq \alpha$. $H^{2\alpha}$ is a Hilbert space with the inner product

$$(u, v)_{H^{2\alpha}} = ((-\Delta)^\alpha u, (-\Delta)^\alpha v) = (2\pi)^n \sum_{k \in \mathbb{Z}^n} |k|^{4\alpha} a_k b_k. \quad (20)$$

For brevity, we introduce $W(x, t) = (u(x, t), n(x, t))$ and $Y(x, t) = (f(x, t), g(x, t))$. We denote the space of functions $W(x, t) = (u(x, t), n(x, t))$ by $E_0 = H^{2\alpha}(\Omega) \cap H^\alpha_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega)$ with norm

$$\|W\|_{E_0} = (\|u\|_{H^{2\alpha}}^2 + \|n\|_{H^1}^2)^{1/2}. \quad (21)$$

Similarly, we denote the space of $Y(x, t)$ by Σ_0 with norm

$$\|Y\|_{\Sigma_0} = (\|f\|_{H^{2\alpha}}^2 + \|g\|_{H^1}^2)^{1/2}. \quad (22)$$

Assumption 10. Suppose that the symbol $Y(x, t)$ belongs to the symbol space Σ , defined by

$$\Sigma = \overline{\{Y_0(x, s+r) \mid r \in \mathbb{R}_+\}}, \quad (23)$$

where $Y_0 = (f_0(x, t), g_0(x, t)) \in L^2_c(\mathbb{R}; E_0)$ and the closure is taken in the sense of local quadratic mean convergence topology in the topological space $L^2_{\text{loc}}(\mathbb{R}; \Sigma_0)$. Moreover, we assume $f_{0t}(x, t) \in L^2_b(\mathbb{R}; H^\alpha)$.

Remark 11. Due to the conception of translation compact/boundedness, we remark that

$$(i) \forall Y_1 \in \Sigma, \|Y_1\|_{L^2_b(\mathbb{R}; \Sigma_0)} \leq \|Y_0\|_{L^2_b(\mathbb{R}; \Sigma_0)};$$

$$(ii) T(t)\Sigma = \Sigma, \forall t \in \mathbb{R}, \text{ where } T(t)\varphi(s) = \varphi(s+t) \text{ is a translation operator.}$$

3. A Uniform A Priori Estimates

In this section, we will get some uniform a priori estimates which hold uniformly independently of initial data, time, and symbols in symbol space ($Y \in \Sigma$). In the following, we denote that $\int \cdot dx = \int_{\Omega} \cdot dx$ and $H = L^2(\Omega)$, which will not cause any confusions.

We first recall the Gagliardo-Nirenberg and the Young inequalities (see [21]).

Lemma 12. Let $u \in L^q(\mathbb{R}^n), D^m u \in L^r(\mathbb{R}^n), 1 \leq q, r < \infty$. Then for $0 \leq j \leq m, j/m \leq a \leq 1$, there exists a constant C such that

$$\|D^j u\|_{L^p} \leq C \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a}, \quad (24)$$

where $1/p = j/n + a(1/r - m/n) + (1-a)(1/q)$.

Lemma 13. Let $a, b > 0$. Then for each (p, q) satisfying $1 < p, q < \infty, 1/p + 1/q = 1$, it holds that

$$a \cdot b \leq \varepsilon \frac{a^p}{p} + \varepsilon^{-q/p} \frac{b^q}{q}, \quad \forall \varepsilon > 0. \quad (25)$$

Lemma 14. Assume that

(i) $Y(x, t)$ satisfy Assumption 10;

(ii) $u_\tau \in L^2(\Omega)$ and $u(t)$ solves problem (1)–(4).

Then there exist positive constants $C_1 = C_1(\delta, \|f_0\|_{L^2_b(\mathbb{R}; H^{2\alpha})})$ and $t_1 = t_1(\delta, \|f_0\|_{L^2_b(\mathbb{R}; H^{2\alpha})}, \|u_\tau\|)$ such that

$$\|u(t)\|^2 \leq C_1, \quad \forall t \geq t_1. \quad (26)$$

Proof. Taking the inner product of (1) with u , we have

$$(iu_t - (-\Delta)^\alpha u - nu + i\delta u, u) = (f(x, t), u). \quad (27)$$

Taking the imaginary part of (27), we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 = \operatorname{Im} (f(x, t), u). \quad (28)$$

By (28) and Remark 11, we have

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + 2\delta \|u\|^2 \\ \leq \delta \|u\|^2 + \frac{1}{\delta} \|f(x, t)\|_{L_b^2(R; H^{2\alpha})}^2 \\ \leq \delta \|u\|^2 + \frac{1}{\delta} \|f_0\|_{L_b^2(R; H^{2\alpha})}^2. \end{aligned} \quad (29)$$

By Gronwall inequality we get the lemma. $\square$

Lemma 15. Assume that

- (i) $\alpha \geq 1$, and $Y(x, t)$ satisfy Assumption 10;
- (ii) $W(\tau) \in H^\alpha \times H$ and $W(t)$ solves problem (1)–(4).

Then there exist positive constants $C_2 = C_2(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)}, \|f_0\|_{L_b^2(R; H^\alpha)})$ and $t_2 = t_2(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)}, \|f_0\|_{L_b^2(R; H^\alpha)}, \|W_\tau\|_{H^\alpha \times H})$ such that

$$\|W(t)\|_{H^\alpha \times H} \leq C_2, \quad \forall t \geq t_2. \quad (30)$$

Proof. Taking the inner product of (1) with u_t , we have

$$(iu_t - (-\Delta)^\alpha u - nu + i\delta u, u_t) = (f(x, t), u_t). \quad (31)$$

Taking the real part of (31), we have

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\alpha/2} u\|^2 - \frac{1}{2} \int n \frac{d}{dt} |u|^2 dx + \operatorname{Re} (i\delta u, u_t) \\ = \operatorname{Re} (f(x, t), u_t), \end{aligned} \quad (32)$$

where

$$\begin{aligned} \int n \frac{d}{dt} |u|^2 \\ = \int \frac{d}{dt} (n|u|^2) dx - \int |u|^2 n_t dx \\ = \int \frac{d}{dt} (n|u|^2) dx - \int |u|^2 (g(x, t) - \beta n - |u_x|^2) dx \\ = \int \frac{d}{dt} (n|u|^2) dx + \int |u|^2 |u_x|^2 dx \\ + \beta \int n|u|^2 dx - \int g(x, t) |u|^2 dx \\ = \int \frac{d}{dt} (n|u|^2) dx + \beta \int n|u|^2 dx - \int g(x, t) |u|^2 dx. \end{aligned} \quad (33)$$

So we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\ + \frac{\beta}{2} \int n|u|^2 dx - \frac{1}{2} \int g(x, t) |u|^2 dx \\ - \operatorname{Re} (i\delta u, u_t) - \operatorname{Re} (f_t(x, t), u) = 0. \end{aligned} \quad (34)$$

Taking the inner product of (1) with δu , we have

$$(iu_t - (-\Delta)^\alpha u - nu + i\delta u, \delta u) = (f(x, t), \delta u). \quad (35)$$

Taking the real part of (35), we have

$$\begin{aligned} \operatorname{Re} (i\delta u, u_t) + \delta \|(-\Delta)^{\alpha/2} u\|^2 \\ + \int \delta n|u|^2 dx + \operatorname{Re} (f(x, t), \delta u) = 0. \end{aligned} \quad (36)$$

By (34) and (36), we have

$$\begin{aligned} \frac{d}{dt} \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\ + \delta \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\ + \delta \|(-\Delta)^{\alpha/2} u\|^2 \\ = \int g(x, t) |u|^2 dx - (\delta + \beta) \int n|u|^2 dx \\ + 2 \operatorname{Re} (f_t(x, t), u). \end{aligned} \quad (37)$$

By Lemmas 12 and 13 and the condition $\alpha \geq 1$, we have

$$\begin{aligned} \int g(x, t) |u|^2 dx \\ \leq \frac{1}{2} \left(\|g(x, t)\|^2 + \|u\|_{L^4}^4 \right) \\ \leq \frac{1}{2} \|g(x, t)\|_{L_b^2(R; \Sigma_0)}^2 \\ + C \|(-\Delta)^{\alpha/2} u\|^{1/\alpha} \|u\|^{4-(1/\alpha)} \\ \leq \rho \|(-\Delta)^{\alpha/2} u\|^2 \\ + C \left(\rho, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right), \\ \left| -(\delta + \beta) \int n|u|^2 dx \right| \\ \leq \rho \|n\|^2 + C(\delta, \beta) \|u\|_{L^4}^4 \\ \leq \rho \|n\|^2 + \rho \|(-\Delta)^{\alpha/2} u\|^2 + C(\rho, \delta, \beta). \end{aligned} \quad (38)$$

So we reduce that

$$\begin{aligned}
& \frac{d}{dt} \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\
& + \delta \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\
& + \delta \|(-\Delta)^{\alpha/2} u\|^2 \\
& \leq \rho \|n\|^2 + 2\rho \|(-\Delta)^{\alpha/2} u\|^2 \\
& + C \left(\rho, \|f_{0t}\|_{L_b^2(R; H^\alpha)}, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right).
\end{aligned} \tag{39}$$

Similarly, we also derive that

$$\begin{aligned}
& \frac{d}{dt} \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\
& + \beta \left(\|(-\Delta)^{\alpha/2} u\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \right) \\
& + (2\delta - \beta) \|(-\Delta)^{\alpha/2} u\|^2 \\
& \leq \rho \|n\|^2 + 2\rho \|(-\Delta)^{\alpha/2} u\|^2 \\
& + C \left(\rho, \|f_{0t}\|_{L_b^2(R; H^\alpha)}, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right).
\end{aligned} \tag{40}$$

Taking the inner product of (2) with n , we have

$$\frac{1}{2} \frac{d}{dt} \|n\|^2 + \int n|u_x|^2 dx + \beta \|n\|^2 - \int g(x, t) n dx = 0. \tag{41}$$

By (1), we have

$$\begin{aligned}
& \int n|u_x|^2 dx \\
& = \int n(u\bar{u})_x = \int (nu\bar{u}_x + n\bar{u}u_x) dx \\
& = \int (iu_t - (-\Delta)^\alpha u + i\delta u - f(x, t)) \bar{u}_x dx \\
& + \int (-i\bar{u}_t - (-\Delta)^\alpha \bar{u} - i\delta \bar{u} - \overline{f(x, t)}) u_x dx \\
& = i \int (u_t \bar{u}_x - \bar{u}_t u_x) dx + 2 \operatorname{Re} \int i\delta u \bar{u}_x dx \\
& - 2 \operatorname{Re} \int f(x, t) \bar{u}_x dx,
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
& \frac{d}{dt} \int (iu\bar{u}_x - i\bar{u}u_x) dx \\
& = i \int (u_t \bar{u}_x + u\bar{u}_{xt} - u_{xt} \bar{u} - u_x \bar{u}_t) dx \\
& = i \int (u_t \bar{u}_x - u_x \bar{u}_t + u_t \bar{u}_x - u_x \bar{u}_t) dx \\
& = 2i \int (u_t \bar{u}_x - u_x \bar{u}_t) dx.
\end{aligned} \tag{43}$$

By (41)~(43), we get

$$\begin{aligned}
& \frac{d}{dt} \left(\|n\|^2 + \int i(u\bar{u}_x - u_x \bar{u}) dx \right) + 2\beta \|n\|^2 \\
& + \delta \int i(u\bar{u}_x - u_x \bar{u}) dx \\
& = \delta \int i(u\bar{u}_x - u_x \bar{u}) dx - 4 \operatorname{Re} \int i\delta u \bar{u}_x dx \\
& + 4 \operatorname{Re} \int f(x, t) \bar{u}_x dx + 2 \int g(x, t) n dx \\
& = -\delta \int i(u\bar{u}_x - u_x \bar{u}) dx + 4 \operatorname{Re} \int f(x, t) \bar{u}_x dx \\
& + 2 \int g(x, t) n dx.
\end{aligned} \tag{44}$$

By Lemmas 12~14 and the condition $\alpha \geq 1$, we have

$$\begin{aligned}
& \left| i\delta \int (u\bar{u}_x - u_x \bar{u}) dx \right| \leq 2\delta \|u\| \|u_x\| \\
& \leq \rho \|(-\Delta)^{\alpha/2} u\|^2 + C(\rho), \\
& \left| 4 \operatorname{Re} \int f(x, t) \bar{u}_x dx \right| \leq 4 \|f(x, t)\| \|\bar{u}_x\| \\
& \leq \rho \|(-\Delta)^{\alpha/2} u\|^2 + C(\rho, \|Y_0\|_{L_b^2(R; \Sigma_0)}), \\
& \left| 2 \int g(x, t) n dx \right| \leq 2 \|g(x, t)\| \|n\| \\
& \leq \rho \|n\|^2 + C(\rho, \|Y_0\|_{L_b^2(R; \Sigma_0)}).
\end{aligned} \tag{45}$$

So we obtain that

$$\begin{aligned}
& \frac{d}{dt} \left(\|n\|^2 + \int i(u\bar{u}_x - u_x \bar{u}) dx \right) \\
& + \delta \left(\|n\|^2 + \int i(u\bar{u}_x - u_x \bar{u}) dx \right) \\
& + \beta \|n\|^2 + (\beta - \delta) \|n\|^2 \\
& = \rho \|n\|^2 + 2\rho \|(-\Delta)^{\alpha/2} u\|^2 + C(\rho, \|Y_0\|_{L_b^2(R; \Sigma_0)}).
\end{aligned} \tag{46}$$

Similarly, we can also get that

$$\begin{aligned}
& \frac{d}{dt} \left(\|n\|^2 + \int i(u\bar{u}_x - u_x \bar{u}) dx \right) \\
& + \beta \left(\|n\|^2 + \int i(u\bar{u}_x - u_x \bar{u}) dx \right) + \beta \|n\|^2 \\
& = \rho \|n\|^2 + 2\rho \|(-\Delta)^{\alpha/2} u\|^2 + C(\rho, \|Y_0\|_{L_b^2(R; \Sigma_0)}).
\end{aligned} \tag{47}$$

Set

$$\begin{aligned}
E & = \|(-\Delta)^{\alpha/2} u\|^2 + \|n\|^2 + \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) \\
& + \int i(u\bar{u}_x - u_x \bar{u}) dx.
\end{aligned} \tag{48}$$

So by (39) and (46), we get

$$\begin{aligned} & \frac{d}{dt}E + \delta E + \delta \|(-\Delta)^{\alpha/2} u\|^2 + \beta \|n\|^2 + (\beta - \delta) \|n\|^2 \\ & \leq 4\rho \|(-\Delta)^{\alpha/2} u\|^2 + 2\rho \|n\|^2 \\ & + C \left(\rho, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|Y_0\|_{L_b^2(R;\Sigma_0)} \right). \end{aligned} \quad (49)$$

By (40) and (47), we also get

$$\begin{aligned} & \frac{d}{dt}E + \beta E + \delta \|(-\Delta)^{\alpha/2} u\|^2 + \beta \|n\|^2 + (\delta - \beta) \|(-\Delta)^{\alpha/2} u\|^2 \\ & \leq 4\rho \|(-\Delta)^{\alpha/2} u\|^2 + 2\rho \|n\|^2 \\ & + C \left(\rho, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|Y_0\|_{L_b^2(R;\Sigma_0)} \right). \end{aligned} \quad (50)$$

Setting $\gamma = \min\{\delta, \beta\}$, $\rho \leq \min\{\delta/4, \beta/2\}$, then we deduce that

$$\frac{d}{dt}E + \gamma E \leq C \left(\rho, \|Y_0\|_{L_b^2(R;\Sigma_0)}, \|f_{0t}\|_{L_b^2(R;H^\alpha)} \right) = C_0, \quad \forall t \geq t_0. \quad (51)$$

By Gronwall inequality, we have

$$E(t) \leq E(t_0) e^{-\gamma(t-t_0)} + \frac{C_0}{\gamma}, \quad \forall t \geq t_0. \quad (52)$$

Obviously for any $t \geq t_0$, we have

$$\begin{aligned} & \left| \int n|u|^2 dx + 2 \operatorname{Re} (f(x, t), u) + \int i(u\bar{u}_x - u_x\bar{u}) dx \right| \\ & \leq \rho \|n\|^2 + \rho \|(-\Delta)^{\alpha/2} u\|^2 + C \left(\rho, \|Y_0\|_{L_b^2(R;\Sigma_0)} \right), \end{aligned} \quad (53)$$

$$\begin{aligned} |E(t_0)| & \leq \|(-\Delta)^{\alpha/2} u(t_0)\|^2 + \|n(t_0)\|^2 \\ & + \left| \int n(t_0) |u(t_0)|^2 dx + 2 \operatorname{Re} (f(t_0), u(t_0)) \right. \\ & \left. + \int i(u(t_0)\bar{u}_x(t_0) - u_x(t_0)\bar{u}(t_0)) dx \right| \\ & \leq C \left(\rho, \|Y_0\|_{L_b^2(R;\Sigma_0)}, \|W_\tau\|_{H^\alpha \times H} \right). \end{aligned} \quad (54)$$

So by (52)~(54), there exists a $t_2 = t_2\{\alpha, \delta, \beta, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|Y_0\|_{L_b^2(R;\Sigma_0)}, \|W_\tau\|_{H^\alpha \times H}\} \geq t_0$ such that

$$E(t) \leq \frac{2C_0}{\gamma}, \quad (55)$$

for any $t \geq t_2$.

By (48), (53), and (55), we get

$$\begin{aligned} & \|(-\Delta)^{\alpha/2} u\|^2 + \|n\|^2 \\ & \leq \rho \|n\|^2 + \rho \|(-\Delta)^{\alpha/2} u\|^2 \\ & + C \left(\rho, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|Y_0\|_{L_b^2(R;\Sigma_0)} \right). \end{aligned} \quad (56)$$

Then setting $\rho = \min\{\delta/4, \beta/2, 1/2\}$, we get

$$\begin{aligned} & \|(-\Delta)^{\alpha/2} u\|^2 + \|n\|^2 \\ & \leq C \left(\alpha, \delta, \beta, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|Y_0\|_{L_b^2(R;\Sigma_0)} \right), \quad \forall t \geq t_2. \end{aligned} \quad (57)$$

By using Lemma 14, we conclude the lemma. $\square$

Lemma 16. Assume that

(i) $\alpha \geq 1$, and $Y(x, t)$ satisfy Assumption 10;

(ii) $W(\tau) \in E_0$ and $W(t)$ solves problem (1)–(4).

Then there exist positive constants $C_3 = C_3(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R;\Sigma_0)}, \|f_{0t}\|_{L_b^2(R;H^\alpha)})$ and $t_3 = t_3(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R;\Sigma_0)}, \|f_{0t}\|_{L_b^2(R;H^\alpha)}, \|W_\tau\|_{E_0})$ such that

$$\|W(t)\|_{E_0} \leq C, \quad \forall t \geq t_3. \quad (58)$$

Proof. Taking the inner product of (1) with $(-\Delta)^\alpha u_t$, we have

$$(iu_t - (-\Delta)^\alpha u - nu + i\delta u, (-\Delta)^\alpha u_t) = (f(x, t), (-\Delta)^\alpha u_t). \quad (59)$$

Taking the real part of (59), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(-\Delta)^\alpha u\|^2 \\ & + \operatorname{Re} \int nu(-\Delta)^\alpha \bar{u}_t dx + \operatorname{Re} \int i\delta(-\Delta)^\alpha \bar{u}_t u_t dx \\ & = -\operatorname{Re} (f(x, t), (-\Delta)^\alpha u_t), \end{aligned} \quad (60)$$

where, by (1) and (2), we have

$$\begin{aligned}
& \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u}_t dx \\
&= \frac{d}{dt} \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx - \operatorname{Re} \int n_t u(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - \operatorname{Re} \int n u_t(-\Delta)^{\alpha} \bar{u} dx \\
&= \frac{d}{dt} \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - \operatorname{Re} \int (g(x, t) - \beta n - |u|_x^2) u(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - \operatorname{Re} \int n(-i f(x, t) - i(-\Delta)^{\alpha} u - i n u - \delta u) (-\Delta)^{\alpha} \bar{u} dx \\
&= \frac{d}{dt} \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx \\
&\quad + \operatorname{Re} \int u(-\Delta)^{\alpha} \bar{u} (\beta n + |u|_x^2 - g(x, t)) dx \\
&\quad + \operatorname{Re} \int n(-\Delta)^{\alpha} \bar{u} (i f(x, t) + i(-\Delta)^{\alpha} u + i n u + \delta u) dx \\
&= \frac{d}{dt} \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx + \operatorname{Re} \int \beta u n(-\Delta)^{\alpha} \bar{u} dx \\
&\quad + \operatorname{Re} \int u^2 \bar{u}_x(-\Delta)^{\alpha} \bar{u} dx + \operatorname{Re} \int |u|^2 u_x(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - \operatorname{Re} \int u g(x, t) (-\Delta)^{\alpha} \bar{u} dx + \operatorname{Re} \int i n f(x, t) (-\Delta)^{\alpha} \bar{u} dx \\
&\quad + \operatorname{Re} \int i n^2 u(-\Delta)^{\alpha} \bar{u} dx + \operatorname{Re} \int \delta n u(-\Delta)^{\alpha} \bar{u} dx, \\
&\operatorname{Re} \int i \delta (-\Delta)^{\alpha} \bar{u} u_t dx \\
&= \operatorname{Re} \int \delta (-\Delta)^{\alpha} \bar{u} (f(x, t) + (-\Delta)^{\alpha} u + n u - i \delta u) dx \\
&= \delta \|(-\Delta)^{\alpha} u\|^2 + \delta \operatorname{Re} \int f(x, t) (-\Delta)^{\alpha} \bar{u} dx \\
&\quad + \delta \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx, \\
&\operatorname{Re} \int f(x, t) (-\Delta)^{\alpha} \bar{u}_t dx \\
&= \frac{d}{dt} \operatorname{Re} \int f(x, t) (-\Delta)^{\alpha} \bar{u} dx - \operatorname{Re} \int f_t(x, t) (-\Delta)^{\alpha} \bar{u} dx.
\end{aligned} \tag{61}$$

By Lemmas 12, 14, and 15 and $\alpha \geq 1$, we see that

$$\|u\|_{L^\infty} \leq C \|(-\Delta)^{\alpha/2} u\|^{1/2\alpha} \|u\|^{1-(1/2\alpha)} < \infty. \tag{62}$$

By (60)~(62), Lemmas 12~15, and Hölder inequality, we can see that

$$\begin{aligned}
& \frac{d}{dt} \left(\|(-\Delta)^{\alpha} u\|^2 + 2 \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx \right. \\
&\quad \left. + 2 \operatorname{Re} \int f(x, t) (-\Delta)^{\alpha} \bar{u} dx \right) + \delta \|(-\Delta)^{\alpha} u\|^2 \\
&\quad + \delta \left(\|(-\Delta)^{\alpha} u\|^2 + 2 \operatorname{Re} \int n u(-\Delta)^{\alpha} \bar{u} dx \right. \\
&\quad \left. + 2 \operatorname{Re} \int f(x, t) (-\Delta)^{\alpha} \bar{u} dx \right) \\
&= 2 \operatorname{Re} \int u g(x, t) (-\Delta)^{\alpha} \bar{u} dx - 2 \operatorname{Re} \int \beta u n(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - 2 \operatorname{Re} \int u^2 \bar{u}_x(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - 2 \operatorname{Re} \int |u|^2 u_x(-\Delta)^{\alpha} \bar{u} dx - 2 \operatorname{Re} \int i n f(x, t) (-\Delta)^{\alpha} \bar{u} dx \\
&\quad - 2 \operatorname{Re} \int i n^2 u(-\Delta)^{\alpha} \bar{u} dx \\
&\quad - 2 \operatorname{Re} \int \delta n u(-\Delta)^{\alpha} \bar{u} dx + 2 \operatorname{Re} \int f_t(x, t) (-\Delta)^{\alpha} \bar{u} dx \\
&\leq 2 \|u\|_{L^\infty} \|g(x, t)\| \|(-\Delta)^{\alpha} u\| + 2 \beta \|u\|_{L^\infty} \|n\| \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 \|u\|_{L^\infty}^2 \|\bar{u}_x\| \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 \|u\|_{L^\infty}^2 \|u_x\| \|(-\Delta)^{\alpha} u\| + 2 \|n\|_{L^4} \|f(x, t)\|_{L^4} \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 \|u\|_{L^\infty} \|n\|_{L^4}^2 \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 \delta \|u\|_{L^\infty} \|n\| \|(-\Delta)^{\alpha} u\| + 2 \|f_t(x, t)\| \|(-\Delta)^{\alpha} u\| \\
&\leq (2 \|u\|_{L^\infty} \|g(x, t)\| + 2 \beta \|u\|_{L^\infty} \|n\| + 2 \delta \|u\|_{L^\infty} \|n\| \\
&\quad + 2 \|f_t(x, t)\|) \|(-\Delta)^{\alpha} u\| \\
&\quad + 4 C \|u\|_{L^\infty}^2 \|(-\Delta)^{\alpha/2} u\|^{1/\alpha} \|u\|^{1-(1/\alpha)} \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 C \|n_x\|^{1/4} \|n\|^{3/4} \|(-\Delta)^{\alpha} u\| \\
&\quad + 2 C \|n_x\|^{1/2} \|n\|^{3/2} \|(-\Delta)^{\alpha} u\| \\
&\leq C \|(-\Delta)^{\alpha} u\| + \frac{\beta}{4} \|n_x\|^2 + C \|(-\Delta)^{\alpha} u\|^{8/7} + \frac{\beta}{4} \|n_x\|^2 \\
&\quad + C \|(-\Delta)^{\alpha} u\|^{4/3} \\
&\leq \frac{\delta}{2} \|(-\Delta)^{\alpha} u\| + \frac{\beta}{2} \|n_x\|^2 \\
&\quad + C \left(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)}, \|f_{0t}\|_{L_b^2(R; H^\alpha)} \right).
\end{aligned} \tag{63}$$

Taking the inner product of (2) with n_{xx} , we have

$$-\frac{1}{2} \frac{d}{dt} \|n_x\|^2 + \int |u|_x^2 n_{xx} dx - \beta \|n_x\|^2 = \int g(x, t) n_{xx} dx, \quad (64)$$

where, by Lemmas 12~15 and $\alpha \geq 1$, we can see that

$$\begin{aligned} & \left| \int |u|_x^2 n_{xx} dx \right| \\ &= \left| \int (u_x \bar{u} + u \bar{u}_x) n_{xx} dx \right| \\ &= \left| -2 \operatorname{Re} \int u \bar{u}_{xx} n_x dx - 2 \int |u_x|^2 n_x dx \right| \\ &\leq \left| -2 \operatorname{Re} \int u \bar{u}_{xx} n_x dx \right| \\ &\quad + \left| 2 \int (u_{xx} \bar{u}_x + u_x \bar{u}_{xx}) n dx \right| \\ &\leq 2 \|u\|_{L^\infty} \|\bar{u}_{xx}\| \|n_x\| + 4 \|u_x\|_{L^4} \|u_{xx}\| \|n\|_{L^4} \\ &\leq C \|(-\Delta)^\alpha u\|^{1/\alpha} \|u\|^{1-(1/\alpha)} \|n_x\| \\ &\quad + C \|(-\Delta)^\alpha u\|^{5/8\alpha} \|u\|^{1-(5/8\alpha)} \|(-\Delta)^\alpha u\|^{1/\alpha} \\ &\quad \times \|u\|^{1-(1/\alpha)} \|n_x\|^{1/4} \|n\|^{3/4} \\ &\leq \frac{\delta}{16} \|(-\Delta)^\alpha u\|^2 + C \|n_x\|^{2\alpha/(2\alpha-1)} \\ &\quad + \frac{\beta}{16} \|n_x\|^2 + C \|(-\Delta)^\alpha u\|^{13/7\alpha} \\ &\leq \frac{\delta}{4} \|(-\Delta)^\alpha u\|^2 + \frac{\beta}{8} \|n_x\|^2 \\ &\quad + C \left(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right). \end{aligned} \quad (65)$$

So by (64) and (65), we get

$$\begin{aligned} & \frac{d}{dt} \|n_x\|^2 + 2\beta \|n_x\|^2 \\ &= -2 \int g(x, t) n_{xx} dx + 2 \int |u|_x^2 n_{xx} dx \\ &= 2 \int g_x(x, t) n_x dx + 2 \int |u|_x^2 n_{xx} dx \\ &\leq 2 \|g_x(x, t)\| \|n_x\| + \frac{\delta}{2} \|(-\Delta)^\alpha u\|^2 \\ &\quad + \frac{\beta}{4} \|n_x\|^2 + C \left(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right) \\ &\leq \frac{\delta}{2} \|(-\Delta)^\alpha u\|^2 + \frac{\beta}{2} \|n_x\|^2 \\ &\quad + C \left(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right). \end{aligned} \quad (66)$$

Set $\gamma = \min\{\delta, \beta\}$ and

$$\begin{aligned} E &= \|(-\Delta)^\alpha u\|^2 + \|n_x\|^2 + 2 \operatorname{Re} \int nu(-\Delta)^\alpha \bar{u} dx \\ &\quad + 2 \operatorname{Re} \int f(x, t) (-\Delta)^\alpha \bar{u} dx. \end{aligned} \quad (67)$$

Then by (63) and (66), we can deduce that

$$\frac{d}{dt} E + \gamma E \leq C \left(\alpha, \delta, \beta, \|f_{0t}\|_{L_b^2(R; H^\alpha)}, \|Y_0\|_{L_b^2(R; \Sigma_0)} \right) \quad \forall t \geq t_3^*, \quad (68)$$

which has the same form with (51) in the proof of Lemma 15. Similar to the study of (51), there exist positive constants $C_3 = C_3(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(R; \Sigma_0)}, \|f_{0t}\|_{L_b^2(R; H^\alpha)})$ and $t_3 = t_3(\alpha, \delta, \beta, \|W_\tau\|_{E_0}, \|f_{0t}\|_{L_b^2(R; H^\alpha)}, \|Y_0\|_{L_b^2(R; \Sigma_0)})$ such that

$$\|(-\Delta)^\alpha u\|^2 + \|n_x\|^2 \leq \frac{1}{2} \left(\|(-\Delta)^\alpha u\|^2 + \|n_x\|^2 \right) + C_3, \quad (69)$$

which conclude the proof of Lemma 16. $\square$

4. Unique Existence of the Solution

In this section, we show the unique existence theorem of the solutions. Since uniform a priori estimates have been established in the above section, one can readily get the existence of the solution by *Galärkin's* method (see [20, 22–24]). We show the theorem and prove it briefly for readers' convenience.

Theorem 17. Set $\alpha \geq 1$, and $Y(x, t)$ satisfy Assumption 10; for each $W_\tau \in E_0$, then system (1)–(4) has a unique global solution $W(x, t) \in L^\infty(\tau, T; E_0)$, $\forall T > \tau$.

Proof. We prove this theorem by two steps.

Step 1. The existence of solution.

By *Galärkin's* method, we construct the approximate solution of the periodic initial value problem (1)–(4). We apply the following approximate solution:

$$\begin{aligned} u^l(x, t) &= \sum_{j=1}^l a_j^l(t) \eta_j(x), \quad n^l(x, t) = \sum_{j=1}^l b_j^l(t) \eta_j(x), \\ \eta_j(x) &= e^{ijx}, \\ W^l(x, t) &= (u^l(x, t), n^l(x, t)), \end{aligned} \quad (70)$$

to approach $W(x, t)$, the solution of the problem (1)–(4). And for $j = 1, 2, \dots, l$, $W^l(x, t)$ satisfies

$$\begin{aligned} & (iu_t^l - (-\Delta)^\alpha u^l - n^l u^l + i\delta u^l - f(x, t), \eta_j) = 0, \\ & (n_t^l + \beta n^l + |u^l|_x^2 - g(x, t), \eta_j) = 0, \\ & (W^l(x, \tau), \eta_j) = (W_\tau, \eta_j), \\ & W^l(x + 2\pi, t) = W^l(x, t). \end{aligned} \quad (71)$$

We see that system (71) is an initial boundary value problem of ordinary differential equations (ODE). By the standard existence theory for ODE and uniform a priori estimates in Section 3, for any l , there exists a unique solution of (71), such that

$$\sup_{\tau \leq t \leq \infty} (\|u'\|_{H^\alpha} + \|n'\|_{H^1} + \|u'_t\|_{L^2} + \|n'_t\|_{H^1}) \leq C(\alpha, \delta, \beta, \|Y_0\|_{L_b^\infty(R; \Sigma_0)}, \|f_{0t}\|_{L_b^2(R; H^\alpha)}, \|W_\tau\|_{E_0}). \quad (72)$$

There is a subsequence $W^{l_k}(x, t)$ of $W^l(x, t)$ and $W(x, t) = (u(x, t), n(x, t))$ such that

$$\begin{aligned} W^{l_k}(x, t) &\longrightarrow W(x, t), \quad * \text{-weakly in } L^\infty(\tau, T; E_0), \\ W_t^{l_k}(x, t) &\longrightarrow W(x, t), \quad (73) \\ &* \text{-weakly in } L^\infty(\tau, T; L^2(\Omega) \times H^1(\Omega)). \end{aligned}$$

Due to the above proof and the continuous extension theorem, $W(x, t)$ is the solution Of (1)~(4).

Step 2. The uniqueness of solution.

Suppose $W_1(x, t) = (u_1(x, t), n_1(x, t))$, $W_2(x, t) = (u_2(x, t), n_2(x, t))$ are two solutions of problem (1)~(4). Let $W(x, t) = W_1(x, t) - W_2(x, t)$, and then $W(x, t) = (u(x, t), n(x, t))$ satisfies

$$iu_t - (-\Delta)^\alpha u - n_1 u_1 + n_2 u_2 + i\delta u = 0, \quad (74)$$

$$n_t + \beta n + |u_1|_x^2 - |u_2|_x^2 = 0, \quad (75)$$

$$W(x, \tau) = 0, \quad (76)$$

$$W(x + 2\pi, t) = W(x, t). \quad (77)$$

Obviously, $\|W(x, t)\|_{E_0}$ is uniformly bounded. Note that $-n_1 u_1 + n_2 u_2 = -nu_1 - n_2 u$, $|u_1|_x^2 - |u_2|_x^2 = u_x \bar{u}_1 + u_{2x} \bar{u} + u \bar{u}_{1x} + u_2 \bar{u}_x$.

Taking the inner product of (74) with u and taking the imaginary part, we can get

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= -2\delta \|u\|^2 + 2 \operatorname{Im}(nu_1 + n_2 u, u) \\ &\leq 2\delta \|u\|^2 + 2\|u_1\|_{L^\infty} \|n\| \|u\| \\ &\leq C(\|u\|^2 + \|n\|^2). \end{aligned} \quad (78)$$

Taking the inner product of (75) with n , we can obtain

$$\begin{aligned} \frac{d}{dt} \|n\|^2 &= -2\beta \|n\|^2 - 2(u_x \bar{u}_1 + u_{2x} \bar{u} + u \bar{u}_{1x} + u_2 \bar{u}_x, n) \\ &\leq 2\beta \|n\|^2 + 2(\|u_x\| \|u_1\|_{L^\infty} + \|u_{2x}\|_{L^\infty} \|u\| \\ &\quad + \|u\| \|u_{1x}\|_{L^\infty} + \|u_2\|_{L^\infty} \|u_x\|) \|n\| \\ &\leq 2\beta \|n\|^2 + C\left(\|(-\Delta)^{\alpha/2} u\|^{1/\alpha} \|u\|^{1-(1/\alpha)} + \|u\|\right) \|n\| \\ &\leq 2\beta \|n\|^2 + C\left(\|(-\Delta)^{\alpha/2} u\| + \|u\|\right) \|n\| \\ &\leq C\left(\|u\|^2 + \|n\|^2 + \|(-\Delta)^{\alpha/2} u\|^2\right). \end{aligned} \quad (79)$$

Taking the inner product of (74) with u_t and taking the real part, we can get

$$\begin{aligned} \frac{d}{dt} \|(-\Delta)^{\alpha/2} u\|^2 &= 2 \operatorname{Re}(i\delta u - nu_1 - n_2 u, u_t) \\ &\leq C(\|u\| + \|u_1\|_{L^\infty} \|n\| + \|n_2\|_{L^\infty} \|u\|) \|u_t\| \\ &\leq C(\|u\|^2 + \|n\|^2 + \|u_t\|^2). \end{aligned} \quad (80)$$

Differentiating (74) with respect to t , taking the inner product of with u_t , and taking the imaginary part, we can get

$$\begin{aligned} \frac{d}{dt} \|u_t\|^2 &= 2 \operatorname{Im}(n_t u_1, u_t) + 2 \operatorname{Im}(nu_{1t}, u_t) + 2 \operatorname{Im}(n_{2t} u, u_t) \\ &\leq C(\|u_1\|_{L^\infty} \|n_t\| + \|u_{1t}\| \|n\|_{L^\infty} \\ &\quad + \|n_{2t}\|_{L^\infty} \|u\|) \|u_t\| \\ &\leq C\|-\beta n - u_x \bar{u}_1 - u_{2x} \bar{u} - u \bar{u}_{1x} - u_2 \bar{u}_x\| \|u_t\| \\ &\quad + C(\|n_x\|^{1/2} \|n\|^{1/2} + \|u\|) \|u_t\| \\ &\leq C(\|n\| + \|n_x\| + \|u_x\| + \|u\|) \|u_t\| \\ &\leq C\left(\|n\| + \|n_x\| + \|(-\Delta)^{\alpha/2} u\|^{1/\alpha} \|u\|^{1-(1/\alpha)} + \|u\|\right) \|u_t\| \\ &\leq C\left(\|n\|^2 + \|n_x\|^2 + \|(-\Delta)^{\alpha/2} u\|^2 + \|u\|^2 + \|u_t\|^2\right). \end{aligned} \quad (81)$$

Taking the inner product of (75) with n_{xx} , we have

$$\begin{aligned} \frac{d}{dt} \|n_x\|^2 &= -2\beta \|n_x\|^2 + 2(u_x \bar{u}_1 + u_{2x} \bar{u} + u \bar{u}_{1x} + u_2 \bar{u}_x, n_{xx}) \end{aligned}$$

$$\begin{aligned}
&= -2\beta\|n_x\|^2 - 2 \int (u_{xx}\bar{u}_1 + u_x\bar{u}_{1x}) n_x dx \\
&\quad - 2 \int (u_{2xx}\bar{u} + u_{2x}\bar{u}_x) n_x dx \\
&\quad - 2 \int (u_x\bar{u}_{1x} + u\bar{u}_{1xx}) n_x dx \\
&\quad - 2 \int (u_{2x}\bar{u}_x + u_2\bar{u}_{xx}) n_x dx \\
&\leq 2\beta\|n_x\|^2 + C(\|u\| + \|u_x\| + \|u_{xx}\|)\|n_x\| \\
&\leq 2\beta\|n_x\|^2 + C\left(\|u\| + \|(-\Delta)^{\alpha/2}u\|^{1/\alpha}\|u\|^{1-(1/\alpha)}\right. \\
&\quad \left.+ \|(-\Delta)^\alpha u\|^{1/\alpha}\|u\|^{1-(1/\alpha)}\right)\|n_x\| \\
&\leq C\left(\|u\|^2 + \|(-\Delta)^{\alpha/2}u\|^2 + \|(-\Delta)^\alpha u\|^2 + \|n_x\|^2\right) \\
&\leq C\left(\|u\|^2 + \|(-\Delta)^{\alpha/2}u\|^2\right. \\
&\quad \left.+ \|iu_t - nu_1 - n_2u + i\delta u\|^2 + \|n_x\|^2\right) \\
&\leq C\left(\|u\|^2 + \|n\|^2 + \|(-\Delta)^{\alpha/2}u\|^2 + \|u_t\|^2 + \|n_x\|^2\right). \tag{82}
\end{aligned}$$

Therefore by (78)~(81), we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2 + \|n\|^2 + \|(-\Delta)^{\alpha/2}u\|^2 + \|n_x\|^2 + \|u_t\|^2\right) \\
&\leq C\left(\|u\|^2 + \|n\|^2 + \|(-\Delta)^{\alpha/2}u\|^2 + \|n_x\|^2 + \|u_t\|^2\right). \tag{83}
\end{aligned}$$

From Gronwall inequality and (76), we have

$$u(x, t) = 0, \quad n(x, t) = 0. \tag{84}$$

Therefore, we complete the proof of the theorem. $\square$

5. Uniform Absorbing Set and Uniform Attractor

In this section, we will prove the existence of the strong compact uniform attractor of problem (1)~(4) applying Ball et al.'s idea (see [19, 22]). Firstly, we construct a bounded uniformly absorbing set. Next, we show the weak uniform attractor of the system. Lastly, we derive that the weak uniform attractor is actually the strong one.

Theorem 18. *Under assumptions of Theorem 17, $\{U_{\sigma \in \Sigma}(t, \tau)\}$ admits a strong compact uniform attractor $\mathcal{A}_\Sigma$.*

Proof. We prove this theorem by three steps.

Step 1. $\{U_{\sigma \in \Sigma}(t, \tau)\}$ possess a bounded uniformly absorbing set in E_0 .

Let $B_0 = \{W \in E_0 \mid \|W\|_{E_0}^2 \leq C(\alpha, \delta, \beta, \|Y_0\|_{L_b^2(\mathbb{R}; \Sigma_0)}, \|f_{0t}\|_{L^2(\mathbb{R}; H^\alpha)}, \|W_\tau\|_{E_0})\}$. By Theorem 17, B_0 is a bounded absorbing set of the process $U_{\sigma=Y_0}$.

By Assumption 10, we know that, for each $Y \in \Sigma$, $\|Y\|_{L_b^2(\mathbb{R}; \Sigma_0)}^2 \leq \|Y_0\|_{L_b^2(\mathbb{R}; \Sigma_0)}^2$ holds. So the solution of (1)~(4) satisfies

$$\begin{aligned}
\|W\|_{E_0} &\leq C(\alpha, \delta, \beta, \|f_{0t}\|_{L^2(\mathbb{R}; H^\alpha)}, \|Y\|_{L_b^2(\mathbb{R}; \Sigma_0)}, \|W_\tau\|_{E_0}) \\
&\leq C(\alpha, \delta, \beta, \|f_{0t}\|_{L^2(\mathbb{R}; H^\alpha)}, \|Y_0\|_{L_b^2(\mathbb{R}; \Sigma_0)}, \|W_\tau\|_{E_0}). \tag{85}
\end{aligned}$$

Then we can get that the set B_0 is a bounded uniformly absorbing set of $\{U_{\sigma \in \Sigma}(t, \tau)\}$.

Step 2. we prove the existence of weakly compact uniform attractor $\mathcal{A}_\Sigma$ in E_0 .

From Lemma 6, Theorem 17, and Step 1, we only need to prove that $\{U_{\sigma \in \Sigma}(t, \tau)\}$ is $(E_0 \times \Sigma, E_0)$ -continuous. We denote weak convergence by $\rightharpoonup$ and $*$ weak convergence by $\overset{*}{\rightharpoonup}$.

For any fixed $t_1 \geq \tau \in \mathbb{R}$, let

$$(W_{\tau k}, \sigma_k) \rightharpoonup (W_\tau, \sigma) \quad \text{in } E_0 \times \Sigma. \tag{86}$$

If we can deduce that

$$W_{\sigma_k}(t_1) \rightharpoonup W_\sigma(t_1) \quad \text{in } E_0, \tag{87}$$

where $W_{\sigma_k}(t_1) = (u_k(t_1), n_k(t_1)) = U_{\sigma_k}(t_1, \tau)W_{\tau k}$, $W_\sigma(t_1) = (u(t_1), n(t_1)) = U_\sigma(t_1, \tau)W_\tau$, we will obtain that $\{U_{\sigma \in \Sigma}(t, \tau)\}$ is $(E_0 \times \Sigma, E_0)$ -continuous. By (86) and Theorem 17, we can get that

$$\|W_{\tau k}\|_{E_0} \leq C, \tag{88}$$

$$\sup_{t \in [\tau, T]} \|W_{\sigma_k}(t)\|_{E_0} \leq C. \tag{89}$$

Then by Lemmas 12~16, we can see that

$$\|W_{\sigma_k}(t)\|_{\infty} \leq C, \quad \forall 0 \leq t \leq T. \tag{90}$$

Note that

$$iu_{kt} = (-\Delta)^\alpha u_k + n_k u_k - i\delta u_k + f_k(x, t), \tag{91}$$

$$n_{kt} = -|u_k|_x^2 - \beta n_k + g_k(x, t), \tag{92}$$

and $\sigma_k = (f_k(x, t), g_k(x, t)) \in \Sigma$. By (89) and (90), we find that $\partial_t W_{\sigma_k}(t) \in L^\infty(\tau, T; L^2(\Omega) \times H^1(\Omega))$ and

$$\|\partial_t W_{\sigma_k}(t)\|_{L^\infty(\tau, T; L^2(\Omega) \times H^1(\Omega))} \leq C. \tag{93}$$

Because of Theorem 17 and (93), we easily see that there exist a subsequence $\{W_{\sigma_{k_l}}(t)\}$ of $\{W_{\sigma_k}(t)\}$ and $\widetilde{W}(t) \triangleq (\widetilde{u}(t), \widetilde{n}(t)) \in L^\infty(\tau, T; E_0)$, such that

$$W_{\sigma_{k_l}}(t) \overset{*}{\rightharpoonup} \widetilde{W}(t) \quad \text{in } L^\infty(\tau, T; E_0), \tag{94}$$

$$\partial_t W_{\sigma_{k_l}}(t) \overset{*}{\rightharpoonup} \partial_t \widetilde{W}(t) \quad \text{in } L^\infty(\tau, T; L^2(\Omega) \times H^1(\Omega)). \tag{95}$$

Besides, for any $t_1 \in [\tau, T]$, by (89) there exists $W^0 \triangleq (u^0(t_1), n^0(t_1)) \in E_0$ such that

$$W_{\sigma_k}(t_1) \rightharpoonup W^0 \quad \text{in } E_0. \tag{96}$$

By (94) and compactness embedding theorem, we can get that

$$u_{k_l}(t) \longrightarrow \tilde{u}(t) \quad \text{in } L^2(\tau, T; H^\alpha). \quad (97)$$

Next, we will obtain that $\tilde{W}(t)$ is a solution of problem (1)~(4).

For all $v \in L^2(\Omega)$, $\forall \psi \in C_0^\infty(\tau, T)$, by (91) we have that

$$\begin{aligned} & \int_\tau^T (iu_{k_l t}, \psi(t)v) dt - \int_\tau^T ((-\Delta)^\alpha u_{k_l}, \psi(t)v) dt \\ & - \int_\tau^T (n_{k_l} u_{k_l}, \psi(t)v) dt + \int_\tau^T (i\delta u_{k_l}, \psi(t)v) dt \\ & - \int_\tau^T (f(x, t), \psi(t)v) dt = 0. \end{aligned} \quad (98)$$

Since

$$\begin{aligned} & \int_\tau^T (n_{k_l} u_{k_l}, \psi(t)v) dt - \int_\tau^T (\tilde{n}\tilde{u}, \psi(t)v) dt \\ & = \int_\tau^T ((u_{k_l} - \tilde{u})n_{k_l}, \psi(t)v) dt \\ & + \int_\tau^T (\tilde{u}(n_{k_l} - \tilde{n}), \psi(t)v) dt, \end{aligned} \quad (99)$$

by (90), (94), and (97),

$$\begin{aligned} & \int_\tau^T ((u_{k_l} - \tilde{u})n_{k_l}, \psi(t)v) dt \\ & \leq \sup_{0 \leq t \leq T} \|n_{k_l}(t)\|_{L^\infty} \|\psi(t)v\|_{L^2(0, T; L^2(\Omega))} \\ & \quad \times \|u_{k_l} - \tilde{u}\|_{L^2(0, T; L^2(\Omega))} \longrightarrow 0, \\ & \int_\tau^T (\tilde{u}(n_{k_l} - \tilde{n}), \psi(t)v) dt \\ & = \int_\tau^T ((n_{k_l} - \tilde{n}), \psi(t)v\tilde{u}) dt \longrightarrow 0. \end{aligned} \quad (100)$$

Then we have

$$\int_\tau^T (n_{k_l} u_{k_l}, \psi(t)v) dt \longrightarrow \int_\tau^T (\tilde{n}\tilde{u}, \psi(t)v) dt. \quad (101)$$

And by (94), we have that

$$\begin{aligned} & \int_\tau^T ((-\Delta)^\alpha u_{k_l}, \psi(t)v) dt - \int_\tau^T ((-\Delta)^\alpha \tilde{u}, \psi(t)v) dt \\ & \leq \|(-\Delta)^\alpha (u_{k_l} - \tilde{u})\|_{L^2(\tau, T; L^2(\Omega))} \\ & \quad \times \|\psi(t)v\|_{L^2(\tau, T; L^2(\Omega))} \longrightarrow 0. \end{aligned} \quad (102)$$

By using the similar methods to the other terms of (98), we have

$$\begin{aligned} & \int_\tau^T (i\tilde{u}_t, v) \psi(t) dt - \int_\tau^T ((-\Delta)^\alpha \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (\tilde{n}\tilde{u}, v) \psi(t) dt + \int_\tau^T (i\delta \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (f(x, t), v) \psi(t) dt = 0. \end{aligned} \quad (103)$$

So, we can get that

$$i\tilde{u}_t - (-\Delta)^\alpha \tilde{u} - \tilde{u}\tilde{n} + i\delta \tilde{u} = f(x, t), \quad (104)$$

which shows that $(\tilde{u}, \tilde{n})$ satisfies (1).

For any $v \in L^2(\Omega)$, $\forall \psi \in C_0^\infty(\tau, T)$ with $\psi(T) = 0$, $\psi(\tau) = 1$, by (91) we find that

$$\begin{aligned} & - \int_\tau^T (iu_{k_l}, v) \psi'(t) dt - \int_\tau^T ((-\Delta)^\alpha u_{k_l}, v) \psi(t) dt \\ & - \int_\tau^T (n_{k_l} u_{k_l}, v) \psi(t) dt + \int_\tau^T (i\delta u_{k_l}, v) \psi(t) dt \\ & - \int_\tau^T (f(x, t), v) \psi(t) dt = i(u_{k_l}(\tau), v). \end{aligned} \quad (105)$$

We know that Assumption (86) implies that

$$u_{k_l}(\tau) = u_{\tau k_l} \rightharpoonup u_\tau \quad \text{in } H^{2\alpha}. \quad (106)$$

Then from (105) and (106), we have

$$\begin{aligned} & - \int_\tau^T (i\tilde{u}, v) \psi'(t) dt - \int_\tau^T ((-\Delta)^\alpha \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (\tilde{n}\tilde{u}, v) \psi(t) dt + \int_\tau^T (i\delta \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (f(x, t), v) \psi(t) dt = i(u_\tau, v), \end{aligned} \quad (107)$$

while by (104) we know that

$$\begin{aligned} & - \int_\tau^T (i\tilde{u}, v) \psi'(t) dt - \int_\tau^T ((-\Delta)^2 \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (\tilde{n}\tilde{u}, v) \psi(t) dt + \int_\tau^T (i\delta \tilde{u}, v) \psi(t) dt \\ & - \int_\tau^T (f(x, t), v) \psi(t) dt = i(\tilde{u}(\tau), v). \end{aligned} \quad (108)$$

So by (107) and (108), we have that

$$(u_\tau, v) = (\tilde{u}(\tau), v), \quad \forall v \in L^2(\Omega), \quad (109)$$

$$\tilde{u}(\tau) = u_\tau. \quad (110)$$

By (104) and (110), we have

$$\tilde{u}(t) = u(t). \quad (111)$$

For any $v \in L^2(\Omega)$, $\forall \psi \in C_0^\infty(\tau, t_1)$, with $\psi(\tau) = 0$, $\psi(t_1) = 1$, then we repeat the procedure of proofs of (105)~(108) by (96) having

$$u^0(t_1) = \tilde{u}(t_1). \quad (112)$$

From (96), (111), and (112), we have that

$$u_k(t_1) \rightharpoonup u(t_1) \quad \text{in } H^{2\alpha}(\Omega). \quad (113)$$

Similarly, we can also derive that

$$n_k(t_1) \rightharpoonup n(t_1) \quad \text{in } H^1(\Omega). \quad (114)$$

From (113) and (114), we deduce (87). We complete the proof of the step.

Step 3. We show the weakly compact uniform attractor $\mathcal{A}_\Sigma$ is actually the strong one.

From the proof of Lemma 16, we know each solution trajectory for problem (1)–(4) satisfies

$$\frac{d}{dt} (\|u_{xx}\|^2 + F) + 2\delta (\|u_{xx}\|^2 + F) = G, \quad (115)$$

$$\frac{d}{dt} \|n_x\|^2 + 2\beta \|n_x\|^2 = G_1, \quad (116)$$

where

$$F = 2 \operatorname{Re} \int nu(-\Delta)^\alpha \bar{u} dx + 2 \operatorname{Re} \int f(x, t) (-\Delta)^\alpha \bar{u} dx,$$

$$\begin{aligned} G &= 2 \operatorname{Re} \int ug(x, t) (-\Delta)^\alpha \bar{u} dx - 2 \operatorname{Re} \int \beta un(-\Delta)^\alpha \bar{u} dx \\ &\quad - 2 \operatorname{Re} \int u^2 \bar{u}_x (-\Delta)^\alpha \bar{u} dx - 2 \operatorname{Re} \int |u|^2 u_x (-\Delta)^\alpha \bar{u} dx \\ &\quad - 2 \operatorname{Re} \int inf(x, t) (-\Delta)^\alpha \bar{u} dx - 2 \operatorname{Re} \int in^2 u (-\Delta)^\alpha \bar{u} dx \\ &\quad + 2 \operatorname{Re} \int f_t(x, t) (-\Delta)^\alpha \bar{u} dx \\ &\quad + 2 \operatorname{Re} \int \delta f(x, t) (-\Delta)^\alpha \bar{u} dx, \end{aligned}$$

$$\begin{aligned} G_1 &= -2 \int g(x, t) n_{xx} dx + 2 \int |u_x|^2 n_{xx} dx \\ &= 2 \int g_x(x, t) n_x dx - 2 \operatorname{Re} \int u \bar{u}_{xx} n_x dx \\ &\quad + 2 \int (u_{xx} \bar{u}_x + u_x \bar{u}_{xx}) n dx. \end{aligned} \quad (117)$$

By the uniform boundedness and the compactness embedding, we have that F , G , and G_1 are all weakly continuous in $E_0 \times \Sigma$.

From Step 2, we can see that the point $(w, m) \in \mathcal{A}_\Sigma$ if and only if there exist two sequences $\{w_k^0, m_k^0\}_{k=1}^\infty$ and $\{t_k\}_{k=1}^\infty$ such that for all $\sigma(t) \in \Sigma$, it uniformly satisfies that

$$U_\sigma(t_k, \tau)(w_k^0, m_k^0) \rightharpoonup (w, m) \quad \text{in } E_0, \quad k \rightarrow \infty, \quad (118)$$

where $t_k \rightarrow \infty$ as $k \rightarrow \infty$. If the weak convergence implies strong one, we obtain $\mathcal{A}_\Sigma$ is actually the strong compact attractor. For each fixed $h > \tau$, because of $t_k \rightarrow \infty$, we consider it as $h < t_k, k \in \mathbb{N}_+$. By Lemma 16 and Theorem 17, $U_\sigma(t_k - h, \tau)(w_k^0, m_k^0)$ is bounded in E_0 . Then there exists a subsequence $U_\sigma(t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0)$ of $U_\sigma(t_k - h, \tau)(w_k^0, m_k^0)$ and a point $(v, p) \in E_0$, such that

$$U_\sigma(t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0) \rightharpoonup (v, p) \quad \text{in } E_0. \quad (119)$$

Let

$$\begin{aligned} &(w_{k_l}(t), m_{k_l}(t)) \\ &= U_{T(t_{k_l} - h - \tau)\sigma}(t + \tau, \tau) U_\sigma(t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0) \\ &= U_\sigma(t + t_{k_l} - h, t_{k_l} - h) U_\sigma(t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0) \\ &= U_\sigma(t + t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0), \end{aligned} \quad (120)$$

where $T(\cdot)$ is the translation operator on Σ . Since $\sigma(t)$ is translation compact symbol, there exists a symbol $\sigma^* \in \Sigma$ such that

$$T(t_{k_l} - h - \tau)\sigma \rightarrow \sigma^* \quad \text{in } \Sigma. \quad (121)$$

Then by (118), (119), and the weak $(E \times \Sigma)$ -continuity of $U_{\sigma \in \Sigma}(t, \tau)$, we can get that

$$(w_{k_l}(t), m_{k_l}(t)) \rightharpoonup U_{\sigma^*}(t, \tau)(v, p) = (w, m) \quad \text{in } E_0, \quad \forall t > \tau. \quad (122)$$

From (119), we can see that the solution trajectory $(w_{k_l}(t), m_{k_l}(t))$ is created by $U_{T(t_{k_l} - h - \tau)\sigma}(t + \tau, \tau)$ starting at $U_\sigma(t_{k_l} - h, \tau)(w_{k_l}^0, m_{k_l}^0)$. By (115), (119), and (122), we have that

$$\begin{aligned} &\|w_{k_l}(t)\|_{H_{\text{per}}^{2\alpha}}^2 + F(w_{k_l}(t), m_{k_l}(t)) \\ &= e^{-2\delta(t-\tau)} \left(\|U_\sigma(t_{k_l} - h, \tau) w_{k_l}^0\|_{H_{\text{per}}^{2\alpha}}^2 \right. \\ &\quad \left. + F(U_\sigma(t_{k_l} - h, \tau) w_{k_l}^0, U_\sigma(t_{k_l} - h, \tau) m_{k_l}^0) \right) \\ &\quad + \int_\tau^t e^{-2\delta(t-s)} G(w_{k_l}(s), m_{k_l}(s)) ds \\ &= e^{-2\delta(t-\tau)} \left(\|U_\sigma(t_{k_l} - h, \tau) w_{k_l}^0\|_{H_{\text{per}}^{2\alpha}}^2 + F(v, p) \right) \\ &\quad + \int_\tau^t e^{-2\delta(t-s)} G(U_{\sigma^*}(s + \tau, \tau)(v, p)) ds. \end{aligned} \quad (123)$$

Let $t = h$ in (122). Since F and G are weakly continuous in E_0 , $\|U_\sigma(t_{k_i} - h, \tau)(w_{k_i}^0, m_{k_i}^0)\|_{H_{\text{per}}^{2\alpha}}^2 \leq C$, and the Lebesgue dominated convergence theorem, we can obtain that

$$\begin{aligned} & \limsup_{k_i \rightarrow \infty} \|U_\sigma(t_{k_i}, \tau)w_{k_i}^0\|_{H_{\text{per}}^{2\alpha}}^2 + F(U_{\sigma^*}(h + \tau, \tau)(v, p)) \\ & \leq e^{-2\delta(h-\tau)} (C + F(v, p)) \\ & \quad + \int_\tau^h e^{-2\delta(h-\tau)} G(U_{\sigma^*}(s + \tau, \tau)(v, p)) ds. \end{aligned} \quad (124)$$

Since $(w, m) = U_{\sigma^*}(S + \tau, \tau)(v, p)$, we can see the solution (w, m) as at h corresponding to the initial data (v, p) and the symbol σ^* . Similarly to (122), we have

$$\begin{aligned} & \|w\|_{H_{\text{per}}^{2\alpha}}^2 + F(w, m) \\ & = e^{-2\delta(h-t)} (\|v\|_{H_{\text{per}}^{2\alpha}}^2 + F(v, p)) \\ & \quad + \int_\tau^h e^{-2\delta(h-t)} G(U_{\sigma^*}(s + \tau, \tau)(v, p)) ds. \end{aligned} \quad (125)$$

Deducting (125) from (124), we can get that

$$\begin{aligned} & \limsup_{k_i \rightarrow \infty} \|U_\sigma(t_{k_i}, \tau)w_{k_i}^0\|_{H_{\text{per}}^{2\alpha}}^2 \\ & \leq \|w\|_{H_{\text{per}}^{2\alpha}}^2 + Ce^{-2\delta(h-t)} - e^{-2\delta(h-t)} \|v\|_{H_{\text{per}}^{2\alpha}}^2 \\ & \leq \|w\|_{H_{\text{per}}^{2\alpha}}^2 + Ce^{-2\delta(h-t)}. \end{aligned} \quad (126)$$

As $h \rightarrow \infty$, we can get that

$$\limsup_{k_i \rightarrow \infty} \|U_\sigma(t_{k_i}, \tau)w_{k_i}^0\|_{H_{\text{per}}^{2\alpha}}^2 \leq \|w\|_{H_{\text{per}}^{2\alpha}}^2. \quad (127)$$

On the other hand, the weak convergence $U_\sigma(t_k, \tau)w_k^0 \rightharpoonup w$ implies that

$$\liminf_{k \rightarrow \infty} \|U_\sigma(t_k, \tau)w_k^0\|_{H_{\text{per}}^{2\alpha}}^2 \geq \|w\|_{H_{\text{per}}^{2\alpha}}^2. \quad (128)$$

From the above two inequalities, we get that

$$\lim_{k \rightarrow \infty} \|U_\sigma(t_k, \tau)w_k^0\|_{H_{\text{per}}^{2\alpha}}^2 = \|w\|_{H_{\text{per}}^{2\alpha}}^2. \quad (129)$$

Similarly to the above arguments, by using (116) we can derive that

$$\lim_{k \rightarrow \infty} \|U_\sigma(t_k, \tau)m_k^0\|_{H_{\text{per}}^1}^2 = \|m\|_{H_{\text{per}}^1}^2. \quad (130)$$

Then we get that $U_\sigma(t_k, \tau)(w_k^0, m_k^0) \rightarrow (w, m)$ in E_0 . We complete the proof of the theorem. $\square$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the NSF of China (nos. 11371183 and 11271050) and the NSF of Shandong Province (no. ZR2013AM004).

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