Blow-Up and Global Existence for a Quasilinear Parabolic System

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The problem of solutions to a class of quasilinear coupling parabolic system was studied. By constructing weak upper-solutions and weak lower-solutions, we obtain the global existence and blow-up of solutions under appropriate conditions.

1. Introduction and Main Result

In this paper, we consider global existence of nonnegative solutions for a class of nonlocal degenerate quasilinear parabolic system as follows:

\[
\begin{align*}
& u_t = \Delta u^m + u^p \|v\|_{\alpha}^p, \\
& v_t = \Delta v^n + v^q \|w\|_{\beta}^q, \\
& w_t = \Delta w^\gamma + w^r \|u\|_{\gamma}^r,
\end{align*}
\]

\[ (x, t) \in \Omega \times (0, T), \quad u(x, t) = v(x, t) = w(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (1) \]

where \( \Omega \in \mathbb{R}^N \) is bounded region; \( \Omega \) has smooth boundary \( \partial \Omega \); the parameters \( m, n, h > 1, \alpha, \beta, \gamma \geq 1, p_1, q_1 \), and \( r_1 \geq 0, p_2, q_2, r_2 > 0 \); the initial functions \( u_0, v_0, \) and \( w_0 \) are nonnegative and bounded, and \( \| \cdot \|_\alpha = \int_\Omega | \cdot |^\alpha dx \).

Quasilinear parabolic system is the model for many problems in the scientific field, for example, gas flow model in some seepage medium, some biological population growth model. In recent years, there are many papers to investigate the nonlinear parabolic equation and many excellent results are obtained (see [1–10] and the references cited therein). In this paper, we expand the equation of [10] into 3 and discuss the global existence and blow-up of the solutions for problem (1), and the main results of this paper are the following.

**Theorem 1.** If one of the following conditions holds, system (1) has global solutions:

1. \( m > p_1, n > q_1, h > r_1, \) and \( p_2 q_2 r_2 < (m - p_1)(n - q_1)(h - r_1) \);
2. \( m > p_1, n > q_1, h > r_1, \) and \( p_2 q_2 r_2 = (m - p_1)(n - q_1)(h - r_1) \), and the magnitude of the region \( \Omega \) is sufficiently small;
3. \( m \leq p_1, n \leq q_1, h \leq r_1, \) and \( p_2 q_2 r_2 > (m - p_1)(n - q_1)(h - r_1) \), and the initial data \( u_0, v_0, \) and \( w_0 \) are sufficiently small.

**Theorem 2.** If one of the following conditions holds, the solution of system (1) blows up in finite time:

1. \( m > p_1, n > q_1, h > r_1, \) and \( p_2 q_2 r_2 < (m - p_1)(n - q_1)(h - r_1) \), and the initial data \( u_0, v_0, \) and \( w_0 \) are sufficiently large;
2. \( m \leq p_1, n \leq q_1, h \leq r_1, \) and the initial data \( u_0, v_0, \) and \( w_0 \) are sufficiently large.

2. Proof of Global Existence

As we know, nonlocal degenerate quasilinear parabolic system may not have classical solutions. Similar to the proof in [10] (see page 388-389), we can obtain that system (1)
has nonnegative weak upper-solutions and nonnegative weak lower-solutions, and the following comparison principle holds.

**Lemma 3** (comparison principle). Suppose \((\overline{u}, \overline{v}, \overline{w}), (u, v, w)\) are the nonnegative weak upper-solutions and nonnegative weak lower-solutions of system (1), if
\[
\overline{u}, \overline{v}, \overline{w} \geq \delta > 0, \quad \text{or} \quad u, v, w \geq \delta > 0,
\]
then one has \((u(x, t), v(x, t), w(x, t)) \leq (\overline{u}(x, t), \overline{v}(x, t), \overline{w}(x, t))\) for every \((x, t) \in \Omega \times (0, T)\).

Therefore, in order to prove Theorem 1, we only show that, for all \(T > 0\), there exists a positive bounded weak upper-solution. Let \(\varphi(x)\) be the unique positive solution of linear elliptic equation as follows:
\[
-\Delta \varphi = 1, \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial \Omega. \quad (3)
\]
Denote \(C = \max_{x \in \Omega} \varphi(x)\), then \(0 \leq \varphi(x) \leq C\). Define \((\overline{u}(x, t), \overline{v}(x, t), \overline{w}(x, t))\) as follows:
\[
\overline{u} = (k(\varphi(x) + 1))^{l_1}, \quad \overline{v} = (k(\varphi(x) + 1))^{l_2},
\]
\[
\overline{w} = (k(\varphi(x) + 1))^{l_3}, \quad (4)
\]
where \(l_1, l_2, l_3 < 1\) and satisfy \(ml_1, nl_2, hl_3 < 1\). \(k\) is a positive constant to be determined suitably. For all \(T > 0\), \(\overline{u}, \overline{v}, \overline{w}\) are bounded, and \(\overline{u} \geq k^{l_1}, \overline{v} \geq k^{l_2}, \overline{w} \geq k^{l_3}\). By deducing, we have
\[
\overline{u}_t - \Delta \overline{u}_t = -k^{l_1} \left[ m l_1 (m l_1 - 1) (\varphi + 1)^{m l_1-2} \right] \| \nabla \varphi \|^2
\]
\[
+ m l_1 (\varphi + 1)^{m l_1-1} \Delta \varphi \geq m l_1 (C + 1)^{m l_1-1} k^{m l_1},
\]
\[
\overline{v}_t - \Delta \overline{v}_t \geq k^{l_1} + l_1; \overline{w}_t - \Delta \overline{w}_t \geq k^{l_1} + l_1; \overline{w}_t - \Delta \overline{w}_t \geq k^{l_1} + l_1.
\]
Similarly
\[
\overline{v}_t - \Delta \overline{v}_t \geq nl_2 (C + 1)^{n l_2-1} k^{n l_2},
\]
\[
\overline{w}_t - \Delta \overline{w}_t \geq hl_3 (C + 1)^{h l_3-1} k^{h l_3},
\]
\[
\overline{w}_t - \Delta \overline{w}_t \geq l_3 (C + 1)^{l_3-1} k^{l_3}.
\]

Denote
\[
C_1 = \left( \frac{|\Omega|^{p_1/\mu}}{m l_1} (C + 1)^{l_1 + l_1 p_1 - m l_1 + 1} \right)^{1/(m l_1 - l_1 - 1 - l_1 p_1)};
\]
\[
C_2 = \left( \frac{|\Omega|^{p_2/\nu}}{n l_2} (C + 1)^{l_2 + l_2 p_2 - n l_2 + 1} \right)^{1/(n l_2 - l_2 - l_2 p_2)};
\]
\[
C_3 = \left( \frac{|\Omega|^{p_3/\gamma}}{h l_3} (C + 1)^{l_3 + l_3 p_3 - h l_3 + 1} \right)^{1/(h l_3 - l_3 - l_3 p_3)}.
\]

(1) If \(m > p_1, n > q_1, h > r_1\), and \(p_2 q_2 r_2 < (m - p_1) (n - q_1) (h - r_1)\), there exist positive constants \(l_1, l_3\) such that
\[
\frac{h - r_1}{r_2} > \frac{l_1}{l_3} > \frac{q_2}{m - p_1} \quad (9)
\]
and \(ml_1, nl_2, hl_3 < 1\). And there exists positive constant \(l_2\) such that
\[
\frac{l_1}{l_2} > \frac{l_2}{m - p_1}, \quad \frac{l_1}{l_2} > \frac{q_2}{n - q_1}. \quad (10)
\]
Thus,
\[
p_2 l_2 < (m - p_1) l_1, \quad q_2 l_2 < (n - q_1) l_2, \quad r_2 l_2 < l_2 (h - r_1).
\]

From (11), we can choose \(k\) sufficiently large, such that \(k > \max\{C_1, C_2, C_3\}\) and
\[
(k(\varphi(x) + 1))^{l_1} \geq u_0(x), \quad (k(\varphi(x) + 1))^{l_2} \geq v_0(x), \quad (k(\varphi(x) + 1))^{l_3} \geq w_0(x).
\]

By (4)–(12) we obtain that \((\overline{u}(x, t), \overline{v}(x, t), \overline{w}(x, t))\) is the weak upper-solutions of system (1).

(2) If \(m > p_1, n > q_1, h > r_1\), and \(p_2 q_2 r_2 = (m - p_1)(n - q_1)(h - r_1)\), there exist positive constants \(l_1, l_3\) such that
\[
\frac{h - r_1}{r_2} > \frac{l_1}{l_3} = \frac{p_2 q_2}{(m - p_1)(n - q_1)}. \quad (13)
\]
And there exists positive constant \(l_2\) such that
\[
\frac{l_1}{l_2} = \frac{p_2}{m - p_1}, \quad \frac{l_1}{l_2} = \frac{q_2}{n - q_1}. \quad (14)
\]
Therefore,
\[
p_2 l_2 = (m - p_1) l_1, \quad q_2 l_2 = (n - q_1) l_2, \quad r_2 l_2 = l_2 (h - r_1).
\]

Without loss of generality, we assume that the domain we discuss is contained in a sufficiently large ball \(B\); denote \(\varphi_0(x)\) is the unique positive solution of the following linear elliptic equation:
\[
-\Delta \varphi = 1, \quad x \in B; \quad \varphi(x) = 0, \quad x \in \partial B. \quad (16)
\]
Let \(C_0 = \max_{x \in B} \varphi_0(x)\); hence \(C \leq C_0\). Suppose that \(|\Omega|\) is sufficiently small such that
\[
|\Omega| < \min \left\{ \left( \frac{ml_1}{C_0 + 1} \right)^{a/p_1}, \left( \frac{nl_2}{C_0 + 1} \right)^{b/q_2}, \left( \frac{hl_3}{C_0 + 1} \right)^{r/2} \right\}. \quad (17)
\]
In addition, we choose \(k\) large enough, such that \(k\) satisfies (12); then from (5), (6), and (12)–(17) we obtain that \((\overline{u}(x, t), \overline{v}(x, t), \overline{w}(x, t))\) is the weak upper-solutions of system (1).
At last, if \( m \leq p_1, n \leq q_1, \) or \( h \leq r_1, \) or if \( m > p_1, n > q_1, h > r_1, \) and \( p_2,q_2,r_2 > (m-p_1)(n-q_1)(h-r_1), \) there exist positive constants \( l_1, l_2, \) and \( l_3 \) such that
\[
\begin{align*}
p_2l_2 &> (m-p_1)l_1, \\
n_2l_2 &> (n-q_1)l_2, \\
r_2l_2 &> (h-r_1), \\
m_l_1, n_l_2, h_l_3 &< 1.
\end{align*}
\] (18)

Therefore, from (18), we can choose \( k \) small enough, such that \( k < \min\{C_1, C_2, C_3\}. \) Furthermore, if \( u_0, v_0, \) and \( w_0 \) are sufficiently small to satisfy (12), then by (5), (6), (12), and (18) we obtain that \((\Omega(x,t), \nu(x,t), \nu(x,t)) \) is the weak upper-solutions of system (1).

This completes the proof of Theorem 1.

### 3. Proof of Blow-Up

In this section we will prove Theorem 2, so we construct blow-up positive weak lower-solutions of system (1). Let \( \lambda_1, \) \( \phi(x) \) be the first eigenvalue and corresponding eigenfunction of the eigenvalue problem as follows:
\[
-\Delta \phi(x) = \lambda \phi(x), \quad x \in \Omega, \quad \phi(x) = 0, \quad x \in \partial \Omega.
\] (19)

Then \( \lambda_1 > 0. \) Standardized \( \phi(x) \), such that \( \phi(x)|_{\Omega} > 0 \) and \( \max_{x \in \Omega} \phi(x) = 1, \) then \( (\partial \phi/\partial n)|_{\partial \Omega} < 0, \) where \( n \) is the outer normal direction of \( \partial \Omega, \) suppose \( \Omega_1 \subset \subset \Omega \) is a compact subset of \( \Omega, \) if any solution \( (u,v,w) \) blows up in \( \Omega_1, \) also \( (u,v,w) \) blows up in \( \Omega. \)

Define functions
\[
\begin{align*}
u(x,t) &= (s(t) \phi(x))^l_1, \\
\psi(x,t) &= (s(t) \phi(x))^l_2, \\
w(x,t) &= (s(t) \phi(x))^l_3,
\end{align*}
\] (20)

where \( l_1, l_2, \) and \( l_3 \) satisfy \( m_l_1, n_l_2, h_l_3 > 1, \) and \( s(t) \) is the solution of the initial value problem as follows:
\[
s'(t) = ks^d, \quad s(0) = b.
\] (21)

Here \( k, b > 0, \) \( d > 1 \) is to be determined suitably. It is clear that \( s(t) \geq b \) and \( s(t) \) are unbounded in finite time, and
\[
\begin{align*}
u(x,t) &\geq (\rho b)^l_1 > 0, \\
\psi(x,t) &\geq (\rho b)^l_2 > 0, \\
w(x,t) &\geq (\rho b)^l_3 > 0, \\
\phi(x) &\geq \Omega_1, \quad t > 0,
\end{align*}
\] (22)

where \( \rho = \min_{x \in \Omega} \phi(x) > 0, \) by computing we obtain that
\[
\begin{align*}
\Delta u^m + \| u^p \|_{L^q}^p &= s^m_1 \left[ m_l_1 (m_l_1 - 1) \phi^{m_l_1 - 2} \nabla \phi \right] \\
+ m_l_1 \phi^{m_l_1 - 1} \Delta \phi \\
+ s^l_1 p_1 s^l_1 p_2 s^l_1 s_1 \| \phi \|^p_1 \| \phi \|^2_1 \\
&\geq -\lambda_1 m_l_1 s^m_1 \phi^{m_l_1} + C_1 s^l_1 p_1 s_1 p_2 s^l_1 \phi^{m_l_1 - 1} \phi^{m_l_1 - l_1} \\
&\geq l_1 s^{l_1 - 1} \phi^{l_1} \left( -\lambda_1 m_l_1 s^m_1 \phi^{m_l_1 - l_1} \\
+ C_1 s^l_1 p_1 s_1 p_2 s^l_1 \phi^{m_l_1 - l_1} \right)
\end{align*}
\]

Similarly
\[
\begin{align*}
\Delta u^m + \| u^p \|_{L^q}^p &\geq l_2 s^{l_2 - 1} \phi^{l_2} \left( -\lambda_1 m_l_2 \phi^{m_l_2} + s^l_2 s_1 p_1 s_1 p_2 \phi^{m_l_2 - m_l_2} \right), \\
\psi &\geq l_2 s^{l_2 - 1} \phi^{l_2} s' (t), \\
\Delta u^m + \| u^p \|_{L^q}^p &\geq l_3 s^{l_3 - 1} \phi^{l_3} \phi^{l_3} \left( -\lambda_1 m_l_3 \phi^{m_l_3} + s^l_3 s_3 s_1 p_1 s_1 p_2 \phi^{m_l_3 - m_l_3} \right), \\
w &\geq l_3 s^{l_3 - 1} \phi^{l_3} s' (t),
\end{align*}
\] (23)

where
\[
\begin{align*}
C_1 &\equiv \phi^{l_2} p_2 \| \phi \|^p_1, \\
C_2 &\equiv \phi^{l_3} s_3 s_1 p_1 s_1 p_2 \| \phi \|^l_1, \\
C_3 &\equiv \phi^{l_3} \phi^{l_3} s_3 s_1 p_1 s_1 p_2 \| \phi \|^l_1
\end{align*}
\] (25)

(1) If \( m > p_1, n > q_1, h > r_1, \) and \( p_2,q_2,r_2 > (m-p_1)(n-q_1)(h-r_1), \) there exist positive constants \( l_1, l_2, \) and \( l_3 \) satisfying \( m_l_1, n_l_2, h_l_3 > 1 \) and
\[
\begin{align*}
p_2l_2 &> (m-p_1)l_1, \\
q_2l_2 &> (n-q_1)l_2, \\
r_2l_2 &> (h-r_1).
\end{align*}
\] (26)

Choose
\[
\begin{align*}
k &\equiv \min \left\{ \frac{C_1}{l_1}, \frac{C_2}{l_2}, \frac{C_3}{l_3} \right\}, \\
d &\equiv \min \left\{ m_l_1 - l_1 + 1, n_l_2 - l_2 + 1, h_l_3 - l_3 + 1 \right\}, \\
b &\equiv \max \left\{ \frac{\lambda_1 m_l_1}{C_1}, \frac{\lambda_1 m_l_2}{C_2}, \frac{\lambda_1 m_l_3}{C_3} \right\}.
\end{align*}
\] (27)

From (26)-(27) and \( m, n, h > 1, \) we have \( k > 0, d > 1, \) and \( b > 1. \) Suppose that \( u_0, v_0, \) and \( w_0 \) are sufficiently large to satisfy
\[
\begin{align*}
u_0 (x) &\geq (b \phi)^l_1, \\
v_0 (x) &\geq (b \phi)^l_2, \\
w_0 (x) &\geq (b \phi)^l_3.
\end{align*}
\] (28)
From (20)–(28), we obtain that $(u, v, w)$ is positive weak lower-solutions of system (1) in $\Omega_1$.

(2) If $m \leq p_1, n \leq q_1, h \leq r_1$, (26) still holds, the results are obtained by the same methods.
Thus completes the proof of Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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