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Research Article

Blow-Up and Global Existence for a Quasilinear Parabolic System

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The problem of solutions to a class of quasilinear coupling parabolic system was studied. By constructing weak upper-solutions and weak lower-solutions, we obtain the global existence and blow-up of solutions under appropriate conditions.

1. Introduction and Main Result

In this paper, we consider global existence of nonnegative solutions for a class of nonlocal degenerate quasilinear parabolic system as follows:

$$\begin{gathered} u_t = \Delta u^m + u^{p_1} \|v\|_\alpha^{p_2}, \\ v_t = \Delta v^n + v^{q_1} \|w\|_\beta^{q_2}, \\ w_t = \Delta w^h + w^{r_1} \|u\|_\gamma^{r_2}, \\ (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \\ w(x,0) = w_0(x), \\ x \in \Omega, \end{gathered} \tag{1}$$

where $\Omega \in R^N$ is bounded region; Ω has smooth boundary $\partial\Omega$; the parameters $m, n, h > 1$, $\alpha, \beta, \gamma \geq 1$, p_1, q_1, and $r_1 \geq 0$, $p_2, q_2, r_2 > 0$; the initial functions u_0, v_0, and w_0 are nonnegative and bounded, and $\|\cdot\|_\alpha^\alpha = \int_\Omega |\cdot|^\alpha dx$.

Quasilinear parabolic system is the model for many problems in the scientific field, for example, gas flow model in some seepage medium, some biological population growth model. In recent years, there are many papers to investigate the nonlinear parabolic equation and many excellent results are obtained (see [1–10] and the references cited therein). In this paper, we expand the equation of [10] into 3 and discuss the global existence and blow-up of the solutions for problem (1), and the main results of this paper are the following.

Theorem 1. *If one of the following conditions holds, system* (1) *has global solutions:*

(1) $m > p_1$, $n > q_1$, $h > r_1$, *and* $p_2 q_2 r_2 < (m - p_1)(n - q_1)(h - r_1)$;

(2) $m > p_1$, $n > q_1$, $h > r_1$, *and* $p_2 q_2 r_2 = (m - p_1)(n - q_1)(h - r_1)$, *and the magnitude of the region* Ω *is sufficiently small;*

(3) *If* $m \leq p_1$, $n \leq q_1$, *or* $h \leq r_1$, *or if* $m > p_1$, $n > q_1$, $h > r_1$, *and* $p_2 q_2 r_2 > (m - p_1)(n - q_1)(h - r_1)$, *and the initial data* u_0, v_0, *and* w_0 *are sufficiently small.*

Theorem 2. *If one of the following conditions holds, the solution of system* (1) *blows up in finite time:*

(1) $m > p_1$, $n > q_1$, $h > r_1$, *and* $p_2 q_2 r_2 > (m - p_1)(n - q_1)(h - r_1)$, *and the initial data* u_0, v_0, *and* w_0 *are sufficiently large;*

(2) $m \leq p_1$, $n \leq q_1$, *or* $h \leq r_1$, *and the initial data* u_0, v_0, *and* w_0 *are sufficiently large.*

2. Proof of Global Existence

As we know, nonlocal degenerate quasilinear parabolic system may not have classical solutions. Similar to the proof in [10] (see page 388-389), we can obtain that system (1)

has nonnegative weak upper-solutions and nonnegative weak lower-solutions, and the following comparison principle holds.

Lemma 3 (comparison principle). *Suppose* $(\overline{u}, \overline{v}, \overline{w})$, $(\underline{u}, \underline{v}, \underline{w})$ *are the nonnegative weak upper-solutions and nonnegative weak lower-solutions of system* (1), *if*

$$\overline{u}, \overline{v}, \overline{w} \geq \delta > 0, \quad or \quad \underline{u}, \underline{v}, \underline{w} \geq \delta > 0, \tag{2}$$

one has $(\underline{u}(x,t), \underline{v}(x,t), \underline{w}(x,t)) \leq (\overline{u}(x,t), \overline{v}(x,t), \overline{w}(x,t))$ *for every* $(x,t) \in \Omega \times (0,T)$.

Therefore, in order to prove Theorem 1, we only show that, for all $T > 0$, there exists a positive bounded weak upper-solutions. Let $\varphi(x)$ be the unique positive solution of linear elliptic equation as follows:

$$-\Delta\varphi = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{3}$$

Denote $C = \max_{x\in\Omega}\varphi(x)$, then $0 \leq \varphi(x) \leq C$. Define $\overline{u}(x,t), \overline{v}(x,t), \overline{w}(x,t)$ as follows:

$$\begin{gathered} \overline{u} = \left(k\left(\varphi(x)+1\right)\right)^{l_1}, \qquad \overline{v} = \left(k\left(\varphi(x)+1\right)\right)^{l_2}, \\ \overline{w} = \left(k\left(\varphi(x)+1\right)\right)^{l_3}, \end{gathered} \tag{4}$$

where $l_1, l_2, l_3 < 1$ and satisfy $ml_1, nl_2, hl_3 < 1$. k is a positive constant to be determined suitably. For all $T > 0$, $\overline{u}$, $\overline{v}$, and $\overline{w}$ are bounded, and $\overline{u} \geq k^{l_1}$, $\overline{v} \geq k^{l_2}$, and $\overline{w} \geq k^{l_3}$. By deducing, we have

$$\begin{aligned} \overline{u}_t - \Delta\overline{u}^m &= -k^{ml_1}\left[ml_1\left(ml_1-1\right)(\varphi+1)^{ml_1-2}|\nabla\varphi|^2 \right. \\ &\qquad \left. + ml_1(\varphi+1)^{ml_1-1}\Delta\varphi\right] \\ &\geq ml_1(C+1)^{ml_1-1}k^{ml_1}, \\ \overline{u}^{p_1}\|\overline{v}\|_\alpha^{p_2} &= \left(k\left(\varphi+1\right)\right)^{l_1p_1}\left\|\left(k\left(\varphi+1\right)\right)^{l_2}\right\|_\alpha^{p_2} \\ &\leq k^{l_1p_1+l_2p_2}(C+1)^{l_1p_1+l_2p_2}|\Omega|^{p_2/\alpha}. \end{aligned} \tag{5}$$

Similarly

$$\begin{gathered} \overline{v}_t - \Delta\overline{v}^n \geq nl_2(C+1)^{nl_2-1}k^{nl_2}, \\ \overline{v}^{q_1}\|\overline{w}\|_\beta^{q_2} \leq k^{l_2q_1+l_3q_2}(C+1)^{l_2q_1+l_3q_2}|\Omega|^{q_2/\beta}, \end{gathered} \tag{6}$$

$$\begin{gathered} \overline{w}_t - \Delta\overline{w}^h \geq hl_3(C+1)^{hl_3-1}k^{hl_3}, \\ \overline{w}^{r_1}\|\overline{u}\|_\gamma^{r_2} \leq k^{l_3r_1+l_1r_2}(C+1)^{l_3r_1+l_1r_2}|\Omega|^{r_2/\gamma}. \end{gathered} \tag{7}$$

Denote

$$\begin{aligned} C_1 &= \left(\frac{|\Omega|^{p_2/\alpha}}{ml_1}(C+1)^{l_1p_1+l_2p_2-ml_1+1}\right)^{1/(ml_1-l_1p_1-l_2p_2)}; \\ C_2 &= \left(\frac{|\Omega|^{q_2/\beta}}{nl_2}(C+1)^{l_2q_1+l_3q_2-nl_2+1}\right)^{1/(nl_2-l_2q_1-l_3q_2)}; \\ C_3 &= \left(\frac{|\Omega|^{r_2/\gamma}}{hl_3}(C+1)^{l_3r_1+l_1r_2-hl_3+1}\right)^{1/(hl_3-l_3r_1-l_1r_2)}. \end{aligned} \tag{8}$$

(1) If $m > p_1, n > q_1, h > r_1$, and $p_2q_2r_2 < (m-p_1)(n-q_1)(h-r_1)$, there exist positive constants l_1, l_3 such that

$$\frac{h-r_1}{r_2} > \frac{l_1}{l_3} > \frac{p_2q_2}{(m-p_1)(n-q_1)} \tag{9}$$

and $ml_1, nl_2, hl_3 < 1$. And there exists positive constant l_2 such that

$$\frac{l_1}{l_2} > \frac{p_2}{m-p_1}, \qquad \frac{l_2}{l_3} > \frac{q_2}{n-q_1}. \tag{10}$$

Thus,

$$\begin{gathered} p_2l_2 < (m-p_1)l_1, \qquad q_2l_3 < (n-q_1)l_2, \\ r_2l_1 < l_3(h-r_1). \end{gathered} \tag{11}$$

From (11), we can choose k sufficiently large, such that $k > \max\{C_1, C_2, C_3\}$ and

$$\begin{gathered} \left(k\left(\varphi(x)+1\right)\right)^{l_1} \geq u_0(x), \qquad \left(k\left(\varphi(x)+1\right)\right)^{l_2} \geq v_0(x), \\ \left(k\left(\varphi(x)+1\right)\right)^{l_3} \geq w_0(x). \end{gathered} \tag{12}$$

By (4)–(12) we obtain that $(\overline{u}(x,t), \overline{v}(x,t), \overline{w}(x,t))$ is the weak upper-solutions of system (1).

(2) If $m > p_1, n > q_1, h > r_1$, and $p_2q_2r_2 = (m-p_1)(n-q_1)(h-r_1)$, there exist positive constants l_1, l_3 such that

$$\frac{h-r_1}{r_2} = \frac{l_1}{l_3} = \frac{p_2q_2}{(m-p_1)(n-q_1)}. \tag{13}$$

And there exists positive constant l_2 such that

$$\frac{l_1}{l_2} = \frac{p_2}{m-p_1}, \qquad \frac{l_2}{l_3} = \frac{q_2}{n-q_1}. \tag{14}$$

Therefore,

$$\begin{gathered} p_2l_2 = (m-p_1)l_1, \qquad q_2l_3 = (n-q_1)l_2, \\ r_2l_1 = l_3(h-r_1). \end{gathered} \tag{15}$$

Without loss of generality, we assume that the domain we discuss is contained in a sufficiently large ball B; denote $\varphi_B(x)$ is the unique positive solution of the following linear elliptic equation:

$$-\Delta\varphi = 1, \quad x \in B; \qquad \varphi(x) = 0, \quad x \in \partial B. \tag{16}$$

Let $C_0 = \max_{x\in B}\varphi_B(x)$; hence $C \leq C_0$. Suppose that $|\Omega|$ is sufficiently small such that

$$|\Omega| < \min\left\{\left(\frac{ml_1}{C_0+1}\right)^{\alpha/p_2}, \left(\frac{nl_2}{C_0+1}\right)^{\beta/q_2}, \left(\frac{hl_3}{C_0+1}\right)^{\gamma/r_2}\right\}. \tag{17}$$

In addition, we choose k large enough, such that k satisfies (12); then from (5), (6), and (12)–(17) we obtain that $(\overline{u}(x,t), \overline{v}(x,t), \overline{w}(x,t))$ is the weak upper-solutions of system (1).

(3) At last, if $m \le p_1$, $n \le q_1$, or $h \le r_1$, or if $m > p_1$, $n > q_1$, $h > r_1$, and $p_2q_2r_2 > (m-p_1)(n-q_1)(h-r_1)$, there exist positive constants l_1, l_2, and l_3 such that

$$\begin{aligned} p_2l_2 &> (m-p_1)\,l_1, \qquad q_2l_3 > (n-q_1)\,l_2, \\ r_2l_1 &> l_3\,(h-r_1), \qquad ml_1, nl_2, hl_3 < 1. \end{aligned} \tag{18}$$

Therefore, from (18), we can choose k small enough,such that $k < \min\{C_1, C_2, C_3\}$. Furthermore, if u_0, v_0, and w_0 are sufficiently small to satisfy (12), then by (5), (6), (12), and (18) we obtain that $(\overline{u}(x,t), \overline{v}(x,t), \overline{w}(x,t))$ is the weak upper-solutions of system (1).

This completes the proof of Theorem 1.

3. Proof of Blow-Up

In this section we will prove Theorem 2, so we construct blow-up positive weak lower-solutions of system (1). Let λ_1, $\phi(x)$ be the first eigenvalue and corresponding eigenfunction of the eigenvalue problem as follows:

$$-\Delta\phi(x) = \lambda\phi(x), \quad x \in \Omega; \qquad \phi(x) = 0, \quad x \in \partial\Omega. \tag{19}$$

Then $\lambda_1 > 0$. Standardized $\phi(x)$, such that $\phi(x)|_\Omega > 0$ and $\max_{x\in\Omega}\phi(x) = 1$, then $(\partial\phi/\partial n)|_{\partial\Omega} < 0$, where n is the outer normal direction of $\partial\Omega$, suppose $\Omega_1 \subset\subset \Omega$ is a compact subset of Ω, if any solution (u,v,w) blows up in Ω_1, also (u,v,w) blows up in Ω.

Define functions

$$\begin{gathered} \underline{u}(x,t) = (s(t)\,\phi(x))^{l_1}, \qquad \underline{v}(x,t) = (s(t)\,\phi(x))^{l_2}, \\ \underline{w}(x,t) = (s(t)\,\phi(x))^{l_3}, \end{gathered} \tag{20}$$

where l_1, l_2, and l_3 satisfy $ml_1, nl_2, hl_3 > 1$, and $s(t)$ is the solution of the initial value problem as follows:

$$s'(t) = ks^d, \qquad s(0) = b. \tag{21}$$

Here $k, b > 0$, $d > 1$ is to be determined suitably. It is clear that $s(t) \ge b$ and $s(t)$ are unbounded in finite time, and

$$\begin{aligned} \underline{u}(x,t) &\ge (\rho b)^{l_1} > 0, \\ \underline{v}(x,t) &\ge (\rho b)^{l_2} > 0, \\ \underline{w}(x,t) &\ge (\rho b)^{l_3} > 0, \\ x \in \Omega_1, &\quad t > 0, \end{aligned} \tag{22}$$

where $\rho = \min_{\overline{\Omega}}\phi(x) > 0$; by computing we obtain that

$$\begin{aligned} \Delta\underline{u}^m + \underline{u}^{p_1}\|\underline{v}\|_\alpha^{p_2} &= s^{ml_1}\left[ml_1(ml_1-1)\,\phi^{ml_1-2}|\nabla\phi|^2 + ml_1\phi^{ml_1-1}\Delta\phi\right] \\ &\quad + s^{l_1p_1+l_2p_2}\phi^{l_1p_1}\|\phi^{l_2}\|_\alpha^{p_2} \\ &\ge -\lambda_1 ml_1 s^{ml_1}\phi^{ml_1} + C_1 s^{l_1p_1+l_2p_2} \\ &= l_1 s^{l_1-1}\phi^{l_1}\left(-\lambda_1 m s^{ml_1-l_1+1}\phi^{ml_1-l_1} + \frac{C_1 s^{l_1p_1+l_2p_2-l_1+1}\phi^{-l_1}}{l_1}\right) \\ &\ge l_1 s^{l_1-1}\phi^{l_1}\frac{C_1}{l_1}s^{ml_1-l_1+1} \times\left(-\frac{\lambda_1 ml_1}{C_1} + s^{l_1p_1+l_2p_2-ml_1}\right), \\ \underline{u}_t &= l_1 s^{l_1-1}\phi^{l_1}s'(t). \end{aligned} \tag{23}$$

Similarly

$$\begin{aligned} &\Delta\underline{v}^n + \underline{v}^{q_1}\|\underline{w}\|_\beta^{q_2} \\ &\quad \ge l_2 s^{l_2-1}\phi^{l_2}\frac{C_2}{l_2}s^{nl_2-l_2+1}\left(-\frac{\lambda_1 nl_2}{C_2} + s^{l_2q_1+l_3q_2-nl_2}\right), \\ &\quad \underline{v}_t = l_2 s^{l_2-1}\phi^{l_2}s'(t), \\ &\Delta\underline{w}^h + \underline{w}^{r_1}\|\underline{u}\|_\gamma^{r_2} \\ &\quad \ge l_3 s^{l_3-1}\phi^{l_3}\frac{C_3}{l_3}s^{hl_3-l_3+1}\left(-\frac{\lambda_1 hl_3}{C_3} + s^{l_3r_1+l_1r_2-hl_3}\right), \\ &\quad \underline{w}_t = l_3 s^{l_3-1}\phi^{l_3}s'(t), \end{aligned} \tag{24}$$

where

$$\begin{gathered} C_1 \equiv \phi^{l_1p_1}\|\phi^{l_2}\|_\alpha^{p_2}, \qquad C_2 \equiv \phi^{l_2q_1}\|\phi^{l_3}\|_\beta^{q_2}, \\ C_3 \equiv \phi^{l_3r_1}\|\phi^{l_1}\|_\gamma^{r_2}. \end{gathered} \tag{25}$$

(1) If $m > p_1$, $n > q_1$, $h > r_1$, and $p_2q_2r_2 > (m-p_1)(n-q_1)(h-r_1)$, there exist positive constants l_1, l_2, and l_3 satisfying $ml_1, nl_2, hl_3 > 1$ and

$$\begin{gathered} p_2l_2 > (m-p_1)\,l_1, \qquad q_2l_3 > (n-q_1)\,l_2, \\ r_2l_1 > l_3\,(h-r_1). \end{gathered} \tag{26}$$

Choose

$$\begin{aligned} k &= \min\left\{\frac{C_1}{l_1}, \frac{C_2}{l_2}, \frac{C_3}{l_3}\right\}, \\ d &= \min\{ml_1 - l_1 + 1, nl_2 - l_2 + 1, hl_3 - l_3 + 1\}, \\ b &= \max\left\{\left(1 + \frac{\lambda_1 ml_1}{C_1}\right)^{1/(l_1p_1+l_2p_2-ml_1)}, \left(1 + \frac{\lambda_1 nl_2}{C_2}\right)^{1/(l_2q_1+l_3q_2-nl_2)}, \left(1 + \frac{\lambda_1 hl_3}{C_3}\right)^{1/(l_3r_1+l_1r_2-hl_3)}\right\}. \end{aligned} \tag{27}$$

From (26)-(27) and $m, n, h > 1$, we have $k > 0$, $d > 1$, and $b > 1$. Suppose that u_0, v_0, and w_0 are sufficiently large to satisfy

$$\begin{gathered} u_0(x) \ge (b\phi)^{l_1}, \qquad v_0(x) \ge (b\phi)^{l_2}, \\ w_0(x) \ge (b\phi)^{l_3}. \end{gathered} \tag{28}$$

From (20)–(28),we obtain that $(\underline{u}, \underline{v}, \underline{w})$ is positive weak lower-solutions of system (1) in Ω_1.

(2) If $m \leq p_1$, $n \leq q_1$, or $h \leq r_1$, (26) still holds, the results are obtained by the same methods.

Thus completes the proof of Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] C. C. Wu, "Global existence of positive solution for a nonlinear parabolic system," *Journal of Fuqing Branch of Fujian Normal University*, no. 2, pp. 22–24, 2012.

[2] C. C. Wu, "The blow-up property of positive solutions to a class of nonlinear parabolic equations with cross-coupling," *Journal of Fujian University of Technology*, vol. 10, no. 6, pp. 533–535, 2012.

[3] C. C. Wu, "Solvability of a nonlinear parabolic system," *Journal of Putian University*, vol. 19, no. 2, pp. 25–27, 2012.

[4] Y.-j. Chen and Y.-p. Zhu, "Blow-up rate and profile for a class of quasilinear parabolic system," *Applied Mathematics and Mechanics*, vol. 30, no. 7, pp. 811–820, 2009.

[5] H. L. Li, "Blow-up estimates for semilinear parabolic systems with exponent reaction terms," *Journal of Southeast University*, vol. 39, no. 6, pp. 1287–1291, 2009.

[6] Y. Z. Xue, "Global existence and blow-up problem of a class of nonlinear parabolic equations with cross coupling," *Journal of Lanzhou University of Technology*, vol. 37, no. 4, pp. 162–164, 2011.

[7] Y. J. Wei and J. Wu, "Existence and blow-up of solutions to a quasilinear degenerate parabolic system," *Journal of Jilin University*, vol. 48, no. 3, pp. 406–408, 2010.

[8] X. S. Wu and W. J. Gao, "Global existence and blow-up of solutions to an evolution p-Laplace system with nonlinear boundary conditions," *Journal of Jilin University*, vol. 47, no. 6, pp. 1105–1111, 2009.

[9] Y. P. Chen and L. H. Liu, "Global blow-up for a localized nonlinear parabolic equation with a nonlocal boundary condition," *Journal of Mathematical Analysis and Applications*, vol. 384, no. 2, pp. 421–430, 2011.

[10] Y. J. Chen, "Blow-up and global existence for a nonlocal degenerate quasilinear parabolic system," *Acta Mathematica Scientia A*, vol. 30, no. 2, pp. 386–396, 2010.