Research Article

On a q -Analogue of the Elzaki Transform Called Mangontarum q -Transform

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Received 21 June 2014; Revised 19 September 2014; Accepted 19 September 2014; Published 13 October 2014

Academic Editor: Seenith Sivasundaram

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Two q -analogues of the Elzaki transform, called Mangontarum q -transforms, are introduced in this paper. Properties such as the transforms of q -trigonometric functions, transform of q -derivatives, duality relation, convolution identity, q -derivative of transforms, and transform of the Heaviside function are derived and presented. Moreover, we will consider applications of the Mangontarum q -transform of the first kind to some ordinary q -differential equations with initial values.

1. Introduction

Given the set

$$S = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{t/k_1}, \\ t \in (-1)^j \times [0, \infty)\}. \quad (1)$$

Elzaki [1] introduced a new integral transform called *Elzaki transform* defined by

$$E\{f(t)\} = T(v) = v \int_0^\infty e^{-t/v} f(t) dt, \quad (2)$$

for $f(t) \in S$, $k_1 \leq v \leq k_2$, and $t \geq 0$. This new transform appears to be a convenient tool in solving various problems related to differential equations as seen in [1–3] and the references cited therein. Fundamental properties of this transform were already established by Elzaki et al. in [1–3] and the references therein. Among these properties are the following:

- (i) the Elzaki transform of special functions such as $f(t) = t^n$ given by

$$E\{t^n\} = n!v^{n+2}; \quad (3)$$

- (ii) the Elzaki transform of $f(t) = \sum_{n \geq 0} a_n t^n$, given by

$$E\{f(t)\} = \sum_{n \geq 0} n! a_n v^{n+2}; \quad (4)$$

- (iii) the Elzaki transform of derivatives given by

$$E\{f^{(n)}\} = \frac{T(v)}{v^n} - \sum_{j=0}^{n-1} v^{2-n+j} f^{(j)}(0); \quad (5)$$

- (iv) the duality relation given by

$$T(v) = vF\left(\frac{1}{v}\right), \quad (6)$$

where $F(v)$ is the Laplace transform of $f(t)$; and

- (v) the convolution identity given by

$$E\{(f * g)(t)\} = \frac{1}{v} M(v) N(v), \quad (7)$$

where $M(v)$ and $N(v)$ are the Elzaki transforms of $f(t)$ and $g(t)$, respectively, and

$$(f * g)(t) = \int_0^\infty f(t) g(t - \tau) d\tau \quad (8)$$

is the convolution of $f(t)$ and $g(t)$.

The study of the q -analogues of classical identities is a popular topic among mathematicians and physicists. One may see [4–8] and the references therein for a brief overview and examples of q -analogues and their applications. The parameter q is known to stand for “quantum” which is widely seen in *quantum calculus* (or *q-calculus*). The book of Kac and Cheung [9] is a good source for further details of quantum calculus. Basically, the term “ q -analogue” refers to a mathematical expression involving a parameter q which generalizes a known identity and reduces back to the known identity when $q \rightarrow 1$. That is, a polynomial $a_k(q)$ is said to be a q -analogue of an integer a_k if by taking its limit as q tends to 1, we recover a_k . For example, for any integers n and k with $0 < q < 1$ and $q \neq 1$, we have the following q -analogues of the integer n , falling factorial $(n)_k$, factorial $n!$, and binomial coefficient $\binom{n}{k}$, respectively:

$$\begin{aligned} [n]_q &= \frac{q^n - 1}{q - 1}, & [n]_{k,q} &= \prod_{j=0}^{k-1} \frac{q^{n-j} - 1}{q - 1}, \\ [n]_q! &= \prod_{j=1}^n [j]_q, & \begin{bmatrix} n \\ k \end{bmatrix}_q &= \prod_{j=1}^k \frac{q^{n-j+1} - 1}{q^j - 1}. \end{aligned} \quad (9)$$

Clearly,

$$\begin{aligned} \lim_{q \rightarrow 1} [n]_q &= n, & \lim_{q \rightarrow 1} [n]_{k,q} &= (n)_k, \\ \lim_{q \rightarrow 1} [n]_q! &= n!, & \lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \binom{n}{k}. \end{aligned} \quad (10)$$

The expressions in (9) are called q -integer, q -falling factorial of n of order k , q -factorial of n , and q -binomial coefficient (or *Gaussian polynomial*), respectively. Furthermore, the q -binomial coefficient can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_{k,q}}{[k]_q!} = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \quad (11)$$

Note that the transition of any classical expression to its q -analogue is not unique. For instance, the two q -analogues for the exponential function e^t denoted by $\hat{e}_q(t)$ and $e_q(t)$ are given, respectively, by

$$\hat{e}_q(t) = \prod_{i=1}^{\infty} (1 + (1-q)q^{i-1}t) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{t^k}{[k]_q!}, \quad (12)$$

$$e_q(t) = \prod_{i=1}^{\infty} (1 - (1-q)q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!}, \quad (13)$$

with $e_q(t)\hat{e}_q(-t) = 1$. Other important tools in this sequel are the *Jackson q -derivative*

$$D_q(f(x)) = \frac{\partial_q}{\partial_q x} f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad (14)$$

and the *definite Jackson q -integral*

$$\int_0^x f(t) d_q t = (1-q) \sum_{i \geq 0} f(q^i x) x q^i. \quad (15)$$

Note that from (14) and (15), we have

$$\int_0^x \frac{\partial_q}{\partial_q t} f(t) d_q t = f(x) - f(0). \quad (16)$$

Furthermore, given the improper q -integral of f

$$\int_0^{\infty/a} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} \frac{q^n}{a} f\left(\frac{q^n}{a}\right), \quad (17)$$

we get (see [10])

$$\int_x^{\infty} f(x) d_q x = \int_0^{x \cdot \infty} f(x) d_q x - \int_0^{\infty} f(x) d_q x. \quad (18)$$

For q -analogues of integral transforms, Chung and Kim [10] defined the q -Laplace transform of the first and second kind given, respectively, by

$$F_q(s) = L_q\{f(t)\} = \int_0^{\infty} \hat{e}_q(-qst) f(t) d_q t, \quad (19)$$

$$\tilde{F}_q(s) = \tilde{L}_q\{f(t)\} = \int_0^{\infty} e_q(-st) f(t) d_q t, \quad (20)$$

for $s > 0$. Other q -analogues of the Laplace transform were earlier considered by Hahn [11] and Abdi [12]. This motivates us to define a q -analogue for the Elzaki transform in [1].

In this paper, we will define two kinds of q -analogues of the Elzaki transform, and to differentiate them from other possible q -analogues, we will refer to these transforms as *Mangontarum q -transforms*. Some properties and interesting formulae are derived and presented.

2. Mangontarum q -Transform of the First Kind

Let

$$\begin{aligned} A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < M \hat{e}_q\left(\frac{|t|}{k_j}\right), \right. \\ \left. t \in (-1)^j \times [0, \infty) \right\}. \end{aligned} \quad (21)$$

Hence, we have the following definition.

Definition 1. The Mangontarum q -transform of the first kind, denoted by E_q , is defined by

$$E_q\{f(t)\} = T_q(v) = v \int_0^{\infty} \hat{e}_q\left(-q\frac{t}{v}\right) f(t) d_q t, \quad (22)$$

over the set in (21), where $k_1 < v < k_2$ and $t \geq 0$.

Clearly,

$$\lim_{q \rightarrow 1} E_q\{f(t)\} = E\{f(t)\}. \quad (23)$$

This makes (22) a q -analogue of the Elzaki transform in (2). Also, from Definition 1, we can easily observe the *linearity relation*

$$E_q \{ \alpha f(t) + \beta g(t) \} = \alpha E_q \{ f(t) \} + \beta E_q \{ g(t) \}, \quad (24)$$

where $f(t), g(t) \in A$, and α, β are real numbers.

A q -analogue of the *gamma function*

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx, \quad (25)$$

for t a complex number, is given by

$$\Gamma_q(t) = \int_0^\infty x^{t-1} \hat{e}_q(-qx) d_q x, \quad (26)$$

for $t > 0$ and

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(n+1) = [n]_q!, \quad (27)$$

where n is a positive integer (see [10, 13, 14]).

Since (22) can be expressed as

$$T_q(v) = v^2 \int_0^\infty \hat{e}_q(-qt) f(vt) d_q t, \quad (28)$$

then by (28) and (27), we have the following:

$$E_q \{1\} = v^2, \quad E_q \{t\} = v^3, \quad E_q \{t^n\} = v^{n+2} [n]_q!. \quad (29)$$

Moreover, by (24) and (29), we have

$$\begin{aligned} E_q \{g(t)\} &= \sum_{n \geq 0} a_n v^{n+2} [n]_q!, & E_q \{e_q(at)\} &= \frac{v^2}{1-av}, \\ E_q \{\hat{e}_q(at)\} &= v^2 \sum_{n \geq 0} q^{\binom{n}{2}} (av)^n, \end{aligned} \quad (30)$$

where $g(t) = \sum_{n \geq 0} a_n t^n$.

Given the q -exponential function

$$e_q(it) = \sum_{n \geq 0} \frac{i^n t^n}{[n]_q!} = \sum_{n \geq 0} \frac{(-1)^n}{[2n]_q!} t^{2n} + i \sum_{n \geq 0} \frac{(-1)^n}{[2n+1]_q!} t^{2n+1}, \quad (31)$$

the q -sine and q -cosine functions can be defined as

$$\begin{aligned} \cos_q(z) &= \sum_{n \geq 0} \frac{(-1)^n}{[2n]_q!} z^{2n} = \frac{1}{2} (e_q(iz) + e_q(-iz)), \\ \sin_q(z) &= \sum_{n \geq 0} \frac{(-1)^n}{[2n+1]_q!} z^{2n+1} = \frac{1}{2i} (e_q(iz) - e_q(-iz)), \end{aligned} \quad (32)$$

where $i = \sqrt{-1}$. Now, by (28) and (27),

$$\begin{aligned} E_q \{\cos_q(t)\} &= v^2 \sum_{n \geq 0} \frac{(-1)^n v^{2n}}{[2n]_q!} \int_0^\infty t^{2n} \hat{e}_q(-qt) d_q t \\ &= v^2 \sum_{n \geq 0} \frac{(-1)^n v^{2n}}{[2n]_q!} \Gamma_q(2n+1) \\ &= v^2 \sum_{n \geq 0} (-v^2)^n \\ &= \frac{v^2}{1+v^2}. \end{aligned} \quad (33)$$

Similarly,

$$E_q \{\sin_q(t)\} = \frac{v^3}{1+v^2}. \quad (34)$$

Now, if we define the *hyperbolic q -sine and q -cosine functions* as

$$\cosh_q(t) = \frac{e_q(t) + e_q(-t)}{2}, \quad \sinh_q(t) = \frac{e_q(t) - e_q(-t)}{2}, \quad (35)$$

we have the next theorem which presents the transforms of the q -trigonometric functions.

Theorem 2 (transform of q -trigonometric functions). *The following identities hold:*

$$\begin{aligned} E_q \{\cos_q(at)\} &= \frac{v^2}{1+a^2 v^2}; & E_q \{\sin_q(at)\} &= \frac{av^3}{1+a^2 v^2}; \\ E_q \{\cosh_q(at)\} &= \frac{v^2}{1-a^2 v^2}; & E_q \{\sinh_q(at)\} &= \frac{av^3}{1-a^2 v^2}. \end{aligned} \quad (36)$$

Proof. $E_q \{\cos_q(at)\}$ and $E_q \{\sin_q(at)\}$ can be proved parallel to (33). For $E_q \{\cosh_q(at)\}$, we have

$$\begin{aligned} E_q \{\cosh_q(at)\} &= \frac{1}{2} [E_q \{e_q(at)\} + E_q \{e_q(-at)\}] \\ &= \frac{1}{2} \left[\frac{v^2}{1-av} + \frac{v^2}{1+av} \right] \\ &= \frac{v^2}{1-a^2 v^2}. \end{aligned} \quad (37)$$

$E_q \{\sinh_q(at)\}$ can be proven similarly. $\square$

The known q -product rule of differentiation is given by

$$D_q(f(t)g(t)) = D_q(f(t))g(t) + f(qt)D_q(g(t)). \quad (38)$$

Thus, we have

$$\begin{aligned} \int_0^\infty D_q(f(t)g(t)) d_q t &= \int_0^\infty D_q(f(t))g(t) d_q t \\ &\quad + \int_0^\infty f(qt)D_q(g(t)) d_q t. \end{aligned} \quad (39)$$

Let $f(t) = \hat{e}_q(-t/v)$ so that $f(qt) = \hat{e}_q(-q(t/v))$. Since

$$D_q\left(\hat{e}_q\left(-q\frac{t}{v}\right)\right) = -\frac{1}{v}\hat{e}_q\left(-q\frac{t}{v}\right), \quad (40)$$

then from (39), we have

$$\begin{aligned} \int_0^\infty \hat{e}_q\left(-q\frac{t}{v}\right) D_q(g(t)) d_q t &= -g(0) \\ &\quad + \frac{1}{v} \int_0^\infty \hat{e}_q\left(-q\frac{t}{v}\right) g(t) d_q t. \end{aligned} \quad (41)$$

Thus, from Definition 1,

$$E_q \{D_q(g(t))\} = -\nu g(0) + \frac{E_q \{g(t)\}}{\nu}. \quad (42)$$

Let $D_q^n(f(t))$ denote the n th q -derivative of the function $f(t) \in A$. Then, we have the following theorem.

Theorem 3 (transform of q -derivatives). *If $f(t)$, $D_q(f(t))$, $\dots$, $D_q^n(f(t)) \in A$, then*

$$E_q \{D_q^n(f(t))\} = \frac{E_q \{f(t)\}}{\nu^n} - \sum_{j=0}^{n-1} \nu^{j-n+2} D_q^j(f(0)). \quad (43)$$

Proof (by induction). The case when $n = 1$ is justified by (42). Now, suppose (43) holds for $n = k > 0$. That is

$$E_q \{D_q^k(f(t))\} = \frac{E_q \{f(t)\}}{\nu^k} - \sum_{j=0}^{k-1} \nu^{j-k+2} D_q^j(f(0)). \quad (44)$$

Let $g(t) = D_q^k(f(t))$ so that $D_q(g(t)) = D_q^{k+1}(f(t))$. Thus, by (42) and (43) we have

$$\begin{aligned} E_q \{D_q^{k+1}(f(t))\} &= E_q \{D_q(g(t))\} \\ &= \frac{E_q \{g(t)\}}{\nu} - \nu g(0) \\ &= \frac{E_q \{f(t)\}}{\nu^{k+1}} - \sum_{j=0}^{k-1} \nu^{j-k+1} D_q^j(f(0)) \\ &\quad - \nu D_q^k(f(0)) \\ &= \frac{E_q \{f(t)\}}{\nu^{k+1}} - \sum_{j=0}^k \nu^{j-k+1} D_q^j(f(0)). \end{aligned} \quad (45)$$

Hence, the proof is done. $\square$

Let $F_q(s)$ be the q -Laplace transform of the first kind. Replacing s with $1/\nu$ in (19) yields

$$F_q\left(\frac{1}{\nu}\right) = \int_0^\infty \widehat{e}_q\left(-q\frac{t}{\nu}\right) f(t) d_q t. \quad (46)$$

The next theorem is obtained by multiplying both sides of this equation by ν .

Theorem 4 (duality relation). *Let $F_q(s)$ be the q -Laplace transform of the first kind of $f(t) \in A$. Then the Mangontarum q -transform of the first kind satisfies the relation*

$$T_q(\nu) = \nu F_q\left(\frac{1}{\nu}\right). \quad (47)$$

Remark 5. Note that when $\nu = 1$ and $s = 1$, $T_q(1) = F_q(1)$.

For $f(t) = t^\alpha$ and $g(t) = t^{\beta-1}$ ($\alpha, \beta > 0$), the q -convolution of f and g is defined in [10] as

$$(f * g)(t) = \int_0^t f(\tau) g(t - q\tau) d_q \tau, \quad (48)$$

where

$$\begin{aligned} g(t - q\tau) &= \prod_{i=0}^{\beta-2} (t - q^i \tau) \\ &= \sum_{k=0}^{\beta-1} \left[\begin{matrix} \beta-1 \\ k \end{matrix} \right]_q (-1)^k q^{\binom{k}{2}} \tau^k \end{aligned} \quad (49)$$

is the q -binomial theorem. If $f(t) = \sum_{i \geq 0} a_i t^{i\alpha}$, then the convolution identity for the q -Laplace transform of the first kind in [10] is

$$L_q \{(f * g)(t)\} = L_q \{f(t)\} \cdot L_q \{g(t)\}. \quad (50)$$

Now, from (47) and (50),

$$\begin{aligned} E_q \{(f * g)(t)\} &= \nu L_q \{(f * g)(t)\} \\ &= \nu \left[F_q\left(\frac{1}{\nu}\right) \cdot G_q\left(\frac{1}{\nu}\right) \right] \\ &= \frac{1}{\nu} M_q(\nu) \cdot N_q(\nu). \end{aligned} \quad (51)$$

Thus, we have the following convolution identity for the Mangontarum q -transform of the first kind.

Theorem 6 (convolution identity). *If $f(t)$, $g(t) \in A$ with q -Laplace transforms of the first kind $F_q(s)$ and $G_q(s)$, respectively, and Mangontarum q -transforms of the first kind $M_q(\nu)$ and $N_q(\nu)$, respectively, then*

$$E_q \{(f * g)(t)\} = \frac{1}{\nu} M_q(\nu) \cdot N_q(\nu). \quad (52)$$

From [10], the q -derivative of the q -Laplace transform of the first kind is

$$L_q \{t^n f(t)\} = (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q s} \right)^n F_q(q^{-n} s). \quad (53)$$

Replacing s with $1/\nu$ and applying Theorem 6 yield

$$\begin{aligned} E_q \{t^n f(t)\} &= (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q \nu} \right)^n \nu \int_0^\infty \widehat{e}_q\left(-q\frac{t}{\nu q^n}\right) f(t) d_q t \\ &= (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q \nu} \right)^n T_q(\nu q^n). \end{aligned} \quad (54)$$

Hence, the following theorem is easily observed.

Theorem 7 (q -derivative of transforms). *For n a positive integer and $f(t) \in A$, one has*

$$E_q \{t^n f(t)\} = (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q \nu} \right)^n T_q(\nu q^n). \quad (55)$$

The next corollary follows directly from (55).

Corollary 8. For n a positive integer and $f(t) \in A$,

$$E_q \{e_q(at) f(t)\} = \sum_{n \geq 0} \frac{(-a)^n}{[n]_q!} q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n T_q(vq^n), \quad (56)$$

$$E_q \{e_q(at) t^n\} = (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \frac{v^2 q^n}{1 - avq^n}.$$

The Heaviside function $H(t - a)$ is defined by

$$H(t - a) = \begin{cases} 1, & \text{if } t \geq a, \\ 0, & \text{if } 0 \leq t \leq a. \end{cases} \quad (57)$$

Now, we have the following theorem.

Theorem 9. Let $H(t - a)$ be the Heaviside function. Then, one has

$$E_q \{H(t - a)\} = v \widehat{e}_q \left(-\frac{a}{v} \right). \quad (58)$$

Proof. From Definition 1 and (18),

$$\begin{aligned} E_q \{H(t - a)\} &= \int_0^\infty \widehat{e}_q \left(-q \frac{t}{v} \right) H(t - a) d_q t \\ &= \int_a^\infty \widehat{e}_q \left(-q \frac{t}{v} \right) d_q t \\ &= \int_0^\infty \widehat{e}_q \left(-q \frac{t}{v} \right) d_q t - \int_0^a \widehat{e}_q \left(-q \frac{t}{v} \right) d_q t. \end{aligned} \quad (59)$$

Since $E_q\{1\} = v^2$ and by (12),

$$\begin{aligned} E_q \{H(t - a)\} &= v - \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n}{2}} (q/v)^n}{[n]_q!} \int_0^a t^n d_q t \\ &= v - \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n+1}{2}}}{[n+1]_q! v^n} a^{n+1}. \end{aligned} \quad (60)$$

Further simplifications lead to (58). $\square$

3. Application of E_q

In this section, we will consider applications of the Mangontarum q -transform of the first kind to some q -differential equations. To obtain this, the following definition is essential.

Definition 10. Let $f(t) \in A$. If $E_q\{f(t)\} = f(v)$, we say that $f(t)$ is an inverse Mangontarum q -transform (of the first kind), or an inverse q -transform of the function $f(v)$, and we write

$$f(t) = E_q^{-1} \{f(s)\}. \quad (61)$$

Observe that linearity also holds for the inverse q -transform of the function $f(v)$. That is,

$$E_q^{-1} \{\alpha f(s) + \beta g(s)\} = \alpha E_q^{-1} \{f(s)\} + \beta E_q^{-1} \{g(s)\}, \quad (62)$$

where $\alpha, \beta \in A$.

The q -difference calculus (or quantum calculus) was first studied by Jackson [15], Carmichael [16], Mason [17], Adams [18], and Trjitzinsky [19] in the early 20th century. Some recent studies involving q -differential equations are those by Ahmad et al. [20], Yu and Wang [21], and Sithiwiratham et al. [22]. In the following examples, the effectiveness of the Mangontarum q -transform of the first kind in solving certain initial value problem involving ordinary q -differential equations is illustrated.

Example 11. Consider the first degree q -differential equation:

$$D_q(f(x)) + f(x) = 0, \quad (63)$$

where $f(x) \in A$ with initial condition $f(0) = 1$. Taking the Mangontarum q -transform (of the first kind) of both sides of this equation yields

$$E_q \{D_q(f(x))\} + E_q \{f(x)\} = 0. \quad (64)$$

Let $T_q(v) = E_q\{f(x)\}$. Applying Theorem 3, we have

$$E_q \{D_q(f(x))\} = \frac{T_q(v)}{v} - v. \quad (65)$$

Hence,

$$\begin{aligned} \frac{T_q(v)}{v} - v + 3T_q(v) &= 0, \\ T_q(v) \left(\frac{1 + 3v}{v} \right) &= v, \\ E_q \{f(x)\} &= \frac{v^2}{1 + 3v}. \end{aligned} \quad (66)$$

Using the inverse q -transform in (61) yields the solution

$$f(x) = E_q^{-1} \left\{ \frac{v^2}{1 + 3v} \right\} = e_q(-3x). \quad (67)$$

Example 12. Find the solution of the second order q -differential equation:

$$\frac{\partial_q y}{\partial_q x} - y = 0, \quad (68)$$

with initial condition $f(0) = 0$ and $f^1(0) = 5$.

Solution. If $T_q(v) = E_q\{y\}$, then by Theorem 3

$$E_q \left\{ \frac{\partial_q y}{\partial_q x} \right\} = \frac{T_q(v)}{v^2} - 5v. \quad (69)$$

Thus, taking the Mangontarum q -transform (of the first kind) of both sides of (68) gives us

$$\begin{aligned}\frac{T_q(v)}{v^2} - 5v - T_q(v) &= 0, \\ T_q(v) \left(\frac{1-v^2}{v^2} \right) &= 5v, \\ E_q\{y\} &= \frac{5v^3}{1-v^2}.\end{aligned}\quad (70)$$

From the inverse q -transform in (61), we have the solution

$$y = E_q^{-1} \left\{ \frac{5v^3}{1-v^2} \right\} = 5\sinh_q(x). \quad (71)$$

Example 13. Find the solution of the equation

$$y'' - 3y' + 2y = 1, \quad (72)$$

where $y'' = (\partial_q/\partial_q x)^2 y$ and $y' = \partial_q y/\partial_q x$ with $y(0) = 0$ and $y'(0) = 1$.

Solution. Let $T_q(v) = E_q\{y\}$. Since

$$\begin{aligned}E_q \left\{ \left(\frac{\partial_q}{\partial_q x} \right)^2 y \right\} &= \frac{T_q(v)}{v^2}, \\ E_q \left\{ \frac{\partial_q y}{\partial_q x} \right\} &= \frac{T_q(v)}{v},\end{aligned}\quad (73)$$

then the Mangontarum q -transform of (72) yields

$$\begin{aligned}\frac{T_q(v)}{v^2} - 3\frac{T_q(v)}{v^2} + 2T_q(v) &= v^2, \\ T_q(v) \left(\frac{1-3v+2v^2}{v^2} \right) &= v^2, \\ T_q(v) &= \frac{v^3}{2v^2-3v+1}, \\ E_q\{y\} &= \frac{-v^2}{1-v} + \frac{2v^2}{1-2v}.\end{aligned}\quad (74)$$

Applying (61) yields the solution

$$y = -e_q(x) + 2e_q(2x). \quad (75)$$

Example 14. Find the solution of

$$y'' + y = 6\sin_q(2x) \quad (76)$$

when $y(0) = 3$ and $y''(0) = 1$.

Solution. Let $T_q(v) = E_q\{y\}$. To solve (76), we make use of the following results:

$$\begin{aligned}E_q \left\{ \left(\frac{\partial_q}{\partial_q x} \right)^2 y \right\} &= \frac{T_q(v)}{v^2} - 3 - v, \\ E_q\{\sin_q(2x)\} &= \frac{2v^3}{1+4v^2}.\end{aligned}\quad (77)$$

Hence, the q -transform of (76) is

$$\begin{aligned}E_q\{y''\} + E_q\{y\} - E_q\{6\sin_q(2x)\} \\ = \frac{T_q(v)}{v^2} - 3 - v + T_q(v) - \frac{12v^3}{1+4v^2}.\end{aligned}\quad (78)$$

Further simplifications yield

$$T_q(v) = \frac{3v^2}{1+v^2} + \frac{5v^3}{1+v^2} - \frac{4v^3}{1+4v^2}. \quad (79)$$

The inverse q -transform in (61) gives the solution

$$y = 3\cos_q(x) + 5\sin_q(x) - 2\sin_q(2x). \quad (80)$$

4. Mangontarum q -Transform of the Second Kind

Since there can be more than one q -analogue of any classical expression, we can define another q -analogue for the Elzaki transform. Let

$$\begin{aligned}\bar{A} = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me_q \left(\frac{|t|}{k_j} \right), \right. \\ \left. t \in (-1)^j \times [0, \infty) \right\}.\end{aligned}\quad (81)$$

Then, we have the following definition.

Definition 15. The Mangontarum q -transform of the second kind, denoted by $\tilde{E}_q$, is defined by

$$\tilde{E}_q\{f(t)\} = \tilde{T}_q(v) = v \int_0^\infty e_q\left(-\frac{t}{v}\right) f(t) d_q t, \quad (82)$$

over the set $\bar{A}$, where $k_1 < v < k_2$ and $t \geq 0$.

Clearly, $\lim_{q \rightarrow 1} \tilde{E}_q\{f(t)\} = E\{f(t)\}$. Moreover, it is easily observed that

$$\tilde{E}_q\{\alpha f(t) + \beta g(t)\} = \alpha \tilde{E}_q\{f(t)\} + \beta \tilde{E}_q\{g(t)\}, \quad (83)$$

where $f(t), g(t) \in \bar{A}$, and α, β are real numbers.

Given the q -gamma function of the second kind

$$\gamma_q(t) = \int_0^\infty x^{t-1} e_q(-x) d_q x, \quad (84)$$

for $t > 0$, we get

$$\begin{aligned}\gamma_q(1) &= 1, \quad \gamma_q(t+1) = q^{-t} [t]_q \gamma_q(t), \\ \gamma_q(n) &= q^{-\binom{n}{2}} \Gamma_q(n),\end{aligned}\quad (85)$$

for n a positive integer. Note that (82) may be expressed as

$$\tilde{T}_q(v) = v^2 \int_0^\infty e_q(-t) f(vt) d_q t. \quad (86)$$

Hence, from (86), we have the following results:

$$\begin{aligned}\tilde{E}_q\{1\} &= v^2, & \tilde{E}_q\{t\} &= \frac{v^3}{q}, \\ \tilde{E}_q\{t^n\} &= \frac{v^{n+2}}{q^{\binom{n+1}{2}}}[n]_q!, \\ \tilde{E}_q\left\{\sum_{n \geq 0} a_n t^n\right\} &= \sum_{n \geq 0} a_n \frac{v^{n+2}}{q^{\binom{n+1}{2}}}[n]_q!, \\ \tilde{E}_q\{e_q(at)\} &= \sum_{n \geq 0} a_n \frac{v^{n+2}}{q^{\binom{n+1}{2}}}, \\ \tilde{E}_q\{\hat{e}_q(at)\} &= \frac{v^2 q}{q - av}.\end{aligned}\quad (87)$$

From (12),

$$\hat{e}_q(it) = \sum_{n \geq 0} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_q!} t^{2n} + i \sum_{n \geq 0} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_q!} t^{2n+1}. \quad (88)$$

Hence, new q -analogues of the sine and cosine functions can be defined, respectively, as (see [10])

$$\begin{aligned}\cos'_q(at) &= \sum_{n \geq 0} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_q!} (at)^{2n} \\ &= \frac{1}{2} (\hat{e}_q(iat) + \hat{e}_q(-iat)); \\ \sin'_q(at) &= \sum_{n \geq 0} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_q!} (at)^{2n+1} \\ &= \frac{1}{2i} (\hat{e}_q(iat) - \hat{e}_q(-iat)),\end{aligned}\quad (89)$$

and the new q -hyperbolic sine and cosine functions as

$$\begin{aligned}\cosh'_q(at) &= \frac{1}{2} (\hat{e}_q(at) + \hat{e}_q(-at)); \\ \sinh'_q(at) &= \frac{1}{2} (\hat{e}_q(at) - \hat{e}_q(-at)).\end{aligned}\quad (90)$$

By application of $\tilde{E}_q\{t^n\} = (v^{n+2}/q^{\binom{n+1}{2}})[n]_q!$, it is easy to obtain the next theorem.

Theorem 16 (transform of q -trigonometric functions). *The following statements are true:*

$$\begin{aligned}\tilde{E}_q\{\cos'_q(at)\} &= \frac{v^2 q^2}{q^2 + a^2 v^2}; \\ \tilde{E}_q\{\sin'_q(at)\} &= \frac{aqv^3}{q^2 + a^2 v^2};\end{aligned}$$

$$\begin{aligned}\tilde{E}_q\{\cosh'_q(at)\} &= \frac{v^2 q^2}{q^2 - a^2 v^2}; \\ \tilde{E}_q\{\sinh'_q(at)\} &= \frac{aqv^3}{q^2 - a^2 v^2}.\end{aligned}\quad (91)$$

From (20),

$$\tilde{F}_q(s) = \int_0^\infty e_q(-st) f(t) d_q t. \quad (92)$$

Replacing s with $1/v$ and multiplying both sides with v yield

$$v\tilde{F}_q\left(\frac{1}{v}\right) = \tilde{T}_q(v). \quad (93)$$

Thus, we have the duality relation between the q -Laplace transform of the second kind and the Mangontarum q -transform of the second kind as stated in the next theorem.

Theorem 17 (duality relation). *Let $\tilde{F}_q(s)$ be the q -Laplace transform of the second kind of $f(t) \in A$. Then the Mangontarum q -transform of the second kind satisfies the relation*

$$\tilde{T}_q(v) = v\tilde{F}_q\left(\frac{1}{v}\right). \quad (94)$$

From [10], for any positive integer n ,

$$\begin{aligned}\tilde{L}_q\{f^{(n)}(t)\} &= s^n q^{-\binom{n+1}{2}} \tilde{F}_q(q^{-n}s) \\ &\quad - \sum_{i=0}^{n-1} s^{n-1-i} q^{-\binom{n-i}{2}} f^{(i)}(0),\end{aligned}\quad (95)$$

where $\tilde{F}_q(q^{-n}s) = \int_0^\infty e_q(-q^{-n}st) f(t) d_q t$, $f^{(n)}(t) = (\partial_q/\partial_q t)^n f(t)$. For $f(t) \in \bar{A}$, let $\tilde{G}_q(s) = \tilde{L}_q\{f^{(n)}(t)\}$ and $\tilde{V}_q(v) = \tilde{E}_q\{f^{(n)}(t)\}$. Then, by Theorem 17,

$$\begin{aligned}\tilde{V}_q(v) &= v\tilde{G}_q\left(\frac{1}{v}\right) \\ &= \frac{1}{v^n} q^{-\binom{n+1}{2}} v \int_0^\infty e_q\left(-\frac{t}{q^n v}\right) \\ &\quad - \sum_{i=0}^{n-1} v^{i-n+2} q^{-\binom{n-i}{2}} f^{(i)}(0) \\ &= q^{-\binom{n+1}{2}} \frac{\tilde{T}_q(q^n v)}{v^n} - \sum_{i=0}^{n-1} v^{i-n+2} q^{-\binom{n-i}{2}} D_q^i(f(0)).\end{aligned}\quad (96)$$

Hence, the following theorem is obtained.

Theorem 18 (transform of q -derivatives). *If $f(t)$, $D_q(f(t))$, $\dots$, $D_q^n(f(t)) \in \bar{A}$, then*

$$\begin{aligned}\tilde{E}_q\{D_q^n f(t)\} &= q^{-\binom{n+1}{2}} \frac{\tilde{T}_q(q^n v)}{v^n} \\ &\quad - \sum_{i=0}^{n-1} v^{i-n+2} q^{-\binom{n-i}{2}} D_q^i(f(0)).\end{aligned}\quad (97)$$

The q -derivative of the q -Laplace transform of the second kind [10] is

$$\tilde{L}_q \{t^n f(t)\} = (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q s} \right)^n \tilde{F}_q(s). \quad (98)$$

Replacing s with $1/v$ and applying Theorem 6 yield

$$\begin{aligned} \tilde{E}_q \{t^n f(t)\} &= (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \nu \int_0^\infty e_q \left(-q \frac{t}{\nu} \right) f(t) d_q t \\ &= (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \tilde{T}_q(\nu). \end{aligned} \quad (99)$$

Thus, the next theorem holds.

Theorem 19 (q -derivative of transforms). *For n a positive integer and $f(t) \in \bar{A}$, one has*

$$\tilde{E}_q \{t^n f(t)\} = (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \tilde{T}_q(\nu). \quad (100)$$

Moreover, we have the following corollary.

Corollary 20. *For n a positive integer and $f(t) \in \bar{A}$,*

$$\begin{aligned} \tilde{E}_q \{\hat{e}_q(at) f(t)\} &= \sum_{n \geq 0} \frac{(-a)^n}{[n]_q!} q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \tilde{T}_q(\nu), \\ \tilde{E}_q \{\hat{e}_q(at) t^n\} &= (-1)^n q^{\binom{n}{2}} \left(\frac{\partial_q}{\partial_q v} \right)^n \frac{v^2 q}{q - av}. \end{aligned} \quad (101)$$

Much is yet to be discovered regarding the Mangontarum q -transforms. The possibility of deriving more properties for these q -transforms is an interesting recommendation. One may read [1–3] and the references therein for guidance regarding this matter. Also, note that there were no applications shown for the Mangontarum q -transform of the second kind in this paper. Researchers are encouraged to further investigate other applications of these q -transforms, especially the second kind.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to thank the academic editor for his invaluable work during the editorial workflow and the referees for their corrections and suggestions which helped improve the clarity of this paper. This research paper is supported by the Mindanao State University-Main Campus, Marawi City, Philippines.

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