Research Article

A Stochastic Dynamic Programming Approach Based on Bounded Rationality and Application to Dynamic Portfolio Choice

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Dynamic portfolio choice is an important problem in finance, but the optimal strategy analysis is difficult when considering multiple stochastic volatility variables such as the stock price, interest rate, and income. Besides, recent research in experimental economics indicates that the agent shows limited attention, considering only the variables with high fluctuations but ignoring those with small ones. By extending the sparse max method, we propose an approach to solve dynamic programming problem with small stochastic volatility and the agent's bounded rationality. This approach considers the agent's behavioral factors and avoids effectively the "Curse of Dimensionality" in a dynamic programming problem with more than a few state variables. We then apply it to Merton dynamic portfolio choice model with stochastic volatility and get a tractable solution. Finally, the numerical analysis shows that the bounded rational agent may pay no attention to the varying equity premium and interest rate with small variance.

1. Introduction

In reality, how to choose an asset's portfolio of consumption and investment is one of the most important decisions for many people. In modern portfolio choice field, Merton [1, 2] provides a general framework for understanding the portfolio demand of long-term investors when investment opportunities change over time. In a classical Merton model [1, 2], however, the riskless interest rate, the risky mean rate of return, and the volatility coefficient are usually assumed to be constant. These assumptions are lack of realism, particularly over long time intervals. A large volume of empirical researches in financial market which indicates the assumption that these variables are stochastic volatile and follow a certain stochastic process (e.g., Ornstein-Uhlenbeck process) is more realistic [3, 4]. But when introducing these stochastic variables into the Merton-style portfolio choice model, the problem becomes increasingly complicated and formidable to solve. Also, this will lead to the "Curse of Dimensionality." Quite a lot of approaches have been developed to deal with this kind of problems, such as martingale methods [5–8] and various approximate numerical algorithms [9–12]. However, these methods have more restrictive assumptions and are too

complex to get a tractable solution of strong explanations. Based on the control of small noise, Judd and Guu [13] proposed a method to solve dynamic programming problems with stochastic disturbance. He makes the simplifying assumption that uncertainty is small and obtains the first- and high-order solutions of complicated dynamic programming model. This method provides a quite suitable solution for dynamic portfolio choice model with stochastic volatility.

On the other hand, a growing body of empirical studies indicate that the agent considers only the variables with high fluctuations but ignores those with small ones [14–16]. Bordalo et al. [17] showed that the agent rationally chooses to be inattentive to news. Kőszegi and Szeidl [18] analyzed the monetary policy and found out that when price is changed, the decision makers are usually unaware of it. There are also many literatures showing that the agent pays attention to salient factors. Sims [19] uses two empirical strategies to analyze how individuals optimize fully with respect to the incentives created by tax policies and shows that tax salience affects agents' behavioral response. Peng and Xiong [20] study the allocation of investors' attention among different information. They find out that investors with limited attention will focus on macroeconomic and industry information rather

than that of a specific firm. Seasholes and Wu [21] demonstrate that attention-grabbing events will attract investors' attention. In their model, they regard them as the proxy variables and their results empirically indicate that these events have a significant impact on the allocation of investor's attention. Maćkowiak and Wiederholt [22] show that decision makers' attention is usually drawn to salient payoffs.

In recent years, Gabaix [23] provides a sparse max operator to model dynamic programming with bounded rationality. In the sparse max, the agent pays less or no attention to some features the fluctuations of which are smaller than some thresholds, and he tries to strike a good balance between the utility loss of inattention and the cognitive cost which can be regarded as the loss for taking time to think about the decisions rather than to enjoy oneself. The sparse max seems more realistic than traditional economic models since it has a very robust psychological foundation. Also, it can deal with problems of maximization with constraints easily and get a tractable solution in a parsimonious way.

However, Gabaix [23] only studies the dynamic programming in a stationary environment without the stochastic volatility terms. But the financial market is strewn with numerous stochastic dynamic programming problems, and these problems are hard to solve due to multitudinous state variables. To address this issue, we extend the sparse max operator and develop a stochastic version of Gabaix's method. The distinctive feature of this method is that it considers the agent's behavioral factors (limited attention) and can effectively preclude the "Curse of Dimensionality" for multiple variables. To verify the validity and practicability of our model, we consider the Merton dynamic portfolio choice problem with stochastic volatility variables (e.g., [24, 25]) and get a tractable solution.

The remainder of this paper is organized as follows. Section 2 presents the sparse dynamic programming method proposed by Gabaix [23]. Section 3 extends this model and gives a general principle for solving continuous-time dynamic programming with stochastic variables. In Section 4, we apply our method to Merton dynamic portfolio choice. Finally, we discuss some implications of our findings and suggest topics for future research in Section 5.

2. The Sparse Max Operator without Constraints

We mainly introduce the sparse max operator proposed by Gabaix [23] in this section. In the traditional version, the agent faces a maximization problem:

$$\begin{aligned} \max_a \quad & u(a, \mathbf{y}) \\ \text{subject to} \quad & b(a, \mathbf{y}) \geq 0, \end{aligned} \quad (1)$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)$, u is a utility function and b is a constraint. Variable a and function b have arbitrary dimensions. For any optimal decision, in principle, thousands of considerations are relevant to the agent. Since it would be too burdensome to take all of these variables into account, the agent is used to discarding most of them. At the same time, his attention is allocated purposefully to important variables.

Hence, the agent might sensibly pick a "sparse" representation of the variables; namely, choose the attention vector $\mathbf{m} = (m_1, m_2, \dots, m_n)$ to replace variable y_i with $y_i^s = m_i y_i$ $i \in (1, 2, \dots, n)$, where the superscript s of y_i^s represents sparse. The optimal attention vector is obtained by weighing the utility losses for imperfect inattention against the cost savings without thinking too much.

The utility losses from imperfect inattention can be expressed as follows [23]:

$$E[v(\mathbf{m}) - v(\mathbf{1})] = -\frac{1}{2} \sum_{i,j} (m_i - 1) \Lambda_{ij} (m_j - 1) + o(\|\mathbf{y}\|^2), \quad (2)$$

where $v(\mathbf{m}) := u(a(\mathbf{y}^s(\mathbf{m})), \mathbf{y})$ is the utility for a sparse agent, $\mathbf{y}^s(\mathbf{m}) = (y_1^s, y_2^s, \dots, y_n^s)$, $\mathbf{1} := (1, 1, \dots, 1)^T$, and $v(\mathbf{1})$ is the utility when the agent is fully attentive. $o(\|\mathbf{y}\|^2)$ denotes the second-order infinitesimal of $\mathbf{y}$. $\Lambda_{ij} = -\sigma_{ij} a_{y_i}^2 u_{aa}$, where $\sigma_{ij} = \text{cov}(y_i, y_j)$, $E(y_i) = 0$, σ_i is the standard deviation of y_i , and $a_{y_i} = -u_{aa}/u_{ay_i}$ which indicates by how much a change y_i should change the action for traditional agent. u_{aa} is the second derivative of u with respect to a . All derivatives above are evaluated at $\mathbf{y} = \mathbf{0}$ and the default action $a^d = \text{argmax}_a u(a, \mathbf{0})$.

Gabaix [23] assumes the cognitive cost is $c(m_i) = \kappa |m_i|^\beta$, where $\beta \geq 0$ and parameter $\kappa \geq 0$ is a penalty for lack of sparsity. If $\kappa = 0$, the agent will be a traditional, rational agent.

Based on above analysis, Gabaix [23] defines the sparse max operator as follows.

Definition 1 (see [23] Sparse max operator without constraints). The sparse max defined by the following procedure.

Step 1. Choose the attention vector $\mathbf{m}^*$

$$\mathbf{m}^* = \arg \min_{\mathbf{m}} \frac{1}{2} \sum_{i,j=1}^n (1 - m_i) \Lambda_{ij} (1 - m_j) + \kappa \sum_{i=1}^n |m_i|^\beta. \quad (3)$$

Define $y_i^s = m_i^* y_i$ as the sparse representation of y_i .

Step 2. Choose the action

$$a^s = \arg \max_a u(a, \mathbf{y}^s), \quad (4)$$

and set the resulting utility to be $u^s = u(a^s, \mathbf{y})$.

Suppose $\mathbf{y}$ is one-dimensional vector; formula (3) can be transformed into $m^* = \min_m (1/2)(m-1)^2 \sigma^2 + \kappa |m|$. Gabaix [23] defines a function A to represent the optimal attention vector, namely, $m^* = A_\beta(\sigma^2/\kappa) = \inf[\arg \min_m (1/2)(m-1)^2(\sigma^2/\kappa) + |m|^\beta]$ and points out when $\beta = 1$, the function $A_\beta(\sigma^2/\kappa)$ satisfies the sparsity and continuity. When $\beta = 1$ and $\kappa = 1$, we have $A_1(\sigma^2) = \max(1 - 1/\sigma^2, 0)$ as shown in Figure 1 [23].

From Figure 1 we know that the agent will not consider the variable when $0 \leq \sigma^2 \leq \kappa$ ($\kappa = 1$).

When the vector $\mathbf{y}$ includes more than one variable and these variables perceived by the agent are uncorrelated, we have $m_i^* = A_1(\sigma_i^2 a_{y_i}^2 |u_{aa}|/\kappa)$ through formula (3). To

FIGURE 1: The attention function A_1 .

analyze the agent's inattention expediently, Gabaix [23] defines the truncation function $\tau(s, p) = s \max(1 - p^2/s^2, 0)$, so we have $m_i^* = \tau(1, \kappa\sigma_a/a_{y_i}\sigma_i)$. Truncation function has more intuitive economic implications: a one-standard-deviation change of the variable y_i makes the agent change his action by $a_{y_i}(\sigma_i/\sigma_a)$. When $a_{y_i}(\sigma_i/\sigma_a)$ is small and satisfies $|a_{y_i}(\sigma_i/\sigma_a)| \leq \kappa$, the agent will not consider this factor. Figure 2 shows the truncation function $\tau(s, p) = s \max(1 - p^2/s^2, 0)$ [23].

From Figure 2, we know that the agent who seeks "sparsity" should sensibly drop relatively unimportant features. In addition, if the features are larger than that cutoff, they are still dampened: in Figure 2, $\tau(s, p)$ is below the 45 degree line (for positive s , in general, $|\tau(s, p)| < |s|$).

Based on the analysis above, we can use the truncation function to represent the sparse agent's optimal action.

Remark 2 (see [23]). If rational optimal action is

$$a^r(\mathbf{y}) = a^d + \sum_i a_{y_i} y_i + o(\|\mathbf{y}\|^2) \quad (5)$$

(r represents the rationality),

which is obtained by the Taylor expansion around the default action a^d , then the sparse agent's optimal action is

$$a^s(\mathbf{y}) = a^d + \sum_i \tau\left(a_{y_i}, \frac{\kappa\sigma_a}{\sigma_i}\right) y_i + o(\|\mathbf{y}\|^2), \quad (6)$$

where σ_a is the standard deviation of a .

3. A Stochastic Dynamic Programming Approach Based on Sparse Max Operator

In order to effectively deal with stochastic dynamic programming in finance in this section, we extend Gabaix [23] sparse max operator and propose a bounded rational stochastic dynamic programming model in continuous time.

The general model of stochastic dynamic programming in continuous time is

$$\begin{aligned} \max \int_0^\infty e^{-\rho t} u(a, \mathbf{x}, \mathbf{y}) dt \\ d\mathbf{x} = g(\mathbf{x}(t), \mathbf{y}(t), a(t)) dt + \sigma(\mathbf{x}(t), a(t)) dZ_{\mathbf{x}}(t), \quad (7) \\ d\mathbf{y} = h(\mathbf{x}(t), \mathbf{y}(t)) dt + \sigma(\mathbf{y}(t)) dB_{\mathbf{y}}(t), \end{aligned}$$

FIGURE 2: The truncation function $\tau(s, p)$ with $p = 1$.

where ρ denotes the discount factor, u is the utility function, a is the decision variable which has an arbitrary dimension, the vector $\mathbf{x}$ represents important factors which are always considered by the agent, and the vector $\mathbf{y}$ defined in Section 2 represents factors that that may not be considered by the sparse agent. $g(\mathbf{x}, \mathbf{y}, a)$, $h(\mathbf{x}, \mathbf{y})$ are the state transition function of $\mathbf{x}$ and $\mathbf{y}$, respectively. And $\sigma(\mathbf{x}, a)$, $\sigma(\mathbf{y})$ represent the stochastic volatility of $\mathbf{x}$ and $\mathbf{y}$, respectively. $dZ_{\mathbf{x}}$, $dB_{\mathbf{y}}$ are independent standard Brownian motions; namely $dZ_{\mathbf{x}} dB_{\mathbf{y}} = 0$. We define the value function as $V(\mathbf{x}, \mathbf{y}) = \int_0^\infty e^{-\rho t} u(a, \mathbf{x}, \mathbf{y}) dt$.

Assumption 3. The utility function u and value function V are n -order continuously differentiable (C^n , $n \geq 3$).

Assumption 4. All state variables are stochastic and they are independent of each other; stochastic volatility of $\mathbf{x}$ is a function of $\mathbf{x}$ and a while stochastic volatility of $\mathbf{y}$ is uncorrelated with a .

Assumption 5. $\mathbf{x}$ is one dimensional; that is, only one variable would be always considered by the agent and other variables may not be considered by the agent.

Assumption 6. According to Judd and Guu [13], we assume the variance of each component of vector $\mathbf{y}$ is small and independent of one another.

To facilitate analysis, we use x to replace $\mathbf{x}$, denote the stochastic differential equation of y_i by $dy_i = h^i(x(t), y_i(t))dt + \sigma^i(y_i(t))dB_{y_i}(t)$, $i \in (1, 2, \dots, n)$, and use the notation $D_x[f] = \partial_x[f] + a_x \partial_a[f]$ as the total derivative with respect to x (i.e., the full impact of a change in x , including the impact it has on a change in the action a).

Based on Remark 2 in Section 2, we have the following proposition.

Proposition 7. The optimal action in bounded rationality model (7) is

$$a^s(x, \mathbf{y}) = a^d(x) + \sum_i \tau\left(a_{y_i}, \frac{\kappa\sigma_a}{\sigma_i}\right) y_i + o(\|\mathbf{y}\|^2), \quad (8)$$

where σ_a is the standard deviation of a .

Proof. See the appendix. $\square$

From Proposition 7, we know that, in order to derive the optimal action a^s , we should get the default action a^d which is related to x and a_{y_i} . The detail process of solving them is described as the following steps, which contain the main results of our method.

Step 1. Solve default action a^d .

By substituting $\mathbf{y} = 0$ into the basic model (7), we get the default model:

$$\max \int_0^\infty e^{-\rho t} u(a, x, \mathbf{0}) dt \quad (9)$$

$$dx = g(x(t), \mathbf{0}, a(t)) dt + \sigma(x(t), a(t)) dZ_x(t).$$

This is a general dynamic programming model in continuous time and the state variable is one dimension, so we can get the optimal default action a^d and the value function $V(x, \mathbf{0}) = \int_0^\infty e^{-\rho t} u(a, x, \mathbf{0}) dt$ easily [26].

Step 2. Solve a_{y_i} .

The following Proposition 8 and its proof in the appendix show the result and the process of obtaining a_{y_i} .

Proposition 8. *The impact of a change in y_i on the value function is*

$$\begin{aligned} V_{xy_i} = & \left(D_x [u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x] \right. \\ & + \sum_{i=1}^n h_{xy_i}^i(x, y_i, a) V_{y_i} + \frac{1}{2} \sigma_x^2(x, a) f(x) V_{y_i} \\ & \left. + \frac{1}{2} \sigma^2(x, a) f_x(x) V_{y_i} \right) \\ & \times \left(\rho - \sum_{i=1}^n h_{y_i}^i(x, y_i, a) - \frac{1}{2} \sigma^2(x, a) f(x) \right)^{-1}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} V_{y_i} = & (u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x) \\ & \times \left(\rho - \sum_{i=1}^n h_{y_i}^i(x, y_i, a) - \frac{1}{2} \sigma^2(x, a) f(x) \right)^{-1}, \quad (11) \\ f(x) = & \frac{V_{xx}}{V}. \end{aligned}$$

By implicit function theorem, the impact of a change in y_i on the optimal action is $a_{y_i} = -\phi_{y_i}/\phi_a$, where

$$\begin{aligned} \phi_a = & u_{aa} + g_{aa}(x, \mathbf{y}, a) V_x + \frac{1}{2} \sigma_{aa}^2(x, a) D_x [V_x], \\ \phi_{y_i} = & u_{ay_i} + g_{ay_i}(x, \mathbf{y}, a) V_x + g_a(x, \mathbf{y}, a) V_{xy_i} \\ & + \frac{1}{2} \sigma_a^2(x, a) D_x [V_{xy_i}]. \end{aligned} \quad (12)$$

Proof. See the appendix. $\square$

Now we can get the optimal action based on the two steps above. By the analysis of Proposition 7, we can see that $\tau(a_{y_i}, \kappa \sigma_a / \sigma_i)$ represents the impact of variable y_i on the action a^s . When a_{y_i} is smaller than $\kappa \sigma_a / \sigma_i$, $\tau(a_{y_i}, \kappa \sigma_a / \sigma_i) = 0$ which means the agent will discard this factor.

4. Application: Dynamic Portfolio Choice

4.1. Merton Portfolio Problem with Stochastic Volatility. In this section, we consider a Merton dynamic portfolio choice problem with stochastic volatility in continuous time [24]. In the traditional version of Merton model [1], the agent's optimal problem is

$$\begin{aligned} \max \quad & E \int_0^\infty e^{-\rho t} u(c(t)) dt, \\ \text{s.t.} \quad & dw(t) = w(t)(r + (b - r)\theta(t)) dt \\ & + \theta(t) \sigma w(t) dW(t) - c(t) dt, \end{aligned} \quad (13)$$

where ρ is the discount factor, u is the utility function, $w(t)$ is the wealth at time t , σ is the standard deviation of $w(t)$, and $c(t)$ is the consumption at time t . The investment control $\theta(t)$ at time t is the fraction of the wealth invested in the risky asset, so $(1 - \theta(t))$ is the fraction of the wealth invested on the riskless asset. r is the riskless interest rate and b is the risky mean rate of return; dW follows standard Brownian motion. We assume the utility function $u(c) = c(t)^{1-\gamma}/(1-\gamma)$, where γ ($0 < \gamma < 1$) is the parameter of risk preference. The goal is to choose consumption $c(t)$ and investment $\theta(t)$ control processes to maximize long-term utility.

In model (13), the riskless interest rate r and the risky mean rate of return b are assumed to be constant [1]. However, this assumption is unrealistic, particularly over long time intervals [27, 28]. Instead, now we assume that these two variables are stochastic and satisfying $r(t) = \bar{r} + \hat{r}(t)$, $b(t) = \bar{b} + \hat{b}(t)$, where $\bar{r}$, $\bar{b}$ represent the long mean of the riskless interest rate and the risky rate of return, respectively, and $\hat{r}(t)$, $\hat{b}(t)$ are their volatile part. $r(t)$ and $b(t)$ depend on some "economic factor" $\eta(t)$ [24]; namely,

$$\begin{aligned} r(t) &= R(\eta(t)), \\ b(t) &= M(\eta(t)). \end{aligned} \quad (14)$$

And $\eta(t)$ satisfies the stochastic differential formula [24]:

$$d\eta = g(\eta(t)) dt + \tilde{\sigma} \left[\lambda dW(t) + (1 - \lambda^2)^{1/2} d\tilde{W}(t) \right], \quad (15)$$

where $\tilde{W}(t)$ and $W(t)$ are independent standard Brownian motions. The parameter λ with $|\lambda| < 1$ allows a correlation between the Brownian motion $W(t)$ driving the short rate and its volatility. $\tilde{\sigma}$ is the standard deviation of η .

The budget equation of model (13) now becomes

$$\begin{aligned} dw = & w(t)(r(t) + (b(t) - r(t))\theta(t)) dt \\ & + \theta(t) \sigma w(t) dW(t) - c(t) dt. \end{aligned} \quad (16)$$

We further assume that the stochastic differential equation of $r(t) = \eta(t)$ follows an Ornstein-Uhlenbeck process [25, 29]:

$$dr = -\zeta_r (\bar{r} - r(t)) dt + \sigma_r \left[\lambda_r dW(t) + (1 - \lambda_r^2)^{1/2} dR(t) \right], \quad (17)$$

where ζ_r is the degree of mean reversion in expected excess returns, σ_r is the standard deviation of $r(t)$, $R(t)$ is a Brownian motion and independent of $W(t)$, and λ_r is the coefficient of w and r . To simplify our model, we assume $\lambda_r = 0$ [30]. By substituting $r(t) = \bar{r} + \hat{r}(t)$ into (17), we obtain $d(\bar{r} + \hat{r}(t)) = -\zeta_r \hat{r}(t) dt + \sigma_r dR(t)$. Since $d\bar{r} = 0$, it follows that $d\hat{r} = -\zeta_r \hat{r}(t) dt + \sigma_r dR(t)$.

We assume that $b(t)$ follows an Ornstein-Uhlenbeck process too, so we obtain $d\hat{b} = -\zeta_b \hat{b}(t) dt + \sigma_b dB(t)$ where ζ_b has the same meaning with ζ_r . σ_b is the standard deviation of $b(t)$, and we assume σ_r and σ_b are small [13]. $B(t)$ is a Brownian motion and independent of $R(t)$ and $W(t)$. Then we get the following model:

$$\max E \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt \quad (18)$$

$$\begin{aligned} \text{s.t. } dw &= w(t) ((r(t) + \theta(t)(b(t) - r(t)))) dt \\ &+ \theta(t) \sigma w(t) dW(t) - c(t) dt, \\ r(t) &= \bar{r} + \hat{r}(t), \\ b(t) &= \bar{b} + \hat{b}(t), \end{aligned} \quad (19)$$

$$d\hat{r} = -\zeta_r \hat{r}(t) dt + \sigma_r dR(t),$$

$$d\hat{b} = -\zeta_b \hat{b}(t) dt + \sigma_b dB(t).$$

From Section 3, we know that, for the bounded rational agent, the optimal consumption c^s and the optimal fraction of wealth allocated to risky market θ^s in model (18) can be expressed as

$$\begin{aligned} c^s &= c^d + \tau \left(c_{\hat{r}}, \frac{\kappa \sigma_c}{\sigma_r} \right) \hat{r} + \tau \left(c_{\hat{b}}, \frac{\kappa \sigma_c}{\sigma_b} \right) \hat{b}, \\ \theta^s &= \theta^d + \tau \left(\theta_{\hat{r}}, \frac{\kappa \sigma_\theta}{\sigma_r} \right) \hat{r} + \tau \left(\theta_{\hat{b}}, \frac{\kappa \sigma_\theta}{\sigma_b} \right) \hat{b}, \end{aligned} \quad (20)$$

where c^d and θ^d are the default actions when $\hat{r}(t) = 0$, $\hat{b}(t) = 0$. σ_c , σ_θ are the standard deviation of c and θ , respectively. $c_{\hat{r}}$, $c_{\hat{b}}$, $\theta_{\hat{r}}$, and $\theta_{\hat{b}}$ are the impact of $\hat{r}$ and $\hat{b}$ on c and θ , respectively. Next we will give the process of solving them using the approach described in Section 3.

Step 1. Solve the default decision c^d and θ^d .

By substituting $\hat{r}(t) = 0$, $\hat{b}(t) = 0$ into the basic model (18), we get the default model:

$$\begin{aligned} \max E \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt \\ dw &= w(t) (\bar{r} + \theta(t)(\bar{b} - \bar{r})) dt \\ &+ \theta(t) \sigma w(t) dW(t) - c(t) dt. \end{aligned} \quad (21)$$

This is the same Merton model as the model in (13); many scholars have solved this problem [1, 2]. We define the value function to be $V(w) = \int_0^\infty e^{-\rho t} (c(t)^{1-\gamma}/(1-\gamma)) dt$; then we have [2]

$$c^d = w(\mathfrak{R}(1-\gamma))^{-1/\gamma}, \quad \theta^d = \frac{\bar{b} - \bar{r}}{\gamma \sigma^2}, \quad (22)$$

$$V(w) = \mathfrak{R} \frac{w^{1-\gamma}}{1-\gamma},$$

where $\mathfrak{R} = [(\rho - \bar{r}(1-\gamma) - (\bar{b} - \bar{r})^2(1-\gamma)/2\sigma^2\gamma)/\gamma]^{-\gamma}/(1-\gamma)$.

Step 2. Solve $c_{\hat{r}}$, $c_{\hat{b}}$, $\theta_{\hat{r}}$, and $\theta_{\hat{b}}$.

Next we will give the results of $c_{\hat{r}}$, $c_{\hat{b}}$, $\theta_{\hat{r}}$, and $\theta_{\hat{b}}$. Proposition 9 shows their expressions, and the proof is the solution process.

Proposition 9. *Based on the results of (22) and implicit function theorem, we have*

$$\begin{aligned} c_{\hat{r}} &= -\frac{(1-\gamma)(1-\theta^d)c^d}{(\rho + \zeta_r + (1/2)\gamma(1-\gamma)\theta^{d^2}\sigma^2)\gamma}, \\ c_{\hat{b}} &= -\frac{(1-\gamma)c^d}{(\rho + \zeta_b + (1/2)\gamma(1-\gamma)\theta^{d^2}\sigma^2)\gamma}, \\ \theta_{\hat{r}} &= -\frac{1}{\gamma\sigma^2}, \quad \theta_{\hat{b}} = \frac{1}{\gamma\sigma^2}. \end{aligned} \quad (23)$$

And the final results of model (18) are

$$\begin{aligned} c^s &= c^d + \tau \left(-\frac{1-\theta^d}{M}, \frac{\kappa\sigma_c}{\sigma_r} \right) \hat{r} + \tau \left(-\frac{1}{N}, \frac{\kappa\sigma_c}{\sigma_b} \right) \hat{b}, \\ \theta^s &= \theta^d + \tau \left(-\frac{1}{\gamma\sigma^2}, \frac{\kappa\sigma_\theta}{\sigma_r} \right) \hat{r} + \tau \left(\frac{1}{\gamma\sigma^2}, \frac{\kappa\sigma_\theta}{\sigma_b} \right) \hat{b}, \end{aligned} \quad (24)$$

where

$$\begin{aligned} M &= \frac{(1-\gamma)c^d}{(\rho + \zeta_r + (1/2)\gamma(1-\gamma)\theta^{d^2}\sigma^2)\gamma}, \\ N &= \frac{(1-\gamma)c^d}{(\rho + \zeta_b + (1/2)\gamma(1-\gamma)\theta^{d^2}\sigma^2)\gamma}, \\ c^d &= w(\mathfrak{R}(1-\gamma))^{-1/\gamma}, \end{aligned} \quad (25)$$

$$\mathfrak{R} = \frac{[(\rho - \bar{r}(1-\gamma) - (\bar{b} - \bar{r})^2(1-\gamma)/2\sigma^2\gamma)/\gamma]^{-\gamma}}{(1-\gamma)},$$

$$\theta^d = \frac{\bar{b} - \bar{r}}{\gamma\sigma^2}.$$

Proof. See the appendix. $\square$

Proposition 9 makes predictions about the sparse agent's choice. When $\kappa = 0$, the agent is the traditional, perfectly rational agent. And when $\kappa > 0$, it is a policy of a sparse agent. Larger κ indicates that the agent is less sensitive to fluctuations of both the riskless interest rate and the risky mean rate of return.

FIGURE 3: (a) Impact of a change in $\hat{r}$ on c^s with $\sigma_r = 1.5$, $\sigma_b = 1.7$. (b) Impact of a change in $\hat{b}$ on c^s with $\sigma_r = 1.5$, $\sigma_b = 1.7$.

FIGURE 4: (a) Impact of a change in $\hat{r}$ on θ^s with $\sigma_r = 1.5$, $\sigma_b = 1.7$. (b) Impact of a change in $\hat{b}$ on θ^s with $\sigma_r = 1.5$, $\sigma_b = 1.7$.

4.2. Numerical Example. The purpose of this numerical analysis is to intuitively understand how the boundedly rational agent changes its decisions with the changing of the variances of factors. Firstly, we set $c_r^s = \tau(-(1 - \theta^d)/M, \kappa\sigma_c/\sigma_r)$, $c_b^s = \tau(-1/N, \kappa\sigma_c/\sigma_b)$, $\theta_r^s = \tau(-1/\gamma\sigma^2, \kappa\sigma_\theta/\sigma_r)$, and $\theta_b^s = \tau(1/\gamma\sigma^2, \kappa\sigma_\theta/\sigma_b)$. Let $\gamma = 0.6$, $\sigma = 2.2$, $\sigma_r = 1.5$, $\sigma_b = 1.7$, $\sigma_\theta = 0.5$, $\sigma_c = 1.3$, $\bar{b} = 3.2$, $\bar{r} = 18$, $\rho = 0.95$, $\zeta_r = 0.2$, $\zeta_b = 0.4$, and $w = 5$; then we have c_r^s, c_b^s as shown in Figures 3 and 4. In these figures, the horizontal axis κ is an index of bounded rationality and $\kappa \in [0, 1]$ which is also applied to Figures 5 and 6.

From Figure 3, we know that whatever κ is, $|c_r^s| > 0$ and $|c_b^s| > 0$, which means when the variances of $\hat{r}$ and $\hat{b}$ are big, the agent will consider them in the process of making a decision.

Figure 3(a) shows that if $\kappa = 0$, then the agent reacts like the rational agent: when $\hat{r}$ goes up by 1%, c_r^s will fall by -2.19% (the agent saves more). For $\kappa = 1$, if $\hat{r}$ goes up by 1%, c_r^s falls by -1.85%. This result indicates that the greater the cognitive cost about the factor is, the less attention will be paid to this factor by the boundedly rational agent. From Figure 3(b), we can reach a similar conclusion.

Figure 4 also shows the agent will always consider $\hat{r}$ and $\hat{b}$, that is, $|\theta_r^s| > 0$ and $|\theta_b^s| > 0$, whatever κ is. It addition, we can obtain that if $\kappa = 0$, $\theta_r^s = -0.34\%$ and $\theta_b^s = 0.34\%$, which means the rational agent has the same sensitivity about the $\hat{r}$ and $\hat{b}$ when deciding θ^s . With the increasing of κ , the absolute values of θ_r^s and θ_b^s both will decrease, which means

that the agent will pay less attention to them. In other words, the impact of $\hat{r}$ and $\hat{b}$ on θ^s will decrease for the increasing cognitive cost.

Next, we assume that the standard deviation of riskless interest rate and the risky mean rate of return is smaller, $\sigma_r = 0.15$, $\sigma_b = 0.25$. By keeping other parameters fixed, we get results shown in Figures 5 and 6.

Figure 5(a) shows that when $\kappa \geq 0.26$, $c_r^s = 0$ which means that if the fluctuation is small the agent may discard $\hat{r}$ when he decides the optimal consumption. We can get a similar conclusion from Figure 5(b): when $\kappa \geq 0.95$, $c_b^s = 0$ with $\sigma_b = 0.25$.

From Figure 5(a), we know that if $\kappa = 0$, $c_r^s = -2.19\%$ while Figure 3(a) also shows if $\kappa = 0$, $c_r^s = -2.19\%$ which means, for the rational agent, the sensitivity of c^s to $\hat{r}$ has nothing to do with $\hat{r}$'s variance. However, the boundedly rational agents have different reactions to $\hat{r}$ as κ increases, such as when $\kappa = 0.26$, $c_r^s = 0$ with $\sigma_r = 0.15$ in Figure 5(a) while $c_r^s = -2.1\%$ with $\sigma_r = 1.5$ in Figure 3(a). This disparity indicates that when the cognitive costs are the same and $\kappa > 0$, that is, the agents have the same boundedly rational degree, more volatile factors will be considered while the factor with smaller variance may be neglected.

Additionally, we can know that when $0.26 \leq \kappa < 0.95$, the agent does not react to $\hat{r}$, namely, $c_r^s = 0$ (in Figure 5(a)), but will react to a change in $\hat{b}$ (in Figure 5(b)), which is more important: the sensitivity of c^s to $\hat{b}$ remains high even for a high cognitive friction κ . Note that this "feature by feature"

FIGURE 5: (a) Impact of a change in $\hat{r}$ on c^s with $\sigma_r = 0.15$, $\sigma_b = 0.25$. (b) Impact of a change in $\hat{b}$ on c^s with $\sigma_r = 0.15$, $\sigma_b = 0.25$.

FIGURE 6: (a) Impact of a change in $\hat{r}$ on θ^s with $\sigma_r = 0.15$, $\sigma_b = 0.25$. (b) Impact of a change in $\hat{b}$ on θ^s with $\sigma_r = 0.15$, $\sigma_b = 0.25$.

selective attention could not be rationalized by just a fixed cost to consumption, which is not feature dependent. But when $\kappa \geq 0.95$, $c_r^s = c_b^s = 0$, which indicates that the agent will pay no attention to both $\hat{r}$ and $\hat{b}$ once their thinking costs are beyond some thresholds.

Considering Figure 6(a), we can see that when $\kappa \geq 0.11$, $\theta_r^s = 0$, while Figure 4(a) shows whatever κ is, $\theta_r^s > 0$ with $\sigma_r = 1.5$ which means the smaller the variance of a factor is, the more likely the agent will ignore it. From Figure 6(b), we can also obtain the same conclusion.

5. Conclusion

Dynamic portfolio choice is an important but complex problem in modern financial field, but extant methods always generate complicated numerical calculations due to numerous state variables. Hence, to address this problem, this paper extends the sparse max operator proposed by Gabaix [23] and proposes a new approach to deal with dynamic programming under stochastic terms under the assumption of the agent's limited attention. We apply this method to Merton dynamic portfolio choice problem and find that it effectively simplifies the model's solution process and avoids the "Curse of Dimensionality." Finally, numerical example shows that this method has significant economic implications and clearly interprets the agent's economic behavior when he makes a portfolio choice.

Our study can be extended in several directions. Future research should consider the condition when the stochastic

factors are correlated with each other for it is more realistic. Besides, information faced by the agent is always imprecise and incomplete, and the fuzzy set theory is an important approach to deal with this kind of problem [31–33]. Hence, using fuzzy set theory to handle imprecise values in dynamic programming may be another direction for further research.

Appendix

Proof of Proposition 7. Based on model (7), we define value function

$$\begin{aligned}
 U^*(a, x, y) = & u(a, x, y) + g(x, y, a) V_x \\
 & + \sum_{i=1}^n h^i(x, y_i) V_{y_i} + \frac{1}{2} \sigma^2(x, a) V_{xx} \\
 & + \sum_{i=1}^n \frac{1}{2} (\sigma^i(y_i(t)))^2 V_{y_i y_i} \\
 & + \sum_{i=1}^n \sigma(x, a) \sigma^i(y_i(t)) \rho_{xy_i} V_{xy_i} \\
 & + \sum_{i,j=1(i \neq j)}^n \sigma^i(y_i(t)) \sigma^j(y_j(t)) \rho_{ij} V_{y_i y_j}.
 \end{aligned} \tag{A.1}$$

For $dZ_x dB_y = 0$, we have $\rho_{xy_i} = 0$, $\rho_{y_i y_j} = 0$ where ρ_{xy_i} , $\rho_{y_i y_j}$ represent coefficient between x and y_i , y_i , and y_j , respectively.

Besides, the volatility of a variable that may not be considered by the agent is assumed to be small in Assumption 6; that is, $\sigma^i(y_i(t)) = 0$; so we have

$$U^*(a, x, \mathbf{y}) = u(a, x, \mathbf{y}) + g(x, \mathbf{y}, a) V_x + \sum_{i=1}^n h^i(x, y_i) V_{y_i} + \frac{1}{2} \sigma^2(x, a) V_{xx}. \quad (\text{A.2})$$

Similarly, we define

$$\begin{aligned} U^{**}(a, x, \mathbf{y}) &= u(a, x, \mathbf{y}) + g(x, \mathbf{y}, a) V_x^s \\ &+ \sum_{i=1}^n h^i(x, y_i) V_{y_i}^s + \frac{1}{2} \sigma^2(x, a) V_{xx}^s \\ &+ \sum_{i=1}^n \frac{1}{2} (\sigma^i(y_i(t)))^2 V_{y_i y_i}^s \\ &+ \sum_{i=1}^n \sigma(x, a) \sigma^i(y_i(t)) \rho_{xy_i} V_{xy_i}^s \\ &+ \sum_{i,j=1(i \neq j)}^n \sigma^i(y_i(t)) \sigma^j(y_j(t)) \rho_{ij} V_{y_i y_j}^s. \end{aligned} \quad (\text{A.3})$$

From the analysis above, we have

$$U^{**}(a, x, \mathbf{y}) = u(a, x, \mathbf{y}) + g(x, \mathbf{y}, a) V_x^s + \sum_{i=1}^n h^i(x, y_i) V_{y_i}^s + \frac{1}{2} \sigma^2(x, a) V_{xx}^s. \quad (\text{A.4})$$

Then the associated optimal actions can be expressed as

$$\begin{aligned} a^*(x, \mathbf{y}) &= \arg \max_a U^*(a, x, \mathbf{y}), \\ a^{**}(x, \mathbf{y}) &= \arg \max_a U^{**}(a, x, \mathbf{y}). \end{aligned} \quad (\text{A.5})$$

First, we will prove $\partial a^*(x, \mathbf{y}) / \partial y_i = \partial a^{**}(x, \mathbf{y}) / \partial y_i$ at $\mathbf{y} = 0$.

From the proof of lemma 1 in Gabaix [23], we know that $V^s(x, \mathbf{y}) = V(x, \mathbf{y}) + x\pi(x, \mathbf{y})x$ where $\pi(x, \mathbf{y})$ is continuous in $(x, \mathbf{y})$ and twice differentiable at $\mathbf{y} = 0$, with $\pi(x, \mathbf{0})$ negative semidefinite. In other words, the $V^s(x, \mathbf{y})$ and $V(x, \mathbf{y}) = x\pi(x, \mathbf{y})x$ differ only by second-order terms in x . This basically generalizes the envelope theorem. It implies that at

$$\begin{aligned} \mathbf{y} = 0, \text{ we have } V_x(x, 0) &= V_x^s(x, 0), \\ V_{xx}(x, 0) &= V_{xx}^s(x, 0), \\ V_{y_i}(x, \mathbf{y}) \Big|_{y_i=0} &= V_{y_i}^s(x, \mathbf{y}) \Big|_{y_i=0}, \\ V_{xy_i}(x, \mathbf{y}) \Big|_{y_i=0} &= V_{xy_i}^s(x, \mathbf{y}) \Big|_{y_i=0}. \end{aligned} \quad (\text{A.6})$$

Differentiating formula (A.2) with respect to a gives

$$U_a^* = u_a + g_a(x, \mathbf{y}, a) V_x + \frac{1}{2} \sigma_a^2(x, a) V_{xx}. \quad (\text{A.7})$$

Differentiating formula (A.7) with respect to y_i and a , respectively, we get

$$\begin{aligned} U_{ay_i}^* &= u_{ay_i} + g_{ay_i}(x, \mathbf{y}, a) V_x + g_a(x, \mathbf{y}, a) V_{xy_i} \\ &+ \frac{1}{2} \sigma_a^2(x, a) D_x [V_{xy_i}], \end{aligned} \quad (\text{A.8})$$

$$U_{aa}^* = u_{aa} + g_{aa}(x, \mathbf{y}, a) V_x + \frac{1}{2} \sigma_{aa}^2(x, a) V_{xx}.$$

Similarly, differentiating formula (A.4) with respect to a gives

$$U_a^{**} = u_a + g_a(x, \mathbf{y}, a) V_x^s + \frac{1}{2} \sigma_a^2(x, a) V_{xx}^s. \quad (\text{A.9})$$

Differentiating formula (A.9) with respect to y_i and a , respectively, gives

$$\begin{aligned} U_{ay_i}^{**} &= u_{ay_i} + g_{ay_i}(x, \mathbf{y}, a) V_x^s + g_a(x, \mathbf{y}, a) V_{xy_i}^s \\ &+ \frac{1}{2} \sigma_a^2(x, a) D_x [V_{xy_i}^s], \end{aligned} \quad (\text{A.10})$$

$$U_{aa}^* = u_{aa} + g_{aa}(x, \mathbf{y}, a) V_x^s + \frac{1}{2} \sigma_{aa}^2(x, a) V_{xx}^s.$$

Hence, we have $U_{ay_i}^* = U_{ay_i}^{**}$ at $y_i = 0$ and $U_{aa}^* = U_{aa}^{**}$ at $y_i = 0$. So

$$\begin{aligned} \frac{\partial a^{**}(x, \mathbf{y})}{\partial y_i} \Big|_{y_i=0} &= - \frac{U_{ay_i}^{**} \Big|_{y_i=0}}{U_{aa}^{**}} \\ &= - \frac{U_{ay_i}^* \Big|_{y_i=0}}{U_{aa}^*} = \frac{\partial a^*(x, \mathbf{y})}{\partial y_i} \Big|_{y_i=0}. \end{aligned} \quad (\text{A.11})$$

Given $a^r(x, \mathbf{y}) = a^d(x) + \sum_i a_{y_i} y_i + o(\|\mathbf{y}\|^2)$, we have $\partial a^r(x, \mathbf{y}) / \partial y_i = a_{y_i}$. According to (A.11), we obtain $\partial a^{**}(x, \mathbf{y}) / \partial y_i = a_{y_i}$, so $a^{**}(x, \mathbf{y}) = a^d(x) + \sum_i a_{y_i} y_i + o(\|\mathbf{y}\|^2)$. Finally

$$\begin{aligned} a^s(x, \mathbf{y}) &= a(x, \mathbf{m}^{*T} \mathbf{y}) = a^d(x) \\ &+ \sum_i a_{y_i} m_i^* y_i + o(\|\mathbf{y}\|^2) \\ &= a^d(x) + \sum_i a_{y_i} A_1 \left(\frac{\Lambda_{ii}}{\kappa} \right) y_i + o(\|\mathbf{y}\|^2) \\ &= a^d(x) + \sum_i \tau \left(a_{y_i}, \frac{\kappa \sigma_a}{\sigma_i} \right) y_i + o(\|\mathbf{y}\|^2), \end{aligned} \quad (\text{A.12})$$

where σ_a is the standard deviation of a . $\square$

Proof of Proposition 8. The laws of motion of model (7) are

$$\begin{aligned} dx &= g(x(t), \mathbf{y}(t), a(t)) dt + \sigma(x(t), a(t)) dZ_x(t) \\ dy_i &= h^i(x(t), y_i(t)) dt + \sigma^i(y_i(t)) dB_{y_i}(t), \end{aligned} \quad (\text{A.13})$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)$. Using Ito formula, Bellman's equation of model (7) can be expressed as follows:

$$\begin{aligned} \rho V = & \max_a u(a, x, \mathbf{y}) + g(x, \mathbf{y}, a) V_x \\ & + \sum_{i=1}^n h^i(x, y_i) V_{y_i} + \frac{1}{2} \sigma^2(x, a) V_{xx} \\ & + \sum_{i=1}^n \frac{1}{2} (\sigma^i(y_i(t)))^2 V_{y_i y_i} \\ & + \sum_{i=1}^n \sigma(x, a) \sigma^i(y_i(t)) \rho_{xy_i} V_{xy_i} \\ & + \sum_{i,j=1(i \neq j)}^n \sigma^i(y_i(t)) \sigma^j(y_j(t)) \rho_{y_i y_j} V_{y_i y_j}. \end{aligned} \quad (\text{A.14})$$

From the proof of Proposition 7 above, we have $\rho_{xy_i} = 0$, $\rho_{y_i y_j} = 0$, and $\sigma^i(y_i(t)) = 0$. So we obtain

$$\begin{aligned} \rho V = & \max_a u(a, x, \mathbf{y}) + g(x, \mathbf{y}, a) V_x \\ & + \sum_{i=1}^n h^i(x, y_i) V_{y_i} + \frac{1}{2} \sigma^2(x, a) V_{xx}. \end{aligned} \quad (\text{A.15})$$

We define function $\phi(x, \mathbf{y}, a)$ as the derivative of the right side in formula (A.15) with respect to a , so a satisfies $\phi = 0$ with

$$\phi(x, \mathbf{y}, a) = u_a + g_a(x, \mathbf{y}, a) V_x + \frac{1}{2} \sigma_a^2(x, a) V_{xx}, \quad (\text{A.16})$$

and we define $f(x) = V_{xx}/V$, where $f(x)$ can be derived from the expression of $V(x, \mathbf{0}) = \int_0^\infty e^{-\rho t} u(a, x, \mathbf{0}) dt$ in Step 1 of Section 3. Hence, differentiating formula (A.15) with respect to y_i gives

$$\begin{aligned} \rho V_{y_i} = & u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x \\ & + \sum_{i=1}^n h^i_{xy_i}(x, y_i, a) V_{y_i} + \frac{1}{2} \sigma^2(x, a) f(x) V_{y_i}. \end{aligned} \quad (\text{A.17})$$

Now we differentiate at $\mathbf{y} = 0$ and evaluate at $(a, x, \mathbf{y}) = (a^d, x, 0)$:

$$\begin{aligned} \rho V_{xy_i} = & D_x [u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x] \\ & + \sum_{i=1}^n h^i_{xy_i}(x, y_i, a) V_{y_i} + \sum_{i=1}^n h^i_{y_i}(x, y_i, a) V_{xy_i} \\ & + \frac{1}{2} \sigma_x^2(x, a) f(x) V_{y_i} \\ & + \frac{1}{2} \sigma^2(x, a) (f_x(x) V_{y_i} + f(x) V_{xy_i}). \end{aligned} \quad (\text{A.18})$$

From (A.17) we get $V_{y_i} = (u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x) / (\rho - \sum_{i=1}^n h^i_{y_i}(x, y_i, a) - (1/2) \sigma^2(x, a) f(x))$.

And from (A.18) we obtain

$$\begin{aligned} V_{xy_i} = & \left(D_x [u_{ay_i} + g_{y_i}(x, \mathbf{y}, a) V_x] \right. \\ & + \sum_{i=1}^n h^i_{xy_i}(x, y_i, a) V_{y_i} + \frac{1}{2} \sigma_x^2(x, a) f(x) V_{y_i} \\ & + \frac{1}{2} \sigma^2(x, a) f_x(x) V_{y_i} \Big) \\ & \times \left(\rho - \sum_{i=1}^n h^i_{y_i}(x, y_i, a) - \frac{1}{2} \sigma^2(x, a) f(x) \right)^{-1}. \end{aligned} \quad (\text{A.19})$$

According to formula (A.16), we know that the impact of y_i on the optimal action can be expressed as $a_{y_i} = -\phi_{y_i}/\phi_a$, where

$$\begin{aligned} \phi_a = & u_{aa} + g_{aa}(x, \mathbf{y}, a) V_x + \frac{1}{2} \sigma_{aa}^2(x, a) D_x [V_x], \\ \phi_{y_i} = & u_{ay_i} + g_{ay_i}(x, \mathbf{y}, a) V_x + g_a(x, \mathbf{y}, a) V_{xy_i} \\ & + \frac{1}{2} \sigma_a^2(x, a) D_x [V_{xy_i}]. \end{aligned} \quad (\text{A.20})$$

□

Proof of Proposition 9. Using Ito formula, Bellman's formula of model (18) is

$$\begin{aligned} \rho V = & \max_c u(c) \\ & + [w(1-\theta)(\bar{r} + \hat{r}) + w\theta(\bar{b} + \hat{b}) - c] V_w \\ & + (-\zeta_r \hat{r}) V_{\hat{r}} + (-\zeta_b \hat{b}) V_{\hat{b}} \\ & + \frac{1}{2} (\theta \sigma w)^2 V_{ww} + \frac{1}{2} \sigma_r^2 V_{\hat{r} \hat{r}} + \frac{1}{2} \sigma_b^2 V_{\hat{b} \hat{b}} \\ & + \sigma \sigma_r \rho_{wr} V_{wr} + \sigma \sigma_b \rho_{wb} V_{wb} + \sigma_r \sigma_b \rho_{rb} V_{rb}, \end{aligned} \quad (\text{A.21})$$

where ρ_{wr} , ρ_{wb} , and ρ_{rb} represent the coefficient between w and r , w and b , and b and r , respectively. For $dWdR = 0$, $dWdB = 0$, and $dRdB = 0$, we have $\rho_{wr} = 0$, $\rho_{wb} = 0$, and $\rho_{rb} = 0$. According to Assumption 6, the variances of r and b are so small that we let $\sigma_r = \sigma_b = 0$. Then formula (A.21) becomes as follows:

$$\begin{aligned} \rho V = & \max_c u(c) + [w(1-\theta)(\bar{r} + \hat{r}) + w\theta(\bar{b} + \hat{b}) - c] V_w \\ & + (-\zeta_r \hat{r}) V_{\hat{r}} + (-\zeta_b \hat{b}) V_{\hat{b}} + \frac{1}{2} (\theta \sigma w)^2 V_{ww}. \end{aligned} \quad (\text{A.22})$$

Differentiating formula (A.22) with respect to $\hat{r}$ and evaluating at $(\hat{r}, \hat{b}) = (0, 0)$, we obtain $\rho V_{\hat{r}} = w(1-\theta) V_w + (-\zeta_r) V_{\hat{r}} + (1/2) (\theta \sigma w)^2 (\partial V_{ww} / \partial \hat{r})$ where $V_{ww} = ((1-\gamma)(-\gamma)/w^2) V$ which can be obtained from (22); then

$$\rho V_{\hat{r}} = w(1-\theta) V_w + (-\zeta_r) V_{\hat{r}} + \frac{1}{2} (1-\gamma)(-\gamma) \theta^2 \sigma^2 V_{\hat{r}}. \quad (\text{A.23})$$

Now differentiating (using the total derivative) formula (A.23) with respect to w and evaluating at $(\hat{r}, \hat{b}) = (0, 0)$, we obtain

$$\begin{aligned} \rho V_{w\hat{r}} &= (1 - \theta) V_w + w (1 - \theta) V_{ww} \\ &+ (-\zeta_r) V_{w\hat{r}} + \frac{1}{2} (1 - \gamma) (-\gamma) \theta^2 \sigma^2 V_{w\hat{r}}. \end{aligned} \quad (\text{A.24})$$

From formula (A.23) we have

$$V_{\hat{r}} = \frac{w (1 - \theta) V_w}{\rho + \zeta_r + (1/2) \gamma (1 - \gamma) \theta^2 \sigma^2}. \quad (\text{A.25})$$

According to formula (A.24) and the term $V_{ww} = (-\gamma/w) V_w$ which can be obtained from (22) we have

$$V_{w\hat{r}} = \frac{(1 - \gamma) (1 - \theta) V_w}{\rho + \zeta_r + (1/2) \gamma (1 - \gamma) \theta^2 \sigma^2}. \quad (\text{A.26})$$

Let $\phi(c, \theta, w, \hat{r}, \hat{b})$ denote the result of derivation of the right side in formula (A.22) with respect to c ; then c satisfies $\phi = 0$ with $\phi(c, \theta, w, \hat{r}, \hat{b}) = u_c - V_w$.

Hence, the impact of $\hat{r}$ on c is

$$c_{\hat{r}} = -\frac{\phi_{\hat{r}}}{\phi_c}, \quad (\text{A.27})$$

where

$$\begin{aligned} \phi_c &= u_{cc}, \\ \phi_{\hat{r}} &= u_{cc} \frac{\partial c}{\partial \hat{r}} - V_{w\hat{r}}. \end{aligned} \quad (\text{A.28})$$

Since all derivatives are evaluated at $(\hat{r}, \hat{b}) = (0, 0)$, we have $c = c^d$, $\theta = \theta^d$. By substituting (A.28) into (A.27) and using the results of (22), (A.25), (A.26), now we can get

$$c_{\hat{r}} = -\frac{(1 - \gamma) (1 - \theta^d) c^d}{(\rho + \zeta_r + (1/2) \gamma (1 - \gamma) \theta^{d^2} \sigma^2) \gamma}. \quad (\text{A.29})$$

Similarly, we have

$$c_{\hat{b}} = -\frac{(1 - \gamma) c^d}{(\rho + \zeta_b + (1/2) \gamma (1 - \gamma) \theta^{d^2} \sigma^2) \gamma}, \quad (\text{A.30})$$

where the concrete expressions of c^d , θ^d are referred to formula (22). $\theta_{\hat{r}}$, $\theta_{\hat{b}}$ can be solved in an analogous way as $\theta_{\hat{r}} = -1/\gamma \sigma^2$, $\theta_{\hat{b}} = 1/\gamma \sigma^2$.

According to Proposition 7, c^s and θ^s can expressed as

$$\begin{aligned} c^s &= c^d + \tau \left(-\frac{1 - \theta^d}{M}, \frac{\kappa \sigma_c}{\sigma_r} \right) \hat{r} + \tau \left(-\frac{1}{N}, \frac{\kappa \sigma_c}{\sigma_b} \right) \hat{b}, \\ \theta^s &= \theta^d + \tau \left(-\frac{1}{\gamma \sigma^2}, \frac{\kappa \sigma_{\theta}}{\sigma_r} \right) \hat{r} + \tau \left(\frac{1}{\gamma \sigma^2}, \frac{\kappa \sigma_{\theta}}{\sigma_b} \right) \hat{b}, \end{aligned} \quad (\text{A.31})$$

where

$$\begin{aligned} M &= \frac{(1 - \gamma) c^d}{(\rho + \zeta_r + (1/2) \gamma (1 - \gamma) \theta^{d^2} \sigma^2) \gamma}, \\ N &= \frac{(1 - \gamma) c^d}{(\rho + \zeta_u + (1/2) \gamma (1 - \gamma) \theta^{d^2} \sigma^2) \gamma}, \\ c^d &= w (\Re (1 - \gamma))^{-1/\gamma} \\ \Re &= \frac{\left[(\rho - \bar{r} (1 - \gamma) - (\bar{b} - \bar{r})^2 (1 - \gamma) / 2 \sigma^2 \gamma) / \gamma \right]^{-\gamma}}{(1 - \gamma)}, \\ \theta^d &= \frac{\bar{b} - \bar{r}}{\gamma \sigma^2}. \end{aligned} \quad (\text{A.32})$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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