Research Article

Parametrically Excited Oscillations of Second-Order Functional Differential Equations and Application to Duffing Equations with Time Delay Feedback

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We study oscillatory behaviour of a large class of second-order functional differential equations with three freedom real nonnegative parameters. According to a new oscillation criterion, we show that if at least one of these three parameters is large enough, then the main equation must be oscillatory. As an application, we study a class of Duffing type quasilinear equations with nonlinear time delayed feedback and their oscillations excited by the control gain parameter or amplitude of forcing term. Finally, some open questions and comments are given for the purpose of further study on this topic.

1. Introduction

Let λ, μ, ρ be three nonnegative parameters and let $A(v)$, $B(t, u, v)$, $F(t, u)$, $G(t, u)$, and $e(t)$ be continuous functions in all their variables satisfying some conditions determined in Section 2. We consider the following large class of second-order functional differential equations:

$$\begin{aligned} & \left(r(t) A(x'(t)) \right)' + B(t, x(t), x'(t)) + \lambda F(t, x(\tau(t))) \\ & + \mu G(t, x(\sigma(t))) = \rho e(t), \quad t \geq t_0, \end{aligned} \quad (1)$$

where as usual the functional terms $\tau(t)$ and $\sigma(t)$ satisfy $\tau(t) \leq t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$, and $\sigma(t) \geq t$. Three cases are studied simultaneously: delay ($\tau(t) \neq t, \sigma(t) = t$), advanced ($\tau(t) = t, \sigma(t) \neq t$), and delay-advanced ($\tau(t) \neq t, \sigma(t) \neq t$), and both $\tau(t)$ and $\sigma(t)$ are increasing functions.

A continuous function $x(t)$ is called nonoscillatory if there is a point $T \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T$. Otherwise, $x(t)$ is an oscillatory function. A function $x = x(t)$, $x \in C^2((t_0, \infty), \mathbb{R})$, is called the (extendable) solution of (1) if it satisfies equality in (1) for all $t > t_0$. Equation (1) is oscillatory if all its solutions are oscillatory.

In the paper, we investigate conditions on $F(t, u)$, $G(t, u)$, and $e(t)$ under which (1) is oscillatory provided at least one of parameters λ , μ , and ρ is large enough. It is shortly called the parametrically excited oscillations of (1). In Section 4 we discuss some known oscillation criteria published in [1–9] which allow the parametrically excited oscillations but only in the form of examples. On various problems concerning the functional differential equations we refer the reader to [10–15] and the references therein.

In Section 2 we state a fundamental lemma proposing a new oscillation criterion that plays a crucial role in the formulation of the main results illustrated on some suitable chosen examples. In Section 3 we consider an application of the main results to the Duffing type quasilinear equations with time delayed feedback, taking into account the known results in applied sciences concerning such kind of nonlinear oscillators without time delay, see [16–37], and with time delay, see [38–48] and the references therein. In Section 5 we present some open questions and comments for further study that can follow our main results. And in Section 6, we describe the method for proving the main results of the paper.

2. Main Assumptions and Results

Let $\mathbb{R}_+ = (0, \infty)$. For the functions $r = r(t)$ and $A = A(v)$ both appearing in the second-order differential operator of (1), we suppose the following:

$$0 < r(t) \leq r_0 \text{ on } [t_0, \infty), \quad r \in C^1([t_0, \infty), \mathbb{R}_+), \quad (2)$$

and $A(v)$ is odd, increasing, and

$$|u|^{-p-1} A(v) v \geq \alpha (|A(v)| |u|^{-p})$$

for some $p > 0$ and all $u \neq 0, v \in \mathbb{R}$,

$$\alpha = \alpha(s), \quad \alpha \in C^1([0, \infty), [0, \infty)), \quad (3)$$

$$\frac{1}{1+\alpha} \in L^1(\mathbb{R}_+, \mathbb{R}_+), \quad \pi_* = \int_0^\infty \frac{2ds}{1+\alpha(s)},$$

$$\alpha(cs) \geq c^\gamma \alpha(s) \text{ for some } \gamma \geq 1 \text{ and all } c > 0, s \geq 0.$$

For instance, it is simple to check that for $\alpha(s) = s^\gamma$, hypothesis (3) is fulfilled in the next three most important cases of $A(v)$: the linear operator $A(v) = v$ if $p = 1$ and $\gamma = 2$, the quasilinear p -Laplacian operator $A(v) = |v|^{p-1}v$ if $p \geq 1$ and $\gamma = (p+1)/p$, and the quasilinear mean curvature operator $A(v) = v(1+v^2)^{-1/2}$ if $p = 1$ and $\gamma = 2$. Some details about the number π_* which depends on the function $\alpha(s)$ given in (3) are presented in Section 6.

The damped term $B = B(t, u, v)$ satisfies the strong condition

$$B(t, u, v) \operatorname{sgn}(u) \geq 0 \quad \forall t \geq t_0, u \neq 0, v \in \mathbb{R}. \quad (4)$$

We hope that (4) can be relaxed with some weaker condition, which is commented as an open problem in Section 5 below.

In the following fundamental lemma which plays a crucial role in the proof of the main results, we are working with such solutions $x(t)$ of (1) that satisfies the inequality

$$\begin{aligned} & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ & \times (|x(t)|^{p-1} x(t))^{-1} \geq k(\lambda, \mu, \rho) Q(t), \end{aligned} \quad (5)$$

for all $t \in J$ and some interval J , where the functions $k(\lambda, \mu, \rho)$ and $Q(t)$ do not depend on $x(t)$ but only on F, G, e and they are determined in the process below. The functions $k(\lambda, \mu, \rho)$ and $Q(t)$ present the key point in the parametrically excited oscillations.

Lemma 1. *Let assumptions (2), (3), and (4) hold. Let (a, b) and (c, d) be two disjoint open intervals such that $\sigma(b) \leq \tau(c)$. Let the functions $k = k(\lambda, \mu, \rho)$, $k \in C([0, \infty)^3, \mathbb{R}_+)$, and $Q = Q(t)$, $Q \in C(\bar{J}, [0, \infty))$, $Q(t) \neq 0$ on J , be such that*

$$\frac{p^{1/\gamma}}{r_0^{1-1/\gamma}} \frac{[k(\lambda, \mu, \rho)]^{1-1/\gamma}}{\pi_*} \int_J Q(t) dt \geq \left(\max_{t \in \bar{J}} Q(t) \right)^{1/\gamma}, \quad (6)$$

for both $J = (a, b)$ and $J = (c, d)$, for all $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and $\rho \geq \rho_0$, and for some $(\lambda_0, \mu_0, \rho_0) \in \mathbb{R}_+^3$, where r_0, p, γ , and π_*

are constants defined, respectively, in (2) and (3). Let $x(t)$ be a solution of (1) satisfying the next two statements:

$$\begin{aligned} & \text{if } x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \\ & \text{on } (\tau(a), \sigma(b)); \end{aligned} \quad (7)$$

then $x(t)$ satisfies (5) on (a, b) ,

$$\begin{aligned} & \text{if } x(t) < 0, \quad x(\tau(t)) < 0, \quad x(\sigma(t)) < 0 \\ & \text{on } (\tau(c), \sigma(d)), \end{aligned} \quad (8)$$

then (t) satisfies (5) on (c, d) .

Then $x(t)$ has at least one zero point in $(\tau(\tau(a)), \sigma(\sigma(d)))$.

This lemma simultaneously holds for all three types of functional arguments: delay, advanced, and delay-advanced. It will be proved in Section 6. In Corollaries 9, 10, and 11 below, we give some simple conditions on two functions $f(t), g(t)$ and two numbers p_1, p_2 such that all solutions of (1) with the functions $F(t, u) = f(t, u)|u|^{p_1} \operatorname{sgn}(u)$ and $G(t, u) = g(t, u)|u|^{p_2} \operatorname{sgn}(u)$ satisfy required statements (7) and (8) with respect to some intervals (a_n, b_n) and the explicitly given functions $k(\lambda, \mu, \rho)$ and $Q_n(t)$ satisfying (6), where $e(t)$ satisfies a basic assumption.

In what follows, (a_n, b_n) , $n \in \mathbb{N}$, denotes a sequence of disjoint open intervals such that $t_0 \leq \tau(a_n) < \sigma(b_n) \leq \tau(a_{n+1}) < \sigma(b_{n+1}) \leq \dots$, and $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Now we present several variations of Lemma 1 in which essential inequality (6) is relaxed with some asymptotic assumptions that are simpler to be verified in several applications.

Lemma 2. *Let assumptions (2), (3), and (4) hold. Let the continuous function $k = k(\lambda, \mu, \rho) > 0$ and the sequence of functions $Q_n = Q_n(t)$, $Q_n \in C([a_n, b_n], [0, \infty))$, $Q_n(t) \neq 0$ on (a_n, b_n) , satisfy the next two inequalities: there are constants $c_0 > 0$, $n_0 \in \mathbb{N}$, and $(\lambda_0, \mu_0, \rho_0) \in \mathbb{R}_+^3$ such that*

$$k(\lambda, \mu, \rho) \geq r_0 p^{-1/(\gamma-1)} \left(\frac{2\pi_*}{c_0} \right)^{\gamma/(\gamma-1)} \quad (9)$$

$$\forall \lambda \geq \lambda_0, \mu \geq \mu_0, \rho \geq \rho_0,$$

where the numbers r_0, p, γ , and π_* are, respectively, from (2) and (3), and

$$\lim_{n \rightarrow \infty} \left(\frac{1}{(\max_{t \in [a_n, b_n]} Q_n(t))^{1/\gamma}} \int_{a_n}^{b_n} Q_n(t) dt \right) \geq c_0 > 0. \quad (10)$$

If $x(t)$ is a solution of (1) that satisfies (7) and (8) with $a = a_{2n-1}$, $b = b_{2n-1}$, $c = a_{2n}$, $d = b_{2n}$, and $Q(t) = Q_n(t)$, then $x(t)$ has at least one zero point in $(\tau(\tau(a_{2n-1})), \sigma(\sigma(b_{2n})))$, $\forall n \geq n_0$.

In the following slightly simpler version of Lemma 2, inequality (9) is replaced with an asymptotic condition and, at the same time, the limit in (10) is relaxed with the limit inferior. Thus, conditions (9) and (10) are replaced with more practical ones.

Lemma 3. Let assumptions (2), (3), and (4) hold. Let the continuous function $k = k(\lambda, \mu, \rho) > 0$ and the sequence of functions $Q_n = Q_n(t)$, $Q_n \in C([a_n, b_n], [0, \infty))$, $Q_n(t) \not\equiv 0$, on (a_n, b_n) satisfy, respectively,

$$\begin{aligned} k(\lambda, \mu, \rho) &\longrightarrow \infty \quad \text{as } \lambda \longrightarrow \infty \\ &\text{or } \mu \longrightarrow \infty \quad \text{or } \rho \longrightarrow \infty, \end{aligned} \quad (11)$$

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{(\max_{t \in [a_n, b_n]} Q_n(t))^{1/\gamma}} \int_{a_n}^{b_n} Q_n(t) dt \right) > 0. \quad (12)$$

If $x(t)$ is a solution of (1) that satisfies (7) and (8) with $a = a_{2n-1}$, $b = b_{2n-1}$, $c = a_{2n}$, $d = b_{2n}$, and $Q(t) = Q_n(t)$, then $x(t)$ has at least one zero point in $(\tau(\tau(a_{2n-1})), \sigma(\sigma(b_{2n})))$, $\forall n \geq n_0$, and for some $n_0 \in \mathbb{N}$.

In some concrete cases, we use the next version of Lemmas 2 and 3, where condition (10) or (12) is replaced with appropriate one that appears in (1) with periodic coefficients.

Lemma 4. Let assumptions (2), (3), and (4) hold. Let the continuous function $k = k(\lambda, \mu, \rho) > 0$ and the sequence of functions $Q_n = Q_n(t)$, $Q_n \in C([a_n, b_n], [0, \infty))$, $Q_n(t) \not\equiv 0$, on (a_n, b_n) satisfy, respectively, (11) and, for some $C_0, c_1 \in \mathbb{R}$,

$$\begin{aligned} 0 < \max_{t \in [a_n, b_n]} Q_n(t) \leq C_0, \quad \int_{a_n}^{b_n} Q_n(t) dt \geq c_1 > 0, \\ \forall n \geq n_0, \end{aligned} \quad (13)$$

and some $n_0 \in \mathbb{N}$. If $x(t)$ is a solution of (1) that satisfies (7) and (8) with $a = a_{2n-1}$, $b = b_{2n-1}$, $c = a_{2n}$, $d = b_{2n}$, and $Q(t) = Q_n(t)$, then $x(t)$ has at least one zero point in $(\tau(\tau(a_{2n-1})), \sigma(\sigma(b_{2n})))$, $\forall n \geq n_0$.

Next, we suppose that the coefficient $r(t)$ additionally satisfies

$$\begin{aligned} r(s) &\leq r(t) \quad \text{in delay,} \\ r(s) &\geq r(t) \quad \text{in advanced case } \forall s \leq t, \end{aligned} \quad (14)$$

$r(t)$ is a constant in delay-advanced case,

and the forcing term $e(t)$ satisfies

$$\begin{aligned} e(t) &\leq 0 \quad \text{on } [\tau(a_{2n-1}), \sigma(b_{2n-1})], \\ e(t) &\geq 0 \quad \text{on } [\tau(a_{2n}), \sigma(b_{2n})], \quad n \in \mathbb{N}. \end{aligned} \quad (15)$$

We remark that $e(t)$ remains arbitrary function outside the set $\bigcup_n (\tau(a_n), \sigma(b_n))$.

The first result of the paper deals with delay equation (1).

Theorem 5. Let assumptions (2), (3), (4), (14), and (15) hold, $\tau(t) = t - \tau$, $\tau \geq 0$, $\sigma(t) \equiv t$, $G(t, u) \equiv 0$, and let $F(t, u)$ satisfy

$$\begin{aligned} F(t, u) \operatorname{sgn}(u) &\geq f(t) |u|^q \quad \forall u \neq 0, \quad t \geq t_0, \\ f(t) &\geq 0, \quad f(t) \not\equiv 0 \quad \text{on } [\tau(a_n), b_n], \end{aligned} \quad (16)$$

where $q \geq p$, number p is from (3), sequence (a_n, b_n) is from (15), and $f \in C([t_0, \infty), \mathbb{R})$ is a periodic function with period $T_* > 0$ such that

$$\begin{aligned} [a_{2n-1} + T_*, b_{2n-1} + T_*] &\subseteq [a_{2n+1}, b_{2n+1}], \\ [a_{2n} + T_*, b_{2n} + T_*] &\subseteq [a_{2n+2}, b_{2n+2}], \quad n \in \mathbb{N}. \end{aligned} \quad (17)$$

Then (1) is oscillatory in the next two cases: $q = p$ and parameter λ is large enough; $q > p$, $\lambda > 0$, $\rho > 0$, and at least one of parameters λ and ρ is large enough.

The proof of Theorem 5 is presented in Section 6 and it is based on Lemma 4, where

$$k(\lambda, \mu, \rho) = \lambda^{p/q} \rho^{1-(p/q)}, \quad q \geq p, \quad (18)$$

$Q_n(t)$

$$\begin{aligned} &\begin{cases} f(t) \left(\frac{t - a_n}{t - a_n + \tau} \right)^p, & \text{if } q = p, \\ \frac{q}{p} \left(\frac{p}{q - p} \right)^{(q/p)-1} |e(t)|^{1-(p/q)} (f(t))^{p/q} \\ \quad \times \left(\frac{t - a_n}{t - a_n + \tau} \right)^p, & \text{if } q > p. \end{cases} \end{aligned} \quad (19)$$

The second result deals with advanced equation (1).

Theorem 6. Let assumptions (2), (3), (4), (14), and (15) hold, $\tau(t) \equiv t$, $\sigma(t) = t + \sigma$, $\sigma \geq 0$, $F(t, u) \equiv 0$, and let $G(t, u)$ satisfy

$$\begin{aligned} G(t, u) \operatorname{sgn}(u) &\geq g(t) |u|^q \quad \forall u \neq 0, \quad t \geq t_0, \\ g(t) &\geq 0, \quad g(t) \not\equiv 0 \quad \text{on } [a_n, \sigma(b_n)], \end{aligned} \quad (20)$$

where $q \geq p$, number p is from (3), sequence (a_n, b_n) is from (15), and $g \in C([t_0, \infty), \mathbb{R})$ is a periodic function with period $T_* > 0$ such that (17) is fulfilled. Then (1) is oscillatory in the next two cases: $q = p$ and parameter μ is large enough; $q > p$, $\mu > 0$, $\rho > 0$, and at least one of parameters μ and ρ is large enough.

The proof of Theorem 6 is based on Lemma 4 (see Section 6), where

$$k(\lambda, \mu, \rho) = \mu^{p/q} \rho^{1-(p/q)}, \quad q \geq p, \quad (21)$$

$Q_n(t)$

$$\begin{aligned} &\begin{cases} g(t) \left(\frac{b_n - t}{b_n - t + \sigma} \right)^p, & \text{if } q = p, \\ \frac{q}{p} \left(\frac{p}{q - p} \right)^{(q/p)-1} |e(t)|^{1-(p/q)} (g(t))^{p/q} \\ \quad \times \left(\frac{b_n - t}{b_n - t + \sigma} \right)^p, & \text{if } q > p. \end{cases} \end{aligned} \quad (22)$$

The third result deals with delay-advanced equation (1).

Theorem 7. Let assumptions (2), (3), (4), (14), and (15) hold, $\tau(t) = t - \tau$, $\tau \geq 0$, $\sigma(t) = t + \sigma$, $\sigma \geq 0$, and $F(t, u)$ and $G(t, u)$ satisfy

$$\begin{aligned} F(t, u) \operatorname{sgn}(u) &\geq f(t) |u|^{p_1}, \quad G(t, u) \operatorname{sgn}(u) \geq g(t) |u|^{p_2} \\ &\forall u \neq 0, \quad t \geq t_0, \\ f(t) &\geq 0, \quad f(t) \not\equiv 0, \quad g(t) \geq 0, \\ g(t) &\not\equiv 0 \text{ on } [\tau(a_n), \sigma(b_n)], \end{aligned} \quad (23)$$

where, additionally, $f, g, e \in C([t_0, \infty), \mathbb{R})$ are three periodic functions having a common period $T_* > 0$ such that (17) is fulfilled, where $e(t)$ is the forcing term in (1). Then (1) is oscillatory provided one of the next two cases is fulfilled, where the number p is from (3): (1) (in superlinear delay-advanced case) $p_1 > p$, $p_2 > p$, $\rho > 0$, and either parameter ρ is large enough or at least one of λ and μ is large enough; (2) (in supersublinear delay-advanced case) $p_1 > p > p_2 > 0$, $\lambda > 0$, $\mu > 0$, $\rho > 0$, and at least one of parameters λ , μ , and ρ is large enough.

The proof of Theorem 7 is based on Lemma 4 (see Section 6), where

$$k(\lambda, \mu, \rho) = \begin{cases} \min \{ \lambda^{p/p_1} \rho^{1-(p/p_1)}, \mu^{p/p_2} \rho^{1-(p/p_2)} \}, & \text{superlinear case,} \\ \lambda^{\eta_1} \mu^{\eta_1} \rho^{\eta_0}, & \text{supersublinear case,} \end{cases} \quad (24)$$

$$\begin{aligned} Q_n(t) &= \begin{cases} \left[\frac{p_1}{p} \left(\frac{p}{2(p_1 - p)} \right)^{(p_1/p)-1} \right. \\ \quad \times |e(t)|^{1-(p/p_1)} (f(t))^{p/p_1} \tau_n(t) \\ \quad + \frac{p_2}{p} \left(\frac{p}{2(p_2 - p)} \right)^{(p_2/p)-1} |e(t)|^{1-(p/p_2)} \\ \quad \times (g(t))^{p/p_2} \sigma_n(t), & \text{superlinear case,} \\ |e(t)|^{\eta_0} (f(t))^{\eta_1} (g(t))^{\eta_2} (\tau_n(t))^{\eta_1(p_1/p)} \\ \quad \times (\sigma_n(t))^{\eta_2(p_2/p)} \prod_{i=0}^2 \eta_i^{-\eta_i}, & \text{supersublinear case,} \end{cases} \end{aligned} \quad (25)$$

where we denote

$$\tau_n(t) = \left(\frac{t - a_n}{t - a_n + \tau} \right)^p, \quad \sigma_n(t) = \left(\frac{b_n - t}{b_n - t + \sigma} \right)^p. \quad (26)$$

Here, the numbers $\eta_0, \eta_1, \eta_2 \in (0, 1)$ are chosen such that $\eta_0 + \eta_1 + \eta_2 = 1$ and $p_1 \eta_1 + p_2 \eta_2 = p$. Let us mention that if $p_1 = 5/2$, $p = 1$ and $p_2 = 1/2$, and $\eta_0 = \eta_1 = \eta_2 = 1/3$, then (p_1, p_2) and (η_0, η_1, η_2) satisfy previous two equalities. About the existence of such $(N+1)$ -tuple $(\eta_0, \eta_1, \dots, \eta_N)$ in a general case, we refer to [49].

Remark 8. A difference between assumptions of Theorems 5, 6, and 7 is that $e(t)$ in Theorems 5 and 6 is not necessarily periodic or bounded function as it is supposed in Theorem 7.

Now, we study an important class of second-order functional differential equations as a particular case of (1):

$$\begin{aligned} &(r(t) A(x'(t)))' + B(t, x(t), x'(t)) \\ &\quad + \lambda f(t) |x(\tau(t))|^{p_1} \operatorname{sgn}(x(\tau(t))) \\ &\quad + \mu g(t) |x(\sigma(t))|^{p_2} \operatorname{sgn}(x(\sigma(t))) = \rho e(t), \quad t \geq t_0, \end{aligned} \quad (27)$$

where $\tau(t) = t - \tau$, $\sigma(t) = t + \sigma$, and $\lambda, \mu, \rho, \tau, \sigma \geq 0$ and $p_1, p_2 > 0$. Using previous theorems, we are able to state the main consequences showing the parametrically excited oscillations in (27).

Corollary 9 (delay equation). Let assumptions (2), (3), (4), (14), and (15) hold. Let $g(t) \equiv 0$, $f(t) \geq 0$, and $f(t) \not\equiv 0$ on $[a_n - \tau, b_n]$, where $f \in C([t_0, \infty), \mathbb{R})$ is a periodic function with period T^* satisfying (16). Equation (27) is oscillatory in the following two cases: $p_1 = p$ and parameter λ is large enough; $p_1 > p$, $\lambda > 0$, $\rho > 0$, and at least one of parameters λ and ρ is large enough.

Corollary 10 (advanced equation). Let assumptions (2), (3), (4), (14), and (15) hold. Let $f(t) \equiv 0$, $g(t) \geq 0$, and $g(t) \not\equiv 0$ on $[a_n, b_n + \sigma]$, where $g \in C([t_0, \infty), \mathbb{R})$ is a periodic function with period T^* satisfying (17). Then (27) is oscillatory in the following two cases: $p_2 = p$ and parameter μ is large enough; $p_2 > p$, $\mu > 0$, $\rho > 0$, and at least one of parameters μ and ρ is large enough.

Corollary 11 (delay-advanced equation). Let assumptions (2), (3), (4), (14), and (15) hold, and $f(t)$ and $g(t)$ satisfy

$$\begin{aligned} f(t) &\geq 0, \quad f(t) \not\equiv 0 \\ g(t) &\geq 0, \quad g(t) \not\equiv 0 \\ &\text{on } [a_n - \tau, b_n + \sigma], \end{aligned} \quad (28)$$

where, additionally, $f, g, e \in C([t_0, \infty), \mathbb{R})$ are three periodic functions having a common period $T_* > 0$ such that (17) is fulfilled, where $e(t)$ is the forcing term in (1). Then (27) is oscillatory in the next two cases, where the number p is from (3): (1) (superlinear case) $p_1 > p$, $p_2 > p$, $\rho > 0$, and either parameter ρ is large enough or at least one of λ and μ is large enough; (2) (supersublinear case) $p_1 > p > p_2 > 0$, $\lambda > 0$, $\mu > 0$, $\rho > 0$, and at least one of parameters λ , μ , and ρ is large enough.

According to previous corollaries, we can derive the following examples.

Example 12 (delay case). Let $p \geq 1$, $\varepsilon \geq 0$, and $m \in \mathbb{N}$ be fixed and $\tau_m \in \mathbb{R}$, $0 \leq \tau_m < \pi/(4m)$. With the help of

Corollary 9, the following two different classes of quasilinear delay differential equations:

$$\begin{aligned} & \left(|x'(t)|^{p-1} x'(t) \right)' + \lambda \sin(mt) |x(t - \tau_m)|^{p+\varepsilon} \\ & \quad \times \operatorname{sgn}(x(t - \tau_m)) = -\rho \cos(2mt), \\ & \left(\frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)' + \lambda \cos(mt) |x(t - \tau_m)|^{1+\varepsilon} \\ & \quad \times \operatorname{sgn}(x(t - \tau_m)) = -\rho \cos(2mt), \end{aligned} \quad (29)$$

are oscillatory provided at least one of $\lambda > 0$ and $\rho > 0$ is large enough (the case $\rho = 0$ is possible if $\varepsilon = 0$). It is because, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} -\cos(2mt) & \begin{cases} \leq 0 & \text{on } \left[\frac{2n\pi}{m}, \frac{2n\pi}{m} + \frac{\pi}{4m} \right], \\ \geq 0 & \text{on } \left[\frac{2n\pi}{m} + \frac{\pi}{4m}, \frac{2n\pi}{m} + \frac{\pi}{2m} \right], \end{cases} \\ \sin(mt) \geq 0, \quad \cos(mt) \geq 0 & \text{on } \left[\frac{2n\pi}{m}, \frac{2n\pi}{m} + \frac{\pi}{2m} \right], \\ & \left[\frac{2n\pi}{m} + \tau_m + T_*, \frac{2n\pi}{m} + \frac{\pi}{4m} + T_* \right] \\ & = \left[\frac{(2n+2)\pi}{m} + \tau_m, \frac{(2n+2)\pi}{m} + \frac{\pi}{4m} \right], \\ & \left[\frac{2n\pi}{m} + \frac{\pi}{4m} + \tau_m + T_*, \frac{2n\pi}{m} + \frac{\pi}{2m} + T_* \right] \\ & = \left[\frac{(2n+2)\pi}{m} + \frac{\pi}{4m} + \tau_m, \frac{(2n+2)\pi}{m} + \frac{\pi}{2m} \right], \end{aligned} \quad (30)$$

where $T_* = 2\pi/m$ is the common period of the functions $\sin(mt)$ and $\cos(mt)$. Thus, in order to apply Corollary 9, we can choose $a_{2n-1} := 2n\pi/m + \tau_m$, $b_{2n-1} := 2n\pi/m + \pi/(4m)$, $a_{2n} := 2n\pi/m + \pi/(4m) + \tau_m$, and $b_{2n} := 2n\pi/m + \pi/(2m)$.

Example 13 (advanced case). Let $p \geq 1$, $\varepsilon \geq 0$, and $m \in \mathbb{N}$ be fixed and $\sigma_m \in \mathbb{R}$, $0 \leq \sigma_m < \pi/(4m)$. With the help of Corollary 10, the following two classes of quasilinear advanced differential equations:

$$\begin{aligned} & \left(|x'(t)|^{p-1} x'(t) \right)' + \mu \sin(mt) |x(t + \sigma_m)|^{p+\varepsilon} \\ & \quad \times \operatorname{sgn}(x(t + \sigma_m)) = -\rho \cos(2mt), \\ & \left(\frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)' + \mu \cos(mt) |x(t + \sigma_m)|^{1+\varepsilon} \\ & \quad \times \operatorname{sgn}(x(t + \sigma_m)) = -\rho \cos(2mt), \end{aligned} \quad (31)$$

are oscillatory provided at least one of $\mu > 0$ and $\rho > 0$ is large enough (the case $\rho = 0$ is possible if $\varepsilon = 0$). In order to apply Corollary 10, we can choose $a_{2n-1} := 2n\pi/m$, $b_{2n-1} := 2n\pi/m + \pi/(4m) - \sigma_m$, $a_{2n} := 2n\pi/m + \pi/(4m)$, and $b_{2n} := 2n\pi/m + \pi/(2m) - \sigma_m$.

Example 14 (delay-advanced case). Let $p \geq 1$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $m \in \mathbb{N}$ be fixed and $\tau_m \geq 0$ and $\sigma_m \geq 0$, $0 \leq \tau_m + \sigma_m < \pi/(4m)$. With the help of Corollary 11, the following class of quasilinear delay-advanced differential equations:

$$\begin{aligned} & \left(|x'(t)|^{p-1} x'(t) \right)' + \lambda \sin(mt) \\ & \quad \times |x(t - \tau_m)|^{p+\varepsilon_1} \operatorname{sgn}(x(t - \tau_m)) \\ & \quad + \mu \cos(mt) |x(t + \sigma_m)|^{p+\varepsilon_2} \\ & \quad \times \operatorname{sgn}(x(t + \sigma_m)) = -\rho \cos(2mt), \end{aligned} \quad (32)$$

is oscillatory provided either $\rho > 0$ is large enough or at least one of $\lambda > 0$ and $\mu > 0$ is large enough. In order to apply Corollary 11, we can choose $a_{2n-1} := 2n\pi/m + \tau_m$, $b_{2n-1} := 2n\pi/m + \pi/(4m) - \sigma_m$, $a_{2n} := 2n\pi/m + \pi/(4m) + \tau_m$, and $b_{2n} := 2n\pi/m + \pi/(2m) - \sigma_m$.

3. Application to Duffing Equations with Time Delay Feedback

Let $\lambda \geq 0$ denote the control gain parameter (often called “displacement feedback coefficient”), $\tau > 0$ the time delay, and $\rho \geq 0$ and $\omega > 0$ the amplitude and frequency of the external force, respectively. Let the function $\Phi = \Phi(t, u)$ that will appear in the delay feedback term $\Phi(t, x(t - \tau))$ satisfy the general condition

$$\begin{aligned} \Phi(t, u) \operatorname{sgn}(u) & \geq \phi_0 |u|^q \quad \forall t \geq t_0, \\ u & \neq 0 \quad \text{and some } q \geq 1, \phi_0 > 0. \end{aligned} \quad (33)$$

For instance, $\Phi(t, u) = u^{2m-1}$, $m \in \mathbb{N}$, or more general, $\Phi(t, u) = \sum_{k=1}^m \phi_k u^{2k-1}$, $\phi_k > 0$, $m \in \mathbb{N}$.

In this section, we consider the following large class of undamped possible nonautonomous and nonconservative Duffing equations without or with the general time delay feedback $\Phi(t, x(t - \tau))$:

$$\begin{aligned} & \left(|x'|^{p-1} x' \right)' + \omega_0^2 x + \frac{\mu_1 |x|^{r_1} \operatorname{sgn}(x)}{(\mu_2 + \mu_3 x^2)^{r_2}} \\ & \quad + \sum_{i=1}^m \beta_i(t) |x|^{\alpha_i-1} x + \lambda \Phi(t, x(t - \tau)) = \rho \cos(\omega t), \end{aligned} \quad (34)$$

where ω_0 is the natural frequency, $\mu_1 \geq 0$ is the density of the nonlinear potential (or rigidity coefficient), and μ_2, μ_3, r_1, r_2 are nonnegative constants, $\beta_i(t) \geq 0$ and $\alpha_i \geq 1$.

When $p = 1$, $\lambda = 0$, and $\beta_i(t) \equiv \beta_i = \text{const.}$, (34) contains many most important classes of undamped autonomous Duffing oscillators such as the following:

(i) the *strongly nonlinear Duffing oscillator with smooth odd nonlinearity* is given in (34) provided $\mu_1 = 0$ and $\alpha_i = 2i + 1$; let us recall some of its known particular cases:

(a) the classic Duffing oscillator $x'' + \omega_0^2 x + \beta x^3 = 0$ has been recently studied in the searching of

solitary wave solutions of classic and generalized Zakharov equations of plasma physics (see [16]) and of nonlinear Schrödinger equation (see [17]); also, it is strongly connected with the Jacobi elliptic equation (see [18]);

- (b) the cubic-quintic oscillator $x'' + \omega_0^2 x + \beta_1 x^3 + \beta_2 x^5 = 0$ is used as a model for the nonlinear dynamics of a slender elastica (see [19]) in nonlinear wave systems (see [20]) for the propagation of a short electromagnetic pulse in a nonlinear medium (see [21]) and in the unimodal Duffing temporal problem (see [22]);
 - (c) the cubic truly nonlinear oscillator $x'' + \beta x^3 = 0$ models the motion of a ball bearing that oscillates in a glass tube that is bent into a curve (see [23]) as well as the motion of a mass attached to identical stretched elastic wires (see [24]);
 - (d) the nonhomogeneous Duffing oscillator $x'' + \omega_0^2 x + \beta x^3 = \rho \cos(\omega t)$ describes various forced vibrations of beams, springs with nonlinear stiffness, cables, plates, shells, and optical fibres in electrical circuits, in nonlinear isolators, and so forth (see, for instance, [25, 26]);
- (ii) the *general Duffing-harmonic oscillator* (with rational or irrational nonlinear restoring-force) is given in (30) if $\mu_1 \neq 0$, $\beta_i = 0$, and $\rho = 0$; the most known subclasses of these oscillators are
- (a) the classic Duffing-harmonic oscillator $x'' + (\mu_1 x^3 / (\mu_2 + \mu_3 x^2)) = 0$ which models many conservative nonlinear oscillatory systems; see [27];
 - (b) the relativistic harmonic oscillator $x'' + (\mu_1 x / \sqrt{1 + x^2}) = 0$; see [28];
 - (c) the nonlinear oscillator $x'' + x - (\mu_1 x / \sqrt{1 + x^2}) = 0$, $\mu_1 \in [0, 1]$, which is typified as a mass attached to a stretched elastic wire; see [29, 30];
 - (d) the nonlinear oscillator $x'' + (\mu_1 x / (\sqrt{1 + x^2})^3) = 0$ which presents nonlinear oscillations of a punctual charge in the electric field of charged ring; see [31].

Finding several explicit forms of periodic approximate solutions for these oscillators has been intensively studied last years by many authors; see, for instance, [28, 30, 32–37] and also the references therein.

When $\lambda \neq 0$ and linear time delay feedback $\Phi(t, x(t-\tau)) = x(t-\tau)$, the following topics have been studied for various types of Duffing oscillators with time delayed feedback: in [38] authors constructed a low-order approximate solution under weak feedback gain parameter; about the low- and high-order approximations see also [39]; in [40] with $\rho = 0$, the Hopf bifurcation diagrams have been explored for the approximate periodic solutions (amplitude versus time delay τ and feedback gain λ versus time delay τ); moreover, in [41]

authors made an analysis on the effect of the control gain and time delay parameters on the amplitude of approximate period solution from the theoretical and numerical points of view; see also [42]; in [43] authors studied the chaotic behaviour with respect to gains and time delay parameters; see also [44].

Equations under time delay control such as (34) (especially with damped term) are used as a model for various controlled physical, mechanical, and engineering systems with time delays; see, for instance, [39, 45–48] and the references therein.

Here, (34) contains very general nonlinear time delay feedback $\Phi(t, x(t-\tau))$ with Φ satisfying (33) and the linear time delay feedback $x(t-\tau)$ is only a particular case of it, and, to the best of our knowledge, the previous topics are not considered for (34), as yet. Moreover, with such an Φ , the oscillations of (34) can be taken under a doubt even with the linear time delay feedback (see the nature of the approximations given in [38, 39]). Hence, we can pose the following question: under what conditions on equation's parameters, (34) is a nonlinear oscillator, that is, possesses only oscillatory solutions? An answer is given in the next result as an easy consequence of the parametrically excited oscillations by Theorem 5.

Theorem 15. *Let $\tau \in (0, \pi/\omega)$ and (33) hold. Equation (34) is oscillatory in the next two cases:*

- (i) $q = p$ and λ is large enough;
- (ii) $q > p$, $\lambda > 0$, $\rho > 0$, and at least one of λ and ρ is large enough.

Proof. Let $r(t) \equiv 1$, $A(v) = |v|^{p-1}$, $F(t, u) = \Phi(t, u)$, $G(t, u) \equiv 0$, $e(t) = \cos(\omega t)$, and

$$B(t, u, v) = \omega_0^2 u + \frac{\mu_1 |u|^{r_1} \operatorname{sgn}(u)}{(\mu_2 + \mu_3 u^2)^{r_2}} + \sum_{i=1}^m \beta_i(t) |u|^{\alpha_i-1} u. \quad (35)$$

It is easy to check that all assumptions of Theorem 5 are fulfilled with respect to the sequence $a_n = -\pi/2\omega + n\pi/\omega + \tau$ and $b_n = \pi/2\omega + n\pi/\omega + \tau$, where $a_n < b_n$ since it is supposed that $\tau < \pi/\omega$. Hence, Theorem 5 proves this theorem. $\square$

Remark 16. Even in the linear forced case ($e(t) \neq 0$), it is not easy to establish the oscillations of all solutions, since the oscillation and nonoscillation can occur simultaneously. The most simple and important example for the coincidence of oscillation and nonoscillation is the following linear forced differential equation: $x'' + (2/t)x' + x = 2/t$, $t > 0$, that allows an oscillatory solution $x_1(t) = (3 \sin t)/t + 2/t$ and a nonoscillatory solution $x_2(t) = 2/t$. This is not possible in the linear case with $e(t) \equiv 0$, because of Sturm's separation theorem.

4. Parametrically Excited Oscillations and Well-Known Oscillation Criteria

In this section, we would like to draw the reader's attention to the fact that the parametrically excited oscillations have

been already appearing in some published papers on the oscillation of functional differential equations, but only in some examples illustrating certain main oscillation criteria. However, with the help of our main results in which the parametrically excited oscillations are studied in a general setting, the equations from these examples are replaced with general ones also having parameters λ and μ .

In [1] (see also [2, Example 3.1] with $\tau = 0$, [3, Example 3.1], and [4, Section 3]), the author considers the oscillation of the second-order delay differential equation:

$$x''(t) + f(t) |x(\tau(t))|^\gamma \operatorname{sgn} x(\tau(t)) = e(t), \quad (36)$$

in the linear case ($\gamma = 1$) and the superlinear ($\gamma > 1$). In the linear case (analogously for the superlinear case see [1, Theorem 2]), the author proved the following oscillation criterion. In what follows, we denote

$$D(a, b) = \{u \in C^1([a, b], \mathbb{R}) : u(t) \neq 0, u(a) = u(b) = 0\}. \quad (37)$$

Theorem 17 ([1, Theorem 1]). *Suppose that, for any $T \geq 0$, there exist constants a_1, b_1, a_2, b_2 such that $T \leq a_1 < b_1$, $T \leq a_2 < b_2$, and $f(t) \geq 0$ on $[\tau(a_1), b_1] \cup [\tau(a_2), b_2]$, $e(t) \leq 0$ on $[\tau(a_1), b_1]$, and $e(t) \geq 0$ on $[\tau(a_2), b_2]$. If there exists $u \in D(a_i, b_i)$, $i = 1, 2$, such that*

$$\int_{a_i}^{b_i} \left[u^2(t) f(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - (u'(t))^2 \right] dt \geq 0, \quad (38)$$

then (36) with $\gamma = 1$ is oscillatory.

Previous criterion has been applied on the following particular equation:

$$\begin{aligned} x''(t) + \lambda \sin(t) \left| x\left(t - \frac{\pi}{4}\right) \right|^\gamma \\ \times \operatorname{sgn} x\left(t - \frac{\pi}{4}\right) = \cos(t), \quad t \geq 0, \end{aligned} \quad (39)$$

where $\lambda \geq 0$ and $\gamma = 1$. Applying Theorem 17 to (39), the author proved that (39) is oscillatory provided the following inequality:

$$\lambda \int_{a_i}^{b_i} \sin^2(2t) \cos^2(2t) \sin(t) \frac{t - a_i}{t - a_i + \pi/4} dt \geq \frac{\pi}{2}, \quad (40)$$

holds for sufficiently large λ . Thus, the oscillation of (39) is excited by the large enough parameter λ . However, according to Theorems 5 and 6, we are able to show that the next parametric equation that corresponds to general equation (36)

$$x''(t) + \lambda f(t) |x(\tau(t))|^\gamma \operatorname{sgn} x(\tau(t)) = e(t) \quad (41)$$

is oscillatory provided λ is large enough, where $p_1 = p_2 = \gamma$, $\mu = 0$, and $\rho = 1$.

Next, in [5] (see also [6–8]), the authors consider the oscillation of the following class of second-order differential equations with delay and advanced arguments:

$$\begin{aligned} (r(t) x'(t))' + f(t) |x(\tau(t))|^{p_1} \operatorname{sgn} x(\tau(t)) \\ + g(t) |x(\sigma(t))|^{p_2} \operatorname{sgn} x(\sigma(t)) = e(t), \quad t \geq 0, \end{aligned} \quad (42)$$

where $p_1, p_2 \geq 1$. When $p_1 = p_2 = 1$ the authors prove the following result (for other cases see [5, Theorems 3.2, 3.3, and 3.4]).

Theorem 18 ([5, Theorem 3.1]). *Suppose that, for any $T \geq 0$, there exist intervals $[\tau(a_1), b_1]$, $[\tau(a_2), b_2]$, $[c_1, \sigma(d_1)]$, and $[c_2, \sigma(d_2)]$ contained in $[T, \infty)$ such that $a_1 < b_1$, $a_2 < b_2$, $c_1 < d_1$, $c_2 < d_2$, and*

$$\begin{aligned} f(t) &\geq 0 \quad \text{on } [\tau(a_1), b_1] \cup [\tau(a_2), b_2], \\ g(t) &\geq 0 \quad \text{on } [c_1, \sigma(d_1)] \cup [c_2, \sigma(d_2)], \\ e(t) &\leq 0 \quad \text{on } [\tau(a_1), b_1] \cup [c_1, \sigma(d_1)], \\ e(t) &\geq 0 \quad \text{on } [\tau(a_2), b_2] \cup [c_2, \sigma(d_2)], \end{aligned} \quad (43)$$

and $c_i = \tau(a_i)$, $d_i = a_i$, and $b_i = \sigma(d_i)$, $i = 1, 2$. If there exist $u_1 \in D(a_i, b_i)$ and $u_2 \in D(c_i, d_i)$ such that either

$$\int_{a_i}^{b_i} \left[u_1^2(t) f(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - (u_1'(t))^2 r(t) \right] dt \geq 0 \quad (44)$$

or

$$\int_{c_i}^{d_i} \left[u_2^2(t) g(t) \frac{\sigma(d_i) - \sigma(t)}{\sigma(d_i) - t} - (u_2'(t))^2 r(t) \right] dt \geq 0, \quad (45)$$

for $i = 1, 2$, then (42) with $p_1 = p_2 = 1$ is oscillatory.

As a consequence of this result, it has been concluded that the particular equation

$$\begin{aligned} (r(t) x'(t))' + \lambda \sin(t) x\left(t - \frac{\pi}{12}\right) \\ + \mu \cos(t) x\left(t + \frac{\pi}{6}\right) = \cos(2t), \quad t \geq 0, \end{aligned} \quad (46)$$

is oscillatory provided either λ or μ is large enough. However, by following Theorems 5 and 6, one can obtain the same conclusion for the following general equation associated with (42):

$$\begin{aligned} (r(t) x'(t))' + \lambda f(t) |x(\tau(t))|^{p_1} \operatorname{sgn} x(\tau(t)) \\ + \mu g(t) |x(\sigma(t))|^{p_2} \operatorname{sgn} x(\sigma(t)) = e(t). \end{aligned} \quad (47)$$

Related observation can be done with [8, Example 3.3] and [9, Example 2.1], where the quasilinear second-order functional differential equations have been considered. It is left to the reader.

5. Some Open Questions and Comments

In this section, we discuss some problems related to our main results that are not studied here.

(1) *Quasiperiodic Case.* In the theory of nonlinear oscillators, a particularly important case occurs when the periodic coefficients in the oscillator do not have any common period. It is called the quasiperiodic (or two-frequency) nonlinear

oscillator and studied, for instance, in [50–52]. Since in Theorems 5, 6, and 7 we assume that the corresponding periodic functions have a common period, it is natural to pose the next question.

Open Question 1. Is it possible to derive sufficient conditions for the oscillation of (27) in the case when $f(t)$ and $g(t)$ (resp., $f(t)$, $g(t)$, and $h(t)$) are two (resp., three) periodic functions not having a common period?

(2) *Equation with More Functional Arguments.* Next, regarding some second-order functional differential equations considered in the references of this paper, more than two nonlinear functional terms are appearing and, therefore, instead of main equation (1) and corresponding particular equation (27) considered in Theorems 5, 6, and 7, we suggest the following classes of equations:

$$\begin{aligned} & \left(r(t) A(x'(t)) \right)' + B(t, x(t), x'(t)) \\ & + \sum_{k=1}^{m_1} \lambda_k F_k(t, x(\tau_k(t))) \\ & + \sum_{k=1}^{m_2} \mu_k G_k(t, x(\sigma_k(t))) = \rho e(t), \end{aligned} \quad (48)$$

where $0 \leq \tau_k(t) \leq t$, $\lim_{t \rightarrow \infty} \tau_k(t) = \infty$, $\sigma_k(t) \geq t$, $m_1, m_2 \in \mathbb{N}$, and

$$\begin{aligned} & \left(r(t) A(x'(t)) \right)' + B(t, x(t), x'(t)) \\ & + \sum_{k=1}^{m_1} \lambda_k f_k(t) |x(t - \tau_k)|^{p_k} \operatorname{sgn}(x(t - \tau_k)) \\ & + \sum_{k=1}^{m_2} \mu_k g_k(t) |x(t + \sigma_k)|^{q_k} \operatorname{sgn}(x(t + \sigma_k)) = \rho e(t), \end{aligned} \quad (49)$$

where $\lambda_k, \mu_k, \rho, \tau_k, \sigma_k \geq 0$ and $p_k, q_k > 0$.

Comment. We suggest the reader to enlarge the main results of this paper to (48) and (49).

(3) *Damped Duffing Equation.* In the application, the Duffing equation (34) is often appearing with the linear damped term $x'(t)$; that is,

$$x'' + d_0 x' + \omega_0^2 x + \beta x^3 + \lambda \Phi(x(t - \tau)) = \rho \cos(\omega t), \quad (50)$$

where d_0 is the damped coefficient which can, in an active way, influence various behaviours of (50). Since $B(t, x(t), x'(t)) = d_0 x'(t)$ does not satisfy the required assumption (4), we are not able to apply our main results to (50). Hence we pose the following question.

Open Question 2. Is it possible to obtain the parametrically excited oscillation for (1) in the case when the damped term $B(t, u, v)$ satisfies a larger condition than (4) in which the linear damped term $\beta x'(t)$ is especially included?

(4) *Functional Argument in Damped Term.* In a class of Duffing equations, we have two time delayed feedback and, hence,

besides the control gain parameter λ_1 another parameter λ_2 appears, the so-called velocity gain parameter. Hence, instead of (34) one can consider

$$\begin{aligned} & x'' + d_0 x' + \omega_0^2 x + \beta x^3 + \lambda_1 x(t - \tau) \\ & + \lambda_2 x'(t - \tau) = \rho \cos(\omega t). \end{aligned} \quad (51)$$

Therefore, we suggest the following problem for further study.

Open Question 3. Is it possible to obtain the parametrically excited oscillation for the following more general functional differential equation than (1) in which the functional argument appears in the damped term too, as follows:

$$\begin{aligned} & \left(r(t) A(x'(t)) \right)' + B(t, x(t), x'(t)) + \lambda_1 F(t, x(\tau(t))) \\ & + \lambda_2 H(t, x'(\tau(t))) = \rho e(t), \quad t \geq t_0, \end{aligned} \quad (52)$$

or

$$\begin{aligned} & \left(r(t) A(x'(t)) \right)' + B(t, x(t), x'(t)) + \mu_1 G(t, x(\sigma(t))) \\ & + \mu_2 H(t, x'(\sigma(t))) = \rho e(t), \quad t \geq t_0? \end{aligned} \quad (53)$$

About known oscillation criteria for the second-order functional differential equations having the functional argument in the damped term, we refer the reader to, for instance, [53] and the references therein.

6. Proofs of Main Results

The proof of Lemma 1 is based on the following three steps: two working forms of condition (6) (see Lemmas 19 and 20), the existence of an explosive solution of a suitable Riccati differential inequality (see Proposition 22), and a comparison principle (see Proposition 24).

Lemma 19 (a necessary condition to (6)). *Let $0 < r(t) \leq r_0$ on $[t_0, \infty)$. If assumption (6) is fulfilled, then there is a positive real number ε such that*

$$\frac{1}{\pi_*} \int_J \min \left\{ \frac{P}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\} dt \geq 1, \quad (54)$$

for all $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and $\rho \geq \rho_0$ and some $(\lambda_0, \mu_0, \rho_0) \in \mathbb{R}_+^3$.

Proof. Since $0 < r(t) \leq r_0$ for $t \geq t_0$, we conclude that, for

$$\varepsilon = \left(\frac{P}{r_0^{\gamma-1} k(\lambda, \mu, \rho) \max_{t \in J} Q(t)} \right)^{1/\gamma}, \quad (\lambda, \mu, \rho) \in \mathbb{R}_+^3, \quad (55)$$

it holds that $p/(\varepsilon r(t))^{\gamma-1} \geq p/(\varepsilon r_0)^{\gamma-1} = \varepsilon k(\lambda, \mu, \rho) \max_{t \in J} Q(t) \geq \varepsilon k(\lambda, \mu, \rho) Q(t)$, $t \in J$, and hence

$$\begin{aligned} & \int_J \min \left\{ \frac{P}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\} dt \\ & = \varepsilon k(\lambda, \mu, \rho) \int_J Q(t) dt. \end{aligned} \quad (56)$$

On the other hand, from (6) we observe

$$\frac{1}{\pi_*} \int_J Q(t) dt \geq \frac{r_0^{1-(1/\gamma)}}{p^{1/\gamma} [k(\lambda, \mu, \rho)]^{1-(1/\gamma)}} \left(\max_{t \in \bar{J}} Q(t) \right)^{1/\gamma}, \quad (57)$$

which, together with (55) and (56), gives

$$\begin{aligned} & \frac{1}{\pi_*} \int_J \min \left\{ \frac{p}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\} dt \\ &= \varepsilon k(\lambda, \mu, \rho) \frac{1}{\pi_*} \int_J Q(t) dt \\ &\geq \varepsilon \frac{r_0^{1-(1/\gamma)}}{p^{1/\gamma}} [k(\lambda, \mu, \rho)]^{1/\gamma} \left(\max_{t \in \bar{J}} Q(t) \right)^{1/\gamma} = 1, \end{aligned} \quad (58)$$

for all $n \geq n_0$, $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and $\rho \geq \rho_0$. It proves this lemma. $\square$

Lemma 20 (an equivalent condition to (54)). *Assumption (54) is fulfilled if and only if there is a real number $\varepsilon > 0$ and a continuous function $K(t) \geq 0$, $t \in \bar{J}$, such that*

$$\begin{aligned} c_0 &:= \int_J K(t) dt > 0, \quad \frac{K(t)}{c_0} \leq \frac{1}{\pi_*} \\ &\times \min \left\{ \frac{p}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\}, \end{aligned} \quad (59)$$

for all $t \in J$, $\lambda \geq \lambda_0$, $\mu \geq \mu_0$, and $\rho \geq \rho_0$ and some $(\lambda_0, \mu_0, \rho_0) \in \mathbb{R}_+^3$.

Proof. This proof is very elementary. Indeed, if (54) holds, then the function $K(t)$ and number c_0 , defined by

$$\begin{aligned} K(t) &= \frac{1}{\pi_*} \min \left\{ \frac{p}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\}, \\ c_0 &:= \int_J K(t) dt, \end{aligned} \quad (60)$$

obviously satisfy $c_0 \geq 1$ and $K(t)/c_0 \leq K(t) = (1/\pi_*) \min\{p/(\varepsilon r(t))^{\gamma-1}, \varepsilon k(\lambda, \mu, \rho) Q(t)\}$ which shows (59). Conversely, if (59) holds, then, integrating both sides of the second inequality in (59), we obtain

$$\int_J \frac{1}{\pi_*} \min \left\{ \frac{p}{(\varepsilon r(t))^{\gamma-1}}, \varepsilon k(\lambda, \mu, \rho) Q(t) \right\} dt \geq \int_J \frac{K(t)}{c_0} dt = 1, \quad (61)$$

which shows (54). $\square$

In conclusion, according to previous two lemmas, we see that supposed condition (6) implies (59), which plays an important role in the proof of the main results.

The second step in the proof of Lemma 1 is to prove the existence of a function $\omega(t)$ which blows up in the finite time and satisfies a generalized Riccati differential lower inequality; we briefly present the existence and properties

of the so-called generalized tangent type function. In what follows, let π_* be a positive real number defined in (3). Let us remark that $\alpha(s) = s^\gamma$, $\gamma > 1$, implies $\pi_* = (2\pi)/(\gamma \sin(\pi/\gamma))$, see, for instance, [54], and obviously for $\gamma = 2$ we have $\pi_* = \pi$.

Lemma 21. *Let $\alpha : [0, \infty) \rightarrow [0, \infty)$ be a continuous function such that*

$$\int_0^\infty \frac{d\tau}{1 + \alpha(\tau)} < \infty. \quad (62)$$

Then there is a real number $\pi_ > 0$ and a function $z = z(s)$, $z \in C^1((-\pi_*/2, \pi_*/2), \mathbb{R})$, such that*

$$\begin{aligned} \frac{dz}{ds} &= 1 + \alpha(|z(s)|), \quad s \in \left(-\frac{\pi_*}{2}, \frac{\pi_*}{2}\right), \\ z(0) &= 0. \end{aligned} \quad (63)$$

Moreover, $z(s)$ is increasing and odd:

$$\begin{aligned} \lim_{s \rightarrow \pi_*/2} z(s) &= \infty, \quad \pi_* = \frac{2\pi}{\gamma \sin(\pi/\gamma)} \quad \text{for } \alpha(s) = s^\gamma, \\ \gamma &> 1. \end{aligned} \quad (64)$$

In particular, for $\alpha(s) = s^2$, one can take $z(s) = \tan(s)$ and $\pi_ = \pi$.*

Proof. Let $Z = Z(t)$, $t \in \mathbb{R}$, be a function defined by

$$Z(t) = \int_0^t \frac{1}{1 + \alpha(|\tau|)} d\tau, \quad t \in \mathbb{R}. \quad (65)$$

The function $Z(t)$ is well defined since $\alpha(s)$ is positive and continuous on $[0, \infty)$, $Z(t)$ is increasing and odd function, and

$$\begin{aligned} \frac{dZ}{dt} &= \frac{1}{1 + \alpha(|t|)}, \quad t \in \mathbb{R}, \\ Z(0) &= 0, \quad Z \in C^1(\mathbb{R}, \mathbb{R}). \end{aligned} \quad (66)$$

Moreover, because of (62), there is a real number $\pi_* > 0$ such that

$$\frac{\pi_*}{2} = \int_0^\infty \frac{d\tau}{1 + \alpha(\tau)}. \quad (67)$$

Thus, $Z : \mathbb{R} \rightarrow (-\pi_*/2, \pi_*/2)$ and there exists an inverse function $Z^{-1} = Z^{-1}(s)$ of the original function $Z = Z(t)$ and $Z^{-1} : (-\pi_*/2, \pi_*/2) \rightarrow \mathbb{R}$. Also, from $Z(Z^{-1}(s)) = s$ and $dZ/dt \neq 0$ on $\mathbb{R}$, we also derive that $dZ^{-1}/ds \neq 0$ on its domain $(-\pi_*/2, \pi_*/2)$ and

$$\frac{dZ}{dt}(Z^{-1}(s)) = \frac{1}{(dZ^{-1}/ds)}, \quad s \in \left(-\frac{\pi_*}{2}, \frac{\pi_*}{2}\right). \quad (68)$$

Putting $t = Z^{-1}(s)$ for $s \in (-\pi_*/2, \pi_*/2)$ into (66) and using (68) we easily obtain

$$\begin{aligned} \frac{dZ^{-1}}{ds} &= 1 + \alpha(|Z^{-1}(s)|), \quad s \in \left(-\frac{\pi_*}{2}, \frac{\pi_*}{2}\right), \\ Z^{-1}(0) &= 0, \quad Z^{-1} \in C^1\left(\left(-\frac{\pi_*}{2}, \frac{\pi_*}{2}\right), \mathbb{R}\right). \end{aligned} \quad (69)$$

Moreover, from (67) we have $\lim_{s \rightarrow \pi_*/2} Z^{-1}(s) = Z^{-1}(\lim_{t \rightarrow \infty} Z(t)) = \lim_{t \rightarrow \infty} Z^{-1}Z(t) = \lim_{t \rightarrow \infty} t = \infty$. Thus, if we set $z(s) := Z^{-1}(s)$, then previous two statements and (67) prove this lemma. $\square$

Next, we prove the main result of this section.

Proposition 22. *Let (2) and (6) hold, where $J = (a, b)$. Let $\varepsilon > 0$ be a real number and let $K(t) \geq 0$, $t \in [a, b]$, be a continuous function, both obtained in Lemma 20. Let π_* be from (3) and c_0 from (59), and let $R_a \in \mathbb{R}$ be an arbitrary real number. If $z = z(s)$ is the generalized tangens function defined in (63) and $V(t)$ is a function defined by*

$$V(t) = \frac{\pi_*}{c_0} \int_a^t K(\tau) d\tau + z^{-1}(R_a), \quad t \in [a, b], \quad (70)$$

then there is a $T_a^* \in [a, b]$ such that

$$V(T_a^*) = \frac{\pi_*}{2}, \quad V([a, T_a^*)) \subset \left(-\frac{\pi_*}{2}, \frac{\pi_*}{2}\right). \quad (71)$$

Moreover, for a function $\omega(t)$ defined by

$$\omega(t) = z(V(t)), \quad t \in [a, T_a^*), \quad (72)$$

one has $\omega(a) = R_a$, $\lim_{t \rightarrow T_a^*} \omega(t) = \infty$, and

$$\begin{aligned} \frac{d\omega}{dt} &\leq \frac{p}{(\varepsilon r(t))^{\gamma-1}} \alpha(|\omega(t)|) \\ &\quad + \varepsilon k(\lambda, \mu, \rho) Q(t), \quad t \in (a, T_a^*), \end{aligned} \quad (73)$$

where the numbers p and γ are from (3) and the functions $k(\lambda, \mu, \rho)$ and $Q(t)$ are from (6).

Proof. Under assumptions (2) and (6) and because of Lemmas 19 and 20, we obtain $\varepsilon > 0$ and $K(t) > 0$, $t \in [a, b]$, satisfying inequality (59).

Next, since $z^{-1}(R_a) \in (-\pi_*/2, \pi_*/2)$ (see Lemma 21), from (70) we directly obtain

$$V(a) = z^{-1}(R_a) < \frac{\pi_*}{2}, \quad V(b) = \pi_* + z^{-1}(R_a) > \frac{\pi_*}{2}. \quad (74)$$

Since $K \in C([a, b], [0, \infty))$ we obtain $V \in C([a, b], \mathbb{R}) \cap C^1((a, b), \mathbb{R})$, and from (74) we observe that there exist numbers $T_a^* \in (a, b)$ such that $V(T_a^*) = \pi_*/2$. Also, $K(t)/c_0 \geq 0$ gives $V([a, T_a^*)) \subset (-\pi_*/2, \pi_*/2)$, which proves statement (71). Moreover, it together with Lemma 21 and (72) proves that

$$\lim_{t \rightarrow T_a^*} \omega(t) = \lim_{t \rightarrow T_a^*} z(V(t)) = z\left(\frac{\pi_*}{2}\right) = \infty. \quad (75)$$

Next, according to (59), (63), and (72), we make the following calculation on the interval $[a, T_a^*)$:

$$\begin{aligned} \omega'(t) &= z'(V(t)) V'(t) = [1 + \alpha(|z(V(t))|)] \frac{\pi_*}{c_0} K(t) \\ &= [1 + \alpha(|\omega(t)|)] \frac{\pi_*}{c_0} K(t) \\ &\leq \frac{p}{(\varepsilon r(t))^{\gamma-1}} \alpha(|\omega(t)|) + \varepsilon k(\lambda, \mu, \rho) Q(t). \end{aligned} \quad (76)$$

Thus, all assertions of this proposition are proved. $\square$

Remark 23. In the proof of the main result, the number R_a is determined by $R_a = \bar{\omega}(a)$, where $\bar{\omega}(t)$ denotes a function associated with a nonoscillatory solution and it is given by (84) below.

The third step in the proof of Lemma 1 is to show the following pointwise comparison principle for the functions $\underline{\omega}$ and $\bar{\omega}$ satisfying, respectively, the lower and upper differential inequalities (73) and

$$\frac{d\bar{\omega}}{dt} \geq \frac{p}{(\varepsilon r(t))^{\gamma-1}} \alpha(|\bar{\omega}(t)|) + \varepsilon k(\lambda, \mu, \rho) Q(t), \quad t \in (a, b). \quad (77)$$

Proposition 24. *Let $[a, b] \subset [t_0, \infty)$ be an arbitrary interval. One supposes that all coefficients of Riccati differential inequalities (73) and (77) are continuous and strictly positive functions. Let $\underline{\omega}, \bar{\omega} \in C^1([a, b], \mathbb{R})$ be two functions satisfying, respectively, (73) and (77) on the interval $[a, b]$. Then*

$$\underline{\omega}(a) \leq \bar{\omega}(a) \text{ implies } \underline{\omega}(t) \leq \bar{\omega}(t) \quad \forall t \in [a, b]. \quad (78)$$

Proof. Let $H(t, u)$ be a function defined by

$$\begin{aligned} H(t, u) &= \frac{p}{(\varepsilon r(t))^{\gamma-1}} \alpha(|u|) + \varepsilon k(\lambda, \mu, \rho) Q(t), \\ t &\in [a, b], \quad u \in \mathbb{R}. \end{aligned} \quad (79)$$

Let $I \subset [a, b]$ and $M > 0$ be arbitrary. For any two u_1, u_2 , $-M \leq u_1 < u_2 \leq M$, let I_{12} be an interval defined by $I_{12} = (\min\{|u_1|, |u_2|\}, \max\{|u_1|, |u_2|\})$. Since $\alpha(s)$ is a C^1 -function on $[0, \infty)$, we know by the Lagrange mean value theorem applied on I_{12} that there is a $\xi \in I_{12}$ such that

$$\begin{aligned} \frac{\alpha(|u_2|) - \alpha(|u_1|)}{u_2 - u_1} &\leq \frac{|\alpha'(\xi)|}{u_2 - u_1} \\ &= |\alpha'(\xi)| \frac{||u_2| - |u_1||}{u_2 - u_1} \leq |\alpha'(\xi)| \\ &\leq \max_{s \in I_{12}} |\alpha'(s)|, \end{aligned} \quad (80)$$

since $||u_2| - |u_1|| \leq u_2 - u_1$. Hence for any $t \in I$ and u_1, u_2 , $-M \leq u_1 < u_2 \leq M$, we have

$$\begin{aligned} \frac{H(t, u_2) - H(t, u_1)}{u_2 - u_1} &= \rho_0(t) \frac{\alpha(|u_2|) - \alpha(|u_1|)}{u_2 - u_1} \\ &\leq \rho_0(t) \max_{s \in I_{12}} |\alpha'(s)| := L_0(t). \end{aligned} \quad (81)$$

Thus, the function $H(t, u)$ from (79) satisfies required condition of [55, Lemma 19] and, applying it to (73) and (77), we prove this proposition. $\square$

Proof of Lemma 1. On the contrary, let $x(t)$ be a solution of (1) such that

$$x(t) \neq 0 \text{ on } (\tau(\tau(a)), \sigma(\sigma(d))); \quad (82)$$

that is, $x(t) > 0$ on $(\tau(\tau(a)), \sigma(\sigma(d)))$ or $x(t) < 0$ on $(\tau(\tau(a)), \sigma(\sigma(d)))$ since $x(t)$ is a continuous function on $[t_0, \infty)$. Let, for instance,

$$x(t) > 0 \quad \text{on } (\tau(\tau(a)), \sigma(\sigma(d))). \quad (83)$$

Another case can be analogously treated; let us see the comment at the end of this proof. In particular from (83) we have $x(t) > 0$ on $(\tau(\tau(a)), \sigma(\sigma(b)))$ which implies (since $\tau(t)$ and $\sigma(t)$ are increasing functions) $x(s) > 0$ for all $s \in (\tau(a), \sigma(b)) \cup (\tau(\tau(a)), \tau(\sigma(b))) \cup (\sigma(\tau(a)), \sigma(\sigma(b)))$, which yields $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ on $(\tau(a), \sigma(b))$. Hence, by assumption (7), we may use inequality (5) on the interval (a, b) .

Firstly, we show that the following classic Riccati transformation of $x(t)$:

$$\bar{\omega}(t) = -\varepsilon \frac{r(t) A(x'(t))}{|x(t)|^{p-1} x(t)}, \quad t \in (a, b), \quad \varepsilon > 0, \quad (84)$$

satisfies upper Riccati differential inequality (77). Let us remark that from (1) we have in particular

$$\begin{aligned} -\left(r(t) A(x'(t))\right)' &= B(t, x(t), x'(t)) + \lambda F(t, x(\tau(t))) \\ &\quad + \mu G(t, x(\sigma(t))) - \rho e(t), \quad t \geq t_0. \end{aligned} \quad (85)$$

Taking the first derivative on both sides of (84) and using assumptions (3), (4), and (5) as well as equality (85) and $(|x(t)|^{p-1} x(t))' = p|x(t)|^{p-1} x'(t)$, we obtain

$$\begin{aligned} \frac{d\bar{\omega}}{dt} &= \varepsilon p r(t) \frac{A(x'(t)) x'(t)}{|x(t)|^{p+1}} \\ &\quad - \varepsilon \frac{1}{|x(t)|^{p-1} x(t)} \left(r(t) A(x'(t)) \right)' \\ &= \varepsilon p r(t) \frac{A(x'(t)) x'(t)}{|x(t)|^{p+1}} + \frac{\varepsilon}{|x(t)|^{p-1} x(t)} \\ &\quad \times [\lambda B(t, x(t), x'(t)) + F(t, x(\tau(t))) \\ &\quad + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ &\geq \varepsilon p r(t) \frac{A(x'(t)) x'(t)}{|x(t)|^{p+1}} + \frac{\varepsilon}{|x(t)|^{p-1} x(t)} \\ &\quad \times [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ &\geq \varepsilon p r(t) \alpha \left(\frac{|A(x'(t))|}{|x(t)|^p} \right) + \varepsilon k(\lambda, \mu, \rho) Q(t) \\ &= \varepsilon p r(t) \alpha \left(\frac{|\bar{\omega}(t)|}{\varepsilon r(t)} \right) + \varepsilon k(\lambda, \mu, \rho) Q(t) \\ &\geq \frac{p}{(\varepsilon r(t))^{\gamma-1}} \alpha(|\bar{\omega}(t)|) + \varepsilon k(\lambda, \mu, \rho) Q(t), \quad t \in (a, b). \end{aligned} \quad (86)$$

Thus, according to inequality (5), it is shown that if $x(t)$ is a solution of (1) which satisfies (83), then the function $\bar{\omega}(t)$ defined by (84) satisfies the Riccati differential inequality (77) and $\bar{\omega} \in C((a, b), \mathbb{R})$. On the other hand, let R_a be a real number defined by $R_a = \bar{\omega}(a)$. According to (6) and Lemma 19 we obtain (54) which together with Lemma 20 ensures that we may use Proposition 22 for such chosen real number R_a . Hence, we obtain a function $\underline{\omega}(t)$ defined by (72) which satisfies the lower Riccati differential inequality (73) on $[a, T_a^*)$, $T_a^* \in (a, b)$, such that $\underline{\omega}(a) = R_a$ and $\lim_{t \rightarrow T_a^*} \underline{\omega}(t) = \infty$. Therefore, by $\underline{\omega}(a) = R_a = \bar{\omega}(a)$ and Proposition 24, we conclude that $\lim_{t \rightarrow T_a^*} \bar{\omega}(t) = \infty$ too, which is a contradiction with the above conclusion saying that $\bar{\omega} \in C((a, b), \mathbb{R})$. Thus, hypothesis (82) is not true and, consequently, Lemma 1 is shown.

For the analogous case $x(t) < 0$ on $(\tau(\tau(a)), \sigma(\sigma(d)))$, we also have $x(t) < 0$ on $(\tau(\tau(c)), \sigma(\sigma(d)))$ which implies (since $\tau(t)$ and $\sigma(t)$ are increasing functions)

$$\begin{aligned} x(s) < 0 \quad \forall s \in &(\tau(c), \sigma(d)) \cup (\tau(\tau(c)), \tau(\sigma(d))) \\ &\cup (\sigma(\tau(c)), \sigma(\sigma(d))), \end{aligned} \quad (87)$$

which yields $x(t) < 0$, $x(\tau(t)) < 0$, and $x(\sigma(t)) < 0$ on $(\tau(c), \sigma(d))$. Now we can repeat the preceding procedure but on interval (c, d) and using (8) instead of (a, b) and (7). $\square$

Proof of Lemma 2. From assumption (10), we obtain the existence of an $n_0 \in \mathbb{N}$ such that

$$\int_{a_n}^{b_n} Q_n(t) dt \geq \frac{c_0}{2} \left(\max_{t \in [a_n, b_n]} Q_n(t) \right)^{1/\gamma}, \quad n \geq n_0; \quad (88)$$

that is,

$$\frac{2}{c_0} \int_{a_n}^{b_n} Q_n(t) dt \geq \left(\max_{t \in [a_n, b_n]} Q_n(t) \right)^{1/\gamma}, \quad n \geq n_0. \quad (89)$$

Now from (9) and previous inequality we deduce that for large enough λ, μ, ρ , and n

$$\begin{aligned} \frac{p^{1/\gamma}}{r_0^{1-1/\gamma}} \frac{[k(\lambda, \mu, \rho)]^{1-1/\gamma}}{\pi_*} \int_{a_n}^{b_n} Q_n(t) dt \\ \geq \frac{2}{c_0} \int_{a_n}^{b_n} Q_n(t) dt \geq \left(\max_{t \in [a_n, b_n]} Q_n(t) \right)^{1/\gamma}, \end{aligned} \quad (90)$$

which shows (6). Thus, all assumptions of Lemma 1 are fulfilled and, hence, Lemma 2 immediately follows from Lemma 1. $\square$

Proof of Lemma 3. Obviously assumption (11) is a particular case of assumption (9). Hence, this proof is very similar to the proof of Lemma 2 and so it is left to the reader. $\square$

Proof of Lemma 4. It is clear that from assumption (13) we obtain

$$\frac{1}{\left(\max_{t \in [a_n, b_n]} Q_n(t) \right)^{1/\gamma}} \int_{a_n}^{b_n} Q_n(t) dt \geq \frac{c_1}{C_0^{1/\gamma}} > 0, \quad \forall n \geq n_0. \quad (91)$$

Thus, hypothesis (12) is fulfilled and, therefore, Lemma 3 proves this lemma. $\square$

Proof of Theorems 5, 6, and 7. This proof is based on Lemma 4. In order to simplify notation, in many places in this proof we set $\tau(t) = t - \tau$ and $\sigma(t) = t + \sigma$. Since assumptions (2), (3), and (4) have been already supposed in Theorems 5, 6, and 7, in order to prove these theorems by Lemma 4, we are going to show that the functions $k(\lambda, \mu, \rho)$ and $Q_n(t)$ explicitly given, respectively, in (18), (21), or (24) and (19), (22), or (25) satisfy required conditions (11) and (13), respectively, and that every solution $x(t)$ of (27) satisfies conditions (7) and (8) with respect to functions $k(\lambda, \mu, \rho)$ and $Q_n(t)$, where $a = a_{2n-1}$, $b = b_{2n-1}$, $c = a_{2n}$, and $d = b_{2n}$.

The proof that the function $k(\lambda, \mu, \rho)$ given in (18), (21), or (24) satisfies (11). Passing to the limit in (18), (21), or (24) it is very simple to show (11).

The proof that the function $Q_n(t)$ given in (19), (22), or (25) satisfies the first claim in (13). From (25) we immediately obtain

$$\begin{aligned} |\tau_n(t)| &= \left| \left(\frac{t - a_n}{t - a_n + \tau} \right)^p \right| \leq 1, \\ |\sigma_n(t)| &= \left| \left(\frac{b_n - t}{b_n - t + \sigma} \right)^p \right| \leq 1 \quad \forall n \in \mathbb{N}. \end{aligned} \quad (92)$$

Next, by assumptions of this corollary we can conclude that there are three positive constants f_0, g_0, e_0 such that $|f(t)| \leq f_0$ and $|g(t)| \leq g_0$ on $[t_0, \infty)$ in cases (i) and (ii) and $|e(t)| \leq e_0$ on $[t_0, \infty)$ in cases (iii) and (iv). Putting previous inequalities into (19), (22), or (25), for all $n \in \mathbb{N}$ and $t \in [t_0, \infty)$, it holds that

$$|Q_n(t)| \leq \begin{cases} \frac{q}{p} \left(\frac{p}{q-p} \right)^{(q/p)-1} e_0^{1-(p/q)} f_0^{p/q}, & \text{delay case with } q > p, \\ \frac{q}{p} \left(\frac{p}{q-p} \right)^{(q/p)-1} e_0^{1-(p/q)} g_0^{p/q}, & \text{advanced case with } q > p, \\ \frac{p_1}{p} \left(\frac{p}{2(p_1-p)} \right)^{(p_1/p)-1} e_0^{1-(p/p_1)} f_0^{p/q} + \frac{p_2}{p} \times \left(\frac{p}{2(p_2-p)} \right)^{(p_2/p)-1} e_0^{1-(p/p_2)} g_0^{p/q}, & \text{delay-advanced case (i),} \\ e_0^{\eta_0} f_0^{\eta_1} g_0^{\eta_2} \prod_{i=0}^2 \eta_i^{-\eta_i}, & \text{delay-advanced case (ii),} \end{cases} \quad (93)$$

which shows the first claim in (13).

The proof that the function $Q_n(t)$ given in (19), (22), or (25) satisfies the second claim in (13). Without loss of generality, we prove this claim only in case (i), since for other cases the proof follows analogously. In this sense, let $Q_n(t) = f(t)\tau_n(t)$. Since $a_{2n+1} - a_{2n-1} \leq T_*$, $b_{2n+1} - b_{2n-1} \geq T_*$, $a_{2n+2} - a_{2n} \leq T_*$, and $b_{2n+2} - b_{2n} \geq T_*$, where $T_* > 0$ is the period of the function $f(t)$, we have $a_{2n-1} \leq a_1 + (n-1)T_*$ and $b_{2n-1} \geq b_1 + (n-1)T_*$, $n \in \mathbb{N}$. Hence

$$\begin{aligned} & \int_{a_{2n-1}}^{b_{2n-1}} Q_n(t) dt \\ &= \int_{a_{2n-1}}^{b_{2n-1}} f(t) \left(\frac{t - a_{2n-1}}{t - a_{2n-1} + \tau} \right)^p dt \\ &\geq \int_{a_1 + (n-1)T_*}^{b_1 + (n-1)T_*} f(t) \left(\frac{t - a_1 - (n-1)T_*}{t - a_1 - (n-1)T_* + \tau} \right)^p dt \quad (94) \\ &= \int_{a_1}^{b_1} f(s + (n-1)T_*) \left(\frac{s - a_1}{s - a_1 + \tau} \right)^p ds \\ &= \int_{a_1}^{b_1} f(s) \left(\frac{s - a_1}{s - a_1 + \tau} \right)^p ds, \end{aligned}$$

which proves that the integral on the left hand side does not depend on $n \in \mathbb{N}$; that is, the second claim in (13) is shown on $[a_{2n-1}, b_{2n-1}]$. This claim follows in the same way on $[a_{2n}, b_{2n}]$. Thus, the second claim in (13) is proved on $[a_n, b_n]$. $\square$

Next, to the end of this proof, let $x(t)$ be a solution of (1). In particular, it implies that $(r(t)A(x'(t)))' = -B(t, x(t), x'(t)) - \lambda F(t, x(\tau(t))) - \mu G(t, x(\sigma(t))) + \rho e(t)$. It together with assumptions (15), (16), (20), and (23) easily gives the next two statements:

$$\begin{aligned} & \text{if } x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \\ & \quad \text{on } (\tau(a_{2n-1}), \sigma(b_{2n-1})), \end{aligned} \quad (95)$$

$$\begin{aligned} & \text{then } x(t) \text{ satisfies } r(t)A(x'(t)) \leq 0 \\ & \quad \text{on } (\tau(a_{2n-1}), \sigma(b_{2n-1})), \quad n \geq n_0; \end{aligned}$$

$$\begin{aligned} & \text{if } x(t) < 0, \quad x(\tau(t)) < 0, \quad x(\sigma(t)) < 0 \\ & \quad \text{on } (\tau(a_{2n}), \sigma(b_{2n})), \end{aligned} \quad (96)$$

$$\begin{aligned} & \text{then } x(t) \text{ satisfies } r(t)A(x'(t)) \geq 0 \\ & \quad \text{on } (\tau(a_{2n}), \sigma(b_{2n})), \quad n \geq n_0. \end{aligned}$$

Now we need the following lemma.

Lemma 25. Let $\tau_{ab}(t)$ and $\sigma_{ab}(t)$ be defined by

$$\begin{aligned}\tau_{ab}(t) &= \left(\frac{\tau(t) - \tau(a)}{t - \tau(a)} \right)^p, \\ \sigma_{ab}(t) &= \left(\frac{\sigma(b) - \sigma(t)}{\sigma(b) - t} \right)^p, \quad t \in (a, b),\end{aligned}\quad (97)$$

and let $x \in C^2([T_0, \infty), \mathbb{R})$ be an arbitrary function. If $(r(t)A(x'(t)) \leq 0$ for all $t \in (\tau(a), \sigma(b))$ or $(r(t)A(x'(t)) \geq 0$ for all $t \in (\tau(a), \sigma(b))$), then

$$\frac{x(\tau(t))}{x(t)} \geq (\tau_{ab}(t))^{1/p}, \quad \frac{x(\sigma(t))}{x(t)} \geq (\sigma_{ab}(t))^{1/p}, \quad t \in (a, b). \quad (98)$$

Since $A(v)$ is supposed to be odd and increasing function, just before (3), and $r(t)$ satisfies (14), the proof of Lemma 25 in the first case, that is, $r(t)A(x'(t)) \leq 0$ for all $t \in (\tau(a), \sigma(b))$, is the same as the proof of [9, Corollaries 17 and 18]. But in the second case, that is, $r(t)A(x'(t)) \geq 0$ for all $t \in (\tau(a), \sigma(b))$, the proof is as follows: if previous inequality holds, then $r(t)A(-x'(t)) \leq 0$ for all $t \in (\tau(a), \sigma(b))$ and, therefore, to the function $-x(t)$ one can apply the first case of this lemma and consequently one obtains

$$\begin{aligned}\frac{x(\tau(t))}{x(t)} &= \frac{-x(\tau(t))}{-x(t)} \geq (\tau_{ab}(t))^{1/p}, \\ \frac{x(\sigma(t))}{x(t)} &= \frac{-x(\sigma(t))}{-x(t)} \geq (\sigma_{ab}(t))^{1/p}, \quad t \in (a, b),\end{aligned}\quad (99)$$

which proves this lemma in the second case.

Now, combining statements (95), (96), and (98), one easily obtains

$$\begin{aligned}\text{if } x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \\ \text{on } (\tau(a_{2n-1}), \sigma(b_{2n-1})), \\ \text{then } x(t) \text{ satisfies } \frac{x(\tau(t))}{x(t)} \geq (\tau_n(t))^{1/p}\end{aligned}\quad (100)$$

$$\begin{aligned}\text{on } (a_{2n-1}, b_{2n-1}), \quad n \geq n_0; \\ \text{if } x(t) < 0, \quad x(\tau(t)) < 0, \quad x(\sigma(t)) < 0 \\ \text{on } (\tau(a_{2n}), \sigma(b_{2n})), \\ \text{then } x(t) \text{ satisfies } \frac{x(\sigma(t))}{x(t)} \geq (\sigma_n(t))^{1/p}\end{aligned}\quad (101)$$

$$\text{on } (a_{2n}, b_{2n}), \quad n \geq n_0,$$

where $\tau_n(t)$ and $\sigma_n(t)$ are defined in (26).

The proof that $x(t)$ satisfies (7) and (8). In this proof, we frequently use assumptions (16), (20), and (23) and statements (100) and (101). Also, because of (15) and $F(t, u) =$

$f(t)|u|^{p_1} \operatorname{sgn}(u)$, $G(t, u) = g(t)|u|^{p_2} \operatorname{sgn}(u)$, in both cases (100) and (101), we can simultaneously use

$$\begin{aligned}-e(t) \left(|x(t)|^{p-1} x(t) \right)^{-1} &= |e(t)| |x(t)|^{-p} \geq 0 \quad \text{on } J_n, \\ F(t, x(\tau(t))) \left(|x(t)|^{p-1} x(t) \right)^{-1} \\ &\geq f(t) |x(\tau(t))|^{p_1} |x(t)|^{-p} \geq 0 \quad \text{on } J_n, \\ G(t, x(\sigma(t))) \left(|x(t)|^{p-1} x(t) \right)^{-1} \\ &\geq g(t) |x(\sigma(t))|^{p_2} |x(t)|^{-p} \geq 0 \quad \text{on } J_n, \\ |x(\tau(t))| |x(t)|^{-1} &= \frac{x(\tau(t))}{x(t)}, \\ |x(\sigma(t))| |x(t)|^{-1} &= \frac{x(\sigma(t))}{x(t)} \quad \text{on } J_n,\end{aligned}\quad (102)$$

where $J_n = (a_{2n-1}, b_{2n-1})$ in the case of (100) and $J_n = (a_{2n}, b_{2n})$ in the case of (101).

(i) *Delay or Advanced Case with $q = p$.* Since $q = p$, we obtain

$$\begin{aligned}[\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ \times \left(|x(t)|^{p-1} x(t) \right)^{-1} \\ \geq [\lambda f(t) |x(\tau(t))|^p + \mu g(t) |x(\sigma(t))|^p \\ + \rho |e(t)|] |x(t)|^{-p} \\ \geq [\lambda f(t) |x(\tau(t))|^p + \mu g(t) |x(\sigma(t))|^p] |x(t)|^{-p} \\ = \lambda f(t) \left(\frac{x(\tau(t))}{x(t)} \right)^p + \mu g(t) \left(\frac{x(\sigma(t))}{x(t)} \right)^p \\ \geq \lambda f(t) \tau_n(t) + \mu g(t) \sigma_n(t), \quad t \in J_n,\end{aligned}\quad (103)$$

where the functions $\tau_n(t)$ and $\sigma_n(t)$ are defined in (26).

(ii) *Delay Case with $q > p$.* In this part we use the next elementary inequality:

$$X^\gamma + (\gamma - 1)Y^\gamma \geq \gamma XY^{\gamma-1}, \quad \gamma > 1, \quad X, Y \geq 0. \quad (104)$$

Since $q > p$ and using (104) especially for

$$\begin{aligned}\gamma &= \frac{q}{p} > 1, \quad X = (\lambda f(t))^{p/q} |x(\tau(t))|^p, \\ Y &= \left(\frac{p}{q-p} \rho |e(t)| \right)^{p/q},\end{aligned}\quad (105)$$

for all $t \in J_n$ we obtain

$$\begin{aligned}
 & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\
 & \quad \times (|x(t)|^{p-1} x(t))^{-1} \\
 &= [\lambda F(t, x(\tau(t))) - \rho e(t)] (|x(t)|^{p-1} x(t))^{-1} \\
 &\geq [\lambda f(t) |x(\tau(t))|^{p_1} + \rho |e(t)|] |x(t)|^{-p} \\
 &= [X^\gamma + (\gamma - 1) Y^\gamma] |x(t)|^{-p} \\
 &\geq \frac{q}{p} (\lambda f(t))^{p/q} |x(\tau(t))|^p \\
 &\quad \times \left(\frac{p}{q-p} \rho |e(t)| \right)^{(p/q)((q/p)-1)} |x(t)|^{-p} \\
 &= \lambda^{p/q} \rho^{1-(p/q)} \frac{q}{p} \left(\frac{p}{q-p} \right)^{1-(p/q)} |e(t)|^{1-(p/q)} \\
 &\quad \times (f(t))^{p/q} \left(\frac{x(\tau(t))}{x(t)} \right)^p \\
 &\geq k(\lambda, \mu, \rho) \frac{q}{p} \left(\frac{p}{q-p} \right)^{1-(p/q)} |e(t)|^{1-(p/q)} \\
 &\quad \times (f(t))^{p/q} \tau_n(t),
 \end{aligned} \tag{106}$$

where the function $k(\lambda, \mu, \rho)$ is from (18).

(iii) *Advanced Case with $q > p$.* Using the same line of arguments as in the proof of the previous case, for all $t \in J_n$ we obtain

$$\begin{aligned}
 & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\
 & \quad \times (|x(t)|^{p-1} x(t))^{-1} \\
 &= [\mu G(t, x(\sigma(t))) - \rho e(t)] (|x(t)|^{p-1} x(t))^{-1} \\
 &\geq k(\lambda, \mu, \rho) \frac{q}{p} \left(\frac{p}{q-p} \right)^{1-(p/q)} |e(t)|^{1-(p/q)} \\
 &\quad \times (g(t))^{p/q} \sigma_n(t),
 \end{aligned} \tag{107}$$

where the function $k(\lambda, \mu, \rho)$ is from (21).

(iv) *Superlinear Delay-Advanced Case.* Since $p_1, p_2 > p$, for all $t \in J_n$ we obtain

$$\begin{aligned}
 & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\
 & \quad \times (|x(t)|^{p-1} x(t))^{-1} \\
 &= \left[\lambda F(t, x(\tau(t))) - \frac{\rho}{2} e(t) \right] (|x(t)|^{p-1} x(t))^{-1} \\
 & \quad + \left[\mu G(t, x(\sigma(t))) - \frac{\rho}{2} e(t) \right] (|x(t)|^{p-1} x(t))^{-1} \\
 &\geq \left[\lambda f(t) |x(\tau(t))|^{p_1} + \frac{\rho}{2} |e(t)| \right] |x(t)|^{-p} \\
 & \quad + \left[\mu g(t) |x(\sigma(t))|^{p_2} + \frac{\rho}{2} |e(t)| \right] |x(t)|^{-p}.
 \end{aligned} \tag{108}$$

Now, just the same as in the proofs of previous delay and advanced cases with $q > p$ and with the help of (104) in particular for

$$\begin{aligned}
 \gamma &= \frac{p_1}{p} > 1, \quad X = (\lambda f(t))^{p/p_1} |x(\tau(t))|^p, \\
 Y &= \left(\frac{p}{p_1 - p} \frac{\rho}{2} |e(t)| \right)^{p/p_1}
 \end{aligned} \tag{109}$$

we have

$$\begin{aligned}
 & \left[\lambda f(t) |x(\tau(t))|^{p_1} + \frac{\rho}{2} |e(t)| \right] |x(t)|^{-p} \\
 &= [X^\gamma + (\gamma - 1) Y^\gamma] |x(t)|^{-p} \\
 &\geq \frac{p_1}{p} (\lambda f(t))^{p/p_1} |x(\tau(t))|^p \\
 &\quad \times \left(\frac{p}{p_1 - p} \frac{\rho}{2} |e(t)| \right)^{(p/p_1)((p_1/p)-1)} |x(t)|^{-p} \\
 &= \lambda^{p/p_1} \rho^{1-(p/p_1)} \frac{p_1}{p} \left(\frac{p}{2(p_1 - p)} \right)^{1-(p/p_1)} \\
 &\quad \times |e(t)|^{1-(p/p_1)} (f(t))^{p/p_1} \left(\frac{x(\tau(t))}{x(t)} \right)^p \\
 &\geq k(\lambda, \mu, \rho) \frac{p_1}{p} \left(\frac{p}{2(p_1 - p)} \right)^{1-(p/p_1)} \\
 &\quad \times |e(t)|^{1-(p/p_1)} (f(t))^{p/p_1} \tau_n(t),
 \end{aligned} \tag{110}$$

where the function $k(\lambda, \mu, \rho)$ is from (24). Analogously, we show that

$$\begin{aligned}
 & \left[\mu g(t) |x(\sigma(t))|^{p_2} + \frac{\rho}{2} |e(t)| \right] |x(t)|^{-p} \\
 &\geq k(\lambda, \mu, \rho) \frac{p_2}{p} \left(\frac{p}{2(p_2 - p)} \right)^{1-(p/p_2)} \\
 &\quad \times |e(t)|^{1-(p/p_2)} (g(t))^{p/p_2} \sigma_n(t).
 \end{aligned} \tag{111}$$

Summarizing previous calculation, we obtain

$$\begin{aligned} & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ & \times (|x(t)|^{p-1} x(t))^{-1} \geq k(\lambda, \mu, \rho) Q_n(t), \quad t \in J_n, \end{aligned} \quad (112)$$

where the function $k(\lambda, \mu, \rho)$ is from (24).

(v) *Supersublinear Delay-Advanced Case*. Since $p_1 > p > p_2$ and the following well-known elementary inequality holds

$$\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2 \geq u_0^{\eta_0} u_1^{\eta_1} u_2^{\eta_2}, \quad \eta_i \geq 0, \quad u_i \geq 0, \quad (113)$$

from $\eta_0, \eta_1, \eta_2 \in (0, 1)$, $\eta_0 + \eta_1 + \eta_2 = 1$ and $p_1 \eta_1 + p_2 \eta_2 = p$, we obtain for all $t \in J_n$, for all $t \in J_n$ we obtain

$$\begin{aligned} & [\lambda F(t, x(\tau(t))) + \mu G(t, x(\sigma(t))) - \rho e(t)] \\ & \times (|x(t)|^{p-1} x(t))^{-1} \\ & \geq [\lambda f(t) |x(\tau(t))|^{p_1} + \mu g(t) |x(\sigma(t))|^{p_2} + \rho |e(t)|] \\ & \times |x(t)|^{-p} \\ & = [\eta_1 [\eta_1^{-1} \lambda f(t) |x(\tau(t))|^{p_1}] + \eta_2 [\eta_2^{-1} \mu g(t) |x(\sigma(t))|^{p_2}] \\ & \quad + \eta_0 [\eta_0^{-1} \rho |e(t)|]] |x(t)|^{-p} \\ & \geq [\eta_1^{-1} \lambda f(t) |x(\tau(t))|^{p_1}]^{\eta_1} [\eta_2^{-1} \mu g(t) |x(\sigma(t))|^{p_2}]^{\eta_2} \\ & \quad \times [\eta_0^{-1} \rho |e(t)|]^{\eta_0} |x(t)|^{-p} \\ & = \lambda^{\eta_1} \mu^{\eta_2} \rho^{\eta_0} |e(t)|^{\eta_0} (f(t))^{\eta_1} (g(t))^{\eta_2} \\ & \quad \times \frac{|x(\tau(t))|^{\eta_1 p_1}}{|x(t)|^{\eta_1 p_1}} \frac{|x(\sigma(t))|^{\eta_2 p_2}}{|x(t)|^{\eta_2 p_2}} \prod_{i=0}^2 \eta_i^{-\eta_i} \\ & = k(\lambda, \mu, \rho) |e(t)|^{\eta_0} (f(t))^{\eta_1} (g(t))^{\eta_2} \\ & \quad \times \left(\frac{x(\tau(t))}{x(t)} \right)^{\eta_1 p_1} \left(\frac{x(\sigma(t))}{x(t)} \right)^{\eta_2 p_2} \prod_{i=0}^2 \eta_i^{-\eta_i} \\ & \geq k(\lambda, \mu, \rho) |e(t)|^{\eta_0} (f(t))^{\eta_1} (g(t))^{\eta_2} (\tau_n(t))^{\eta_1(p_1/p)} \\ & \quad \times (\sigma_n(t))^{\eta_2(p_2/p)} \prod_{i=0}^2 \eta_i^{-\eta_i} \\ & = k(\lambda, \mu, \rho) Q_n(t), \end{aligned} \quad (114)$$

where $k(\lambda, \mu, \rho)$ and $Q_n(t)$ are given, respectively, in (24) and (25). Thus, it is shown that required condition (5) in the cases (i)–(iv) is fulfilled with respect to $k(\lambda, \mu, \rho)$ and $Q_n(t)$ determined by (18), (21), or (24) and (19), (22), or (25).

In conclusion, according to the previous observation, we see that all assumptions of Lemma 4 are fulfilled and, hence, Lemma 4 proves Theorems 5, 6, and 7.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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