Research Article

The Smoothness of Fractal Interpolation Functions on $\mathbb{R}$ and on p -Series Local Fields

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Received 6 November 2013; Revised 5 February 2014; Accepted 10 February 2014; Published 7 April 2014

Academic Editor: Bing Xu

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A fractal interpolation function on a p -series local field K_p is defined, and its p -type smoothness is shown by virtue of the equivalent relationship between the Hölder type space $C^\sigma(K_p)$ and the Lipschitz class $\text{Lip}(\sigma, K_p)$. The orders of the p -type derivatives and the fractal dimensions of the graphs of Weierstrass type function on local fields are given as an example. The α -fractal function on $\mathbb{R}$ is introduced and the conclusion of its smoothness is improved in a more general case; some examples are shown to support the conclusion. Finally, a comparison between the fractal interpolation functions defined on $\mathbb{R}$ and K_p is given.

1. Introduction

As we know, the traditional method for analyzing given experiment data

$$\{(x_n, F_n) : x_n \in [a, b], n = 0, 1, \dots, N, x_0 = a, x_N = b\} \quad (1)$$

is by representing data graphically as a subset of $\mathbb{R}^2$; then the graphical data are analyzed by some geometrical or analytical tools to seek a function $f(x)$ with graph $G \subset \mathbb{R}^2$, whose values are a good fit to the data over the interval $[x_0, x_N]$; this is an interpolation problem. Generally, a function $f(x)$ need to satisfy the following: (1) it fits the data at each point x_n ; that is, $f(x_n) = F_n$; (2) it is some simple function, such as polynomial; (3) it has some order smoothness; for example, “spline interpolant” is smooth in one or two order.

In this paper, we present a new idea: to establish interpolation theory on local fields. We have found that there are not only new concepts but also many new methods in this new branch—interpolation theory on local fields. We may construct a fractal interpolation function $f(x)$ that satisfies the above (1), (2), (3), by virtue of the harmonic analysis theory and fractal analysis theory on local fields [1]. However, “some simple function” in (2) and “some order smoothness”

in (3) all have new senses which are quite different from those in $\mathbb{R}$ case.

Barnsley et al. [2–5] defined the fractal interpolation functions on $\mathbb{R}$ by virtue of the iterated function system.

Definition 1 (see [2]) (IFS, HIFS). Let M be a complete metric space with distance $d(x, y)$ for $x, y \in M$. Let $N > 1$ be a given positive integer and let $\omega_n : M \rightarrow M$ for $n = 1, 2, \dots, N$ be continuous mappings; then we call $\{M, \omega_n : n = 1, 2, \dots, N\}$ an iterated function system (IFS for short). If, for all $n = 1, 2, \dots, N$, there exists an $s \in [0, 1)$ such that

$$d(\omega_n(x), \omega_n(y)) \leq s \cdot d(x, y), \quad \forall x, y \in M; \quad (2)$$

that is, each ω_n is contractive with contractive factor s ; then the IFS $\{M, \omega_n : n = 1, 2, \dots, N\}$ is termed a hyperbolic iterated function system (HIFS for short).

Let H be the set of all nonempty compact subsets of the complete metric space M . Then H is a complete metric space with the Hausdorff metric

$$h(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y) \right\}, \quad (3)$$

$$\forall A, B \in H.$$

Define $W : H \rightarrow H$ by

$$W(A) = \bigcup_{n=1}^N \omega_n(A), \quad \forall A \in H, \quad (4)$$

where $\omega_n(A) = \{\omega_n(x) : x \in A\}$. We call $G \in H$ the attractor of the IFS $\{M, \omega_n : n = 1, 2, \dots, N\}$ if

$$W(G) = G. \quad (5)$$

The attractor G of a HIFS is the unique set satisfying

$$\lim_{m \rightarrow \infty} h(W^m(B), G) = 0, \quad \forall B \in H, \quad (6)$$

where $W^m = \underbrace{W \circ \dots \circ W}_m$ is the m -fold composition of W .

Let $x_0 < x_1 < \dots < x_N$ be a partition of a closed interval $I = [x_0, x_N]$. Set $I_n = [x_{n-1}, x_n]$ for $n \in \{1, 2, \dots, N\}$. Suppose that a set of interpolation points $T = \{(x_n, y_n) \in I \times [a, b] : n = 0, 1, \dots, N\}$ is given, where $-\infty < a < b < \infty$. Let $L_n : I \rightarrow I_n$, $n = 1, 2, \dots, N$, be contractive homeomorphisms such that

$$\begin{aligned} L_n(x_0) &= x_{n-1}, & L_n(x_N) &= x_n, \\ |L_n(c_1) - L_n(c_2)| &\leq r |c_1 - c_2|, & \forall c_1, c_2 \in I, \end{aligned} \quad (7)$$

with some $0 < r < 1$.

Let $F_n : I \times [a, b] \rightarrow [a, b]$ be N continuous mappings satisfying

$$\begin{aligned} F_n(x_0, y_0) &= y_{n-1}, & F_n(x_N, y_N) &= y_n, \\ |F_n(x, t_1) - F_n(x, t_2)| &\leq q |t_1 - t_2|, \end{aligned} \quad (8)$$

where $x \in I$, $t_1, t_2 \in [a, b]$, $0 < q < 1$, and $n = 1, 2, \dots, N$.

Set $M = I \times \mathbb{R}$. Define the mappings $\omega_n : M \rightarrow M$ for $n \in \{1, 2, \dots, N\}$ by

$$\omega_n(x, y) = (L_n(x), F_n(x, y)). \quad (9)$$

Then $\{M, \omega_n : n = 1, 2, \dots, N\}$ constitutes an IFS.

Theorem 2 (see [2, Theorem 1]). *The IFS $\{M, \omega_n : n = 1, 2, \dots, N\}$ defined above admits a unique attractor G , which is the graph of a continuous function $f : I \rightarrow \mathbb{R}$ passing through the interpolation points T .*

Definition 3 (see [2]) (FIF). The function f described in Theorem 2 is referred to as a fractal interpolation function (FIF for short), associated with the IFS $\{M, \omega_n : n = 1, 2, \dots, N\}$.

Several conclusions on the differentiability of the FIFs defined on $\mathbb{R}$ have been given out. Barnsley and Harrington [5] introduced the calculus of FIFs. Chen [6] gave some conditions under which the equidistant FIFs, defined by the affine IFS, are nowhere differentiable. Sha and Chen [7] investigated the Hölder smoothness of a class of FIFs and their logical derivatives of order α . Chen [8] investigated the smoothness of nonequidistant FIF and obtained the

Hölder exponents of such FIFs. Wang [9] investigated the differentiability of the equidistant FIFs generated by the nonlinear IFS. Li et al. [10] obtained the sufficient conditions of Hölder continuity of two kinds of FIFs and proved the sufficient and necessary condition for their differentiability. Navascués [11] introduced α -fractal functions and gave some conditions under which the set of their nondifferentiable points is dense in the domain when the scaling factors have the same value. Certainly, differentiability of FIF is important and so that attracts eyes of mathematicians.

By the construction of FIFs, it is reasonable to establish the interpolation theory on local fields since the structures of local fields are suitable to construct some FIFs.

The main purpose of this work is to establish the interpolation theory on the p -series local field K_p and to investigate the difference of smoothness between the two FIFs defined on $\mathbb{R}$ and on K_p . In Section 2 of this paper, some results on the smoothness of a so-called α -fractal functions are obtained and some examples supporting corresponding conclusions are given. In Section 3, the p -type smoothness on local fields is introduced. In Section 4, a definition of the fractal interpolation functions on K_p is given and the p -type smoothness of the fractal interpolation function on K_p is obtained by virtue of the equivalent relationship between the Hölder type space $C^\sigma(K_p)$ and the Lipschitz class $\text{Lip}(\sigma, K_p)$. As a special example of the fractal interpolation functions on local fields, Weierstrass type function on local fields is shown. A linear relationship between the orders of the p -type derivatives and the fractal dimensions of the graphs of Weierstrass function on local fields is concluded. Finally, in Section 5, we give a comparison between the FIF on $\mathbb{R}$ and on K_p .

2. Smoothness of α -Fractal Functions

In this section, we focus our research on the smoothness of the α -fractal functions on $\mathbb{R}$.

Let $I = [0, 1]$ and $x_n = n/N$ for every $n \in \{0, 1, \dots, N\}$. Define an IFS on $\mathbb{R}$ as follows:

$$\begin{aligned} \omega_n \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} L_n(x) \\ F_n(x, y) \end{pmatrix} = \begin{pmatrix} \frac{x}{N} + \frac{(n-1)}{N} \\ \alpha_n y + g \circ L_n(x) - \alpha_n b(x) \end{pmatrix}, \\ n &= 1, 2, \dots, N, \end{aligned} \quad (10)$$

where the α_n obey $-1 < \alpha_n < 1$, and they are called the vertical scaling factors, $g \in \mathcal{C}(I)$ is a continuous function satisfying $g(x_n) = y_n$ for every $n \in \{0, 1, \dots, N\}$, and $b(x) = y_0 + (y_N - y_0)x$ is the line passing through (x_0, y_0) and (x_N, y_N) .

In [2, 11], the FIF associated with the IFS (10) is called an α -fractal function associated with g with respect to the equidistant partition, denoted by $g^\alpha(x)$.

By [6], it is easy to obtain the explicit representation of g^α . We agree on $\prod_{j=0}^1 \alpha_{n_j} = 1$. Then, for any $x \in I$, there is a sequence $\{n_j\}_{j=1}^{+\infty}, n_j \in \{1, 2, \dots, N\}$ such that x satisfies

$$x = (n_1 n_2 n_3 \dots) = \sum_{j=1}^{+\infty} \frac{n_j - 1}{N^j}, \quad (11)$$

and then

$$g^\alpha(x) = \sum_{k=0}^{+\infty} \left(\prod_{j=1}^k \alpha_{n_j} \right) (g(\sigma^k x) - \alpha_{n_{k+1}} b(\sigma^{k+1} x)), \quad (12)$$

where the shift map $\sigma^k : [0, 1] \rightarrow [0, 1]$ is defined by

$$\sigma^k(n_1 n_2 \dots n_k \dots) = (n_{k+1} n_{k+2} \dots) = \sum_{l=1}^{+\infty} \frac{n_{k+l} - 1}{N^l}, \quad (13)$$

$$k \in \mathbb{N}.$$

Several conclusions on the smoothness of α -fractal function have been given like, for instance, the following one.

Theorem 4 (see [9]). *If there exists some $n \in \{1, 2, \dots, N\}$ such that $\alpha_n = 0$ and $g \in \mathcal{C}^1[0, 1]$, then g^α is a.e. differentiable in $(0, 1)$.*

In [11], the FIF (11) and (12) has constant scaling factors $\alpha_1 = \alpha_2 = \dots = \alpha_N = a$. However, in this paper, one assumes that FIF (12) has different scaling factors and then finds conditions for nonsmoothness of g^α . The following lemma is a generalization of Lemma 5.1 of [11].

Lemma 5. *If $g \in \mathcal{C}^1[0, 1]$, $|\alpha_n| \geq 1/N$, $n = 1, 2, \dots, N$, and g^α is differentiable at $x \in [0, 1]$, then*

$$\lim_{k \rightarrow +\infty} g'(\sigma^k x) = y_N - y_0. \quad (14)$$

Proof. Given $x = (i_1 i_2 \dots i_r \dots) = \sum_{j=1}^{+\infty} (i_j - 1)/N^j$, we define

$$x_r = (i_1 i_2 \dots i_r 11 \dots) = \sum_{j=1}^r \frac{i_j - 1}{N^j}, \quad (15)$$

$$x'_r = (i_1 i_2 \dots i_r N N \dots) = \sum_{j=1}^r \frac{i_j - 1}{N^j} + \frac{1}{N^r}.$$

Then $x_r \leq x \leq x'_r$ and $x'_r - x_r \rightarrow 0$ as $r \rightarrow \infty$.

For $\forall k \geq r$,

$$\sigma^k x_r = (11 \dots) = 0, \quad \sigma^k x'_r = (N N \dots) = 1. \quad (16)$$

From (12),

$$\begin{aligned} g^\alpha(x_r) &= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k \alpha_{i_j} \right) (g(\sigma^k x_r) - \alpha_{i_{k+1}} b(\sigma^{k+1} x_r)) \\ &\quad + \sum_{k=r}^{+\infty} \left(\prod_{j=1}^k \alpha_{i_j} \right) (g(\sigma^k x_r) - \alpha_{i_{k+1}} b(\sigma^{k+1} x_r)). \end{aligned} \quad (17)$$

By (16), the second term of (17) deduces to

$$\begin{aligned} &\sum_{k=r}^{+\infty} \left(\prod_{j=1}^k \alpha_{i_j} \right) (g(0) - \alpha_1 b(0)) \\ &= \left(\prod_{j=1}^r \alpha_{i_j} \right) (y_0 - \alpha_1 y_0) (1 + \alpha_1 + \alpha_1^2 + \dots) \\ &= y_0 \left(\prod_{j=1}^r \alpha_{i_j} \right). \end{aligned} \quad (18)$$

Similarly,

$$\begin{aligned} g^\alpha(x'_r) &= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k \alpha_{i_j} \right) (g(\sigma^k x'_r) - \alpha_{i_{k+1}} b(\sigma^{k+1} x'_r)) \\ &\quad + y_N \left(\prod_{j=1}^r \alpha_{i_j} \right). \end{aligned} \quad (19)$$

Then

$$\begin{aligned} &g^\alpha(x'_r) - g^\alpha(x_r) \\ &= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k \alpha_{i_j} \right) \{ [g(\sigma^k x'_r) - g(\sigma^k x_r)] \\ &\quad - \alpha_{i_{k+1}} [b(\sigma^{k+1} x'_r) - b(\sigma^{k+1} x_r)] \} \\ &\quad + (y_N - y_0) \left(\prod_{j=1}^r \alpha_{i_j} \right). \end{aligned} \quad (20)$$

For any $0 \leq k < r$,

$$\sigma^k x'_r - \sigma^k x_r = \sum_{j=r-k+1}^{+\infty} \frac{(N-1)}{N^j} = \frac{1}{N^{r-k}}, \quad (21)$$

and, then, for $g \in \mathcal{C}^1[0, 1]$, there exists $\xi_k^r \in (\sigma^k x_r, \sigma^k x'_r)$, so that

$$\begin{aligned} &g(\sigma^k x'_r) - g(\sigma^k x_r) = \frac{g'(\xi_k^r)}{N^{r-k}}, \\ &b(\sigma^{k+1} x'_r) - b(\sigma^{k+1} x_r) = \frac{(y_N - y_0)}{N^{r-k-1}}. \end{aligned} \quad (22)$$

Thus we have

$$\begin{aligned}
& \frac{g^\alpha(x'_r) - g^\alpha(x_r)}{x'_r - x_r} \\
&= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k \alpha_{i_j} \right) [g'(\xi_k^r) - (y_N - y_0) N \alpha_{i_{k+1}}] \\
&\quad \cdot N^k + (y_N - y_0) \left(\prod_{j=1}^r \alpha_{i_j} \right) \cdot N^r \\
&= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k (N \alpha_{i_j}) \right) (g'(\xi_k^r) - (y_N - y_0) N \alpha_{i_{k+1}}) \\
&\quad + (y_N - y_0) \prod_{j=1}^r (N \alpha_{i_j}).
\end{aligned} \tag{23}$$

Notice that

$$\prod_{j=1}^r (N \alpha_{i_j}) = 1 + \sum_{k=0}^{r-1} \left(\prod_{j=1}^{k+1} (N \alpha_{i_j}) - \prod_{j=1}^k (N \alpha_{i_j}) \right), \tag{24}$$

and hence we have

$$\begin{aligned}
& \frac{g^\alpha(x'_r) - g^\alpha(x_r)}{x'_r - x_r} \\
&= \sum_{k=0}^{r-1} \left(\prod_{j=1}^k (N \alpha_{i_j}) \right) (g'(\xi_k^r) - (y_N - y_0) N \alpha_{i_{k+1}}) \\
&\quad + (y_N - y_0) + (y_N - y_0) \\
&\quad \times \sum_{k=0}^{r-1} \left(\prod_{j=1}^{k+1} (N \alpha_{i_j}) - \prod_{j=1}^k (N \alpha_{i_j}) \right) \\
&= y_N - y_0 + \sum_{k=0}^{r-1} \left(\prod_{j=1}^k (N \alpha_{i_j}) \right) (g'(\xi_k^r) - (y_N - y_0)).
\end{aligned} \tag{25}$$

Since g^α is differentiable at x , it follows that

$$\begin{aligned}
& (g^\alpha)'(x) = y_N - y_0 \\
& \quad + \lim_{r \rightarrow +\infty} \sum_{k=0}^{r-1} \left(\prod_{j=1}^k (N \alpha_{i_j}) \right) (g'(\xi_k^r) - (y_N - y_0)).
\end{aligned} \tag{26}$$

Note that $\xi_k^r \rightarrow \sigma^k x$ as $r \rightarrow +\infty$, and, by the hypothesis $|\alpha_n| \geq 1/N$, $n = 1, 2, \dots, N$, then

$$\lim_{k \rightarrow +\infty} g'(\sigma^k x) = y_N - y_0. \tag{27}$$

This completes the proof. $\square$

By Lemma 5, we get a conclusion.

Theorem 6. If $g \in \mathcal{C}^1[0, 1]$, $|\alpha_n| \geq 1/N$, $n = 1, 2, \dots, N$, and $g'(x)$ does not agree with $y_N - y_0$ in a nonempty open subinterval of $I = [0, 1]$, then the set of points at which g^α is not differentiable is dense on I .

Example 7. Let W_λ be the Weierstrass function defined by

$$W_\lambda(x) = \sum_{j=0}^{+\infty} 2^{-\lambda j} \cos(2^j x), \quad x \in [0, 2\pi], \tag{28}$$

where $0 < \lambda \leq 1$. Then $W_\lambda(x)$ is continuous, and the set of its nondifferentiable points is dense on $[0, 2\pi]$.

Here we set $N = 2$, $I = [0, 2\pi]$. The set of data points is given as

$$\left\{ \left(0, \frac{2^\lambda}{(2^\lambda - 1)} \right), \left(\pi, \frac{(2 - 2^\lambda)}{(2^\lambda - 1)} \right), \left(2\pi, \frac{2^\lambda}{(2^\lambda - 1)} \right) \right\}. \tag{29}$$

We see that $W_\lambda(x)$ is generated by the IFS

$$\begin{aligned}
w_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{x}{2} \\ 2^{-\lambda} y + \cos\left(\frac{x}{2}\right) \end{pmatrix}, \\
w_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{x}{2} + \pi \\ 2^{-\lambda} y - \cos\left(\frac{x}{2}\right) \end{pmatrix},
\end{aligned} \tag{30}$$

in which the scaling factors are $\alpha_1 = \alpha_2 = 2^{-\lambda}$.

In this case, the line passing through (x_0, y_0) and (x_N, y_N) is $b(x) = 2^\lambda / (2^\lambda - 1)$. Then we get

$$g(x) = \cos x + \frac{1}{(2^\lambda - 1)}. \tag{31}$$

Since $g'(x) = -\sin x$, we may say that $g'(x)$ does not agree with $0 (= y_N - y_0)$ in a nonempty open subinterval of $[0, 2\pi]$. Moreover, by hypothesis $0 < \lambda \leq 1$, we get that $|\alpha_n| \geq 1/2$, $n = 1, 2$. Then we replace $W_\lambda(x)$ with $W_\lambda(2\pi x)$. Using Theorem 6, we get the result (see Figure 1).

By Example 7, we conclude that we may choose $\sin x$ or some other periodic functions on $[0, 1]$ instead of $\cos x$ in (28) to construct a function, the set of whose nondifferentiable points is dense on the domain.

Example 8. Let φ be the tent map defined by

$$\varphi(x) = \begin{cases} x, & x \in \left[0, \frac{1}{2}\right], \\ 1 - x, & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{32}$$

One can extend φ to a periodic function on the line. For the sake of simplicity we will use the same symbol to represent the extension. Then we have the following: when $0 < \lambda \leq 1$, the set of points at which the function

$$\varphi^\alpha(x) = \sum_{j=0}^{+\infty} 2^{-\lambda j} \varphi(2^j x), \quad x \in [0, 1] \tag{33}$$

is not differentiable is dense on $[0, 1]$ (see Figure 2).

FIGURE 1: The graph of $g(x)$ in Example 7 with $\lambda = 0.5$ and the graph of the corresponding Weierstrass function $W_\lambda(x)$ with $\lambda = 0.5$.

FIGURE 2: The graph of $\varphi(x)$ in Example 8 and the graph of $\varphi^\alpha(x)$ with $\lambda = 0.55$.

In Examples 7 and 8, we set the scaling factors to the same value. However, we may give different values to the factors, and these two conclusions are still tenable.

Example 9. Suppose $W(x)$, $x \in I = [0, 2\pi]$ is generated by the IFS

$$\begin{aligned} w_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{x}{2} \\ 2^{-\lambda_1} y + \sin\left(\frac{x}{2}\right) \end{pmatrix}, \\ w_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{x}{2} + \pi \\ 2^{-\lambda_2} y - \sin\left(\frac{x}{2}\right) \end{pmatrix}. \end{aligned} \quad (34)$$

When $\lambda_1, \lambda_2 \in (0, 1]$, $W(x)$ is continuous, and the set of its nondifferentiable points is dense on I .

Also we set $N = 2$; the data points are $\{(0, 0), (\pi, 0), (2\pi, 0)\}$. The scale vector is $\alpha = (2^{-\lambda_1}, 2^{-\lambda_2})$. Then we have

$$b(x) = 0, \quad g(x) = \sin x. \quad (35)$$

By (11) and (12), we conclude that if

$$\frac{x}{2\pi} = \sum_{j=1}^{+\infty} \frac{n_j - 1}{2^j}, \quad n_j \in \{1, 2\}, \quad j \in \mathbb{N}, \quad (36)$$

then

$$W(x) = g^\alpha(x) = \sum_{k=0}^{+\infty} \left(\prod_{j=1}^k 2^{-\lambda_{n_j}} \right) \sin(2^k x). \quad (37)$$

We call this kind of functions Weierstrass-like functions (see Figure 3).

FIGURE 3: The graph of $g(x)$ in Example 9 and the graph of $g^\alpha(x)$ with $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$.

3. p -Type Smoothness on Local Fields

Let K be a local field; that is, it is a locally compact, totally disconnected, nondiscrete, completed topological field [12] with addition $\oplus$ and multiplication $\otimes$.

Moreover, if the character of K is finite, then K must contain a prime field which is isomorphic to the Galois field; otherwise, if the character of K is infinite, then K must contain a prime field which is isomorphic to the rational number field.

Further, such K can be given a valuation by a non-Archimedean norm $|\cdot| : K \rightarrow [0, +\infty)$ which satisfies (1) $|x| \geq 0$, $|x| = 0 \Leftrightarrow x = 0$; (2) $|xy| = |x||y|$; (3) $|x + y| \leq \max\{|x|, |y|\}$. Thus $K \equiv (K, \oplus, \otimes, |\cdot|)$ becomes a valued field. So one can prove that there exist a prime $p \geq 2$ and a prime element $\beta \in K$ with $|\beta| = p^{-1}$, such that every element $x \in K$ can be uniquely expressed as

$$x = (x_{-s}x_{-s+1} \cdots x_{-1}x_0x_1 \cdots) = \sum_{j=-s}^{+\infty} x_j \beta^j, \quad (38)$$

$$x_j \in \{0, 1, \dots, p-1\}, \quad j = -s, -s+1, \dots, s \in \mathbb{N},$$

with $|x| = p^s$. And we can define a distance on K :

$$d_K(x, y) = \begin{cases} ||x| - |y||, & |x| \neq |y|, \\ |x - y|, & |x| = |y|, \\ 0, & x = y, \end{cases} \quad \forall x, y \in K, \quad (39)$$

such that K becomes an ultrametric space with ultrametric $d_K(x, y)$.

Then, there are 4 cases for a local field $K \equiv (K, \oplus, \otimes, |\cdot|)$.

(1) When the character of K is finite,

- (i) K is a p -series field which is isomorphic to the Galois field $\text{GF}(p)$;
- (ii) K is a c -degree finite algebraic extension of a p -series field with $c \in \mathbb{N}$.

At these cases, the operation $\oplus$ is mod p addition term by term, no carrying, and so is $\otimes$.

(2) When the character of K is infinite,

- (i) K is a p -adic field which is isomorphic to the rational number field $\mathbb{Q}$;
- (ii) K is a c -degree finite algebraic extension of a p -adic field with $c \in \mathbb{N}$.

At these cases, the operation $\oplus$ is mod p addition term by term, carrying from left to right, and so is $\otimes$.

In this section, we concentrate to study the fractal interpolation functions on a p -series field, denoted by K_p with a prime $p \geq 2$.

Let Γ_p be the character group of K_p ; then K_p is isomorphic to Γ_p .

Denote by $D = \{x \in K_p : |x| \leq 1\}$ the ring of integers; $B_k = \{x \in K_p : |x| \leq p^{-k}\}$ the ball with center $0 \in K_p$ and radius p^{-k} , $k \in \mathbb{Z}$; $B_1 = \beta D = \{x \in K_p : |x| \leq p^{-1}\}$ the prime ideal in K_p . Moreover, $\Gamma^k = \{\xi \in \Gamma_p : |\xi| \leq p^k\}$, $k \in \mathbb{Z}$.

There are Haar measures on K_p and Γ_p , such that $\mu(B_k) \equiv |B_k| = p^{-k}$ and $\nu(\Gamma^k) \equiv |\Gamma^k| = p^k$ for $k \in \mathbb{Z}$, respectively.

Let $\chi(x)$ be the base character of K_p which is trivial on D but nontrivial on B_{-1} [13]; that is,

$$\chi(\beta^{-j}) = \begin{cases} e^{2\pi i/p}, & j = 1, \\ 1, & \text{otherwise.} \end{cases} \quad (40)$$

For $x = \sum_{j=-s}^{+\infty} x_j \beta^j$ and $\lambda = \sum_{j=-t}^{+\infty} \lambda_j \beta^j \in K_p$, a character $\chi_\lambda \in \Gamma_p$ of K_p has the form

$$\chi_\lambda(x) = \chi(\lambda x) = e^{(2\pi i/p) \sum_j x_j \lambda_{-j-1}}, \quad (41)$$

where $x_j, \lambda_k \in \{0, 1, \dots, p-1\}$, $j = -s, -s+1, \dots$; $k = -t, -t+1, \dots$; $s, t \in \mathbb{N}$.

Definition 10 (see [14]). Let $\langle \xi \rangle = \max\{1, |\xi|\}$, $m \geq 0$. If, for a Borel measurable function $f : K_p \rightarrow \mathbb{C}$ on K_p , the pseudodifferential operator

$$T_{\langle \cdot \rangle^m} f(x) = \int_{\Gamma_p} \left\{ \int_{K_p} \langle \xi \rangle^m f(t) \bar{\chi}_\xi(t-x) dt \right\} d\xi \quad (42)$$

exists at $x \in K_p$, it is said to be a p -type pointwise derivative of order m of f at x , denoted by $f^{(m)}(x)$. Similarly, if, for a Borel measurable function $f : K_p \rightarrow \mathbb{C}$ on K_p , the pseudodifferential operator for $m \geq 0$

$$T_{\langle \cdot \rangle^{-m}} f(x) = \int_{\Gamma_p} \left\{ \int_{K_p} \langle \xi \rangle^{-m} f(t) \bar{\chi}_\xi(t-x) dt \right\} d\xi \quad (43)$$

exists at $x \in K_p$, it is said to be a p -type pointwise integral of order m of f at x , denoted by $f_{(m)}(x)$.

4. IFS and FIF on Local Fields

We now turn to consider the interpolation theory on a p -series field K_p .

Suppose the set of interpolating points $\{(x^{(n)}, y^{(n)}) \in D \times \mathbb{R} : n = 0, 1, \dots, p-1\}$ is given, and

$$x^{(n)} = \sum_{j=0}^{+\infty} n \cdot \beta^j \in n \cdot \beta^0 + B_1 \quad (44)$$

for every $n \in \{0, 1, \dots, p-1\}$, where $D = \{x \in K_p : |x| \leq 1\}$. Let $L = D \times \mathbb{R}$.

Definition 11. Let K_p be the p -series field, let (K_p, d) be the metric space with d in (39), and let $w_n : L \rightarrow L$ be contractive mappings with contractive factors s_n , $n = 0, 1, \dots, p-1$, respectively. Then $\{L, w_n : n = 0, 1, \dots, p-1\}$ is called a hyperbolic iterated function system (HIFS, for short), with contractive factor $s \in [0, 1]$, where $s = \max_{0 \leq n \leq p-1} \{s_n\}$.

We construct IFS on K_p as follows.

For $n = 0, 1, \dots, p-1$, take p mappings $L_n : D \rightarrow D$ by $L_n(x) = n\beta^0 + x\beta^1$ and p mappings $F_n : L \rightarrow \mathbb{R}$ by $F_n(x, y) = \alpha_n y + \varphi_n(x)$, with $|\alpha_n| < 1$ and $\varphi_n : D \rightarrow \mathbb{R}$ in Lipschitz class $\text{Lip}(\gamma_n, K_p)$ [15], where $\gamma_n \geq 1$. Then determine p mappings $w_n : L \rightarrow L$ by

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_n(x) \\ F_n(x, y) \end{pmatrix} = \begin{pmatrix} n\beta^0 + x\beta^1 \\ \alpha_n y + \varphi_n(x) \end{pmatrix}. \quad (45)$$

Thus, $\{L, w_n : n = 0, 1, \dots, p-1\}$ forms an IFS on K_p with a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{p-1})$, α_n is called a vertical scaling factor of w_n , and let $\bar{\gamma} = \max_{0 \leq n \leq p-1} \gamma_n$.

Lemma 12. Suppose $\rho \geq 1$, $x \in [0, 1]$; then $x^\rho \leq \rho x$.

Proof. Let $f(x) = x^\rho - \rho x$, $x \in [0, 1]$. Then $f'(x) = \rho(x^{\rho-1} - 1)$. When $\rho > 1$, we have $f'(x) < 0$; hence, $f(x) \leq f(0) = 0$ for any $x \in [0, 1]$; when $\rho = 1$, $f(x) \equiv 0$ for any $x \in [0, 1]$. $\square$

Similar to that in [3], we have the following theorem.

Theorem 13. Let $\{L, w_n : n = 0, 1, \dots, p-1\}$ be the IFS defined in (45) with $|\alpha_n| < 1$ and $\varphi_n \in \text{Lip}(\gamma_n, K_p)$ with $\gamma_n \geq 1$, $n \in \{0, 1, \dots, p-1\}$. Then there exists an ultrametric d_L^* on L , such that the IFS in (45) is hyperbolic with respect to d_L^* . Moreover, there exists a unique nonempty compact set $G \subset L$ such that

$$G = \bigcup_{n=0}^{p-1} w_n(G). \quad (46)$$

Proof. We denote $d_L^* : L \times L \rightarrow [0, +\infty)$ defined by

$$\begin{aligned} d_L^*((x_1, y_1), (x_2, y_2)) &= d_{K_p}(x_1, x_2) + \theta d_{\mathbb{R}}(y_1, y_2), \\ x_1, x_2 &\in D, \quad y_1, y_2 \in \mathbb{R}, \end{aligned} \quad (47)$$

where $d_{K_p} : K_p \times K_p \rightarrow [0, +\infty)$ is defined in (39), and factor $\theta > 0$ will be determined later.

For every $n \in \{0, 1, \dots, p-1\}$,

$$\begin{aligned} d_L^*(w_n(x_1, y_1), w_n(x_2, y_2)) &= d_{K_p}(L_n(x_1), L_n(x_2)) \\ &\quad + \theta d_{\mathbb{R}}(F_n(x_1, y_1), F_n(x_2, y_2)) \\ &\leq p^{-1} d_{K_p}(x_1, x_2) \\ &\quad + \theta \left\{ |\alpha_n| |y_1 - y_2| + [d_{K_p}(x_1, x_2)]^{\gamma_n} \right\}. \end{aligned} \quad (48)$$

Since $\gamma_n \geq 1$, $|\alpha_n| < 1$, by Lemma 12, we have

$$\begin{aligned} d_L^*(w_n(x_1, y_1), w_n(x_2, y_2)) &\leq p^{-1} d_{K_p}(x_1, x_2) + \theta \left\{ |\alpha_n| |y_1 - y_2| + \gamma_n d_{K_p}(x_1, x_2) \right\} \\ &= (p^{-1} + \theta \gamma_n) d_{K_p}(x_1, x_2) + \theta |\alpha_n| d_{\mathbb{R}}(y_1, y_2). \end{aligned} \quad (49)$$

By $\bar{\gamma} = \max_{0 \leq n \leq p-1} \gamma_n$, we set $\theta = (p-1)/p\bar{\gamma}$, $a = 1/p + (p-1)\gamma_n/p\bar{\gamma} < 1$, and $b = \max\{|\alpha_n|\} < 1$, then

$$\begin{aligned} d_L^*(w_n(x_1, y_1), w_n(x_2, y_2)) &\leq a d_{K_p}(x_1, x_2) + \theta b d_{\mathbb{R}}(y_1, y_2) \\ &\leq \max\{a, b\} d_L^*((x_1, y_1), (x_2, y_2)), \\ n &= 0, 1, \dots, p-1. \end{aligned} \quad (50)$$

The proof completes. $\square$

Definition 14. One calls G in Theorem 13 an attractor of the HIFS $\{L, w_n : n = 0, 1, \dots, p-1\}$ in (45). The set $G \subset L$ is the graph of a function $f : D \rightarrow \mathbb{R}$ which obeys $f(x^{(n)}) = y^{(n)}$ for $n = 0, 1, \dots, p-1$. One refers to such function f as a fractal interpolation function on K_p (FIF, for short), associated with the HIFS $\{L, w_n : n = 0, 1, \dots, p-1\}$.

Proposition 15. Let $f : D \rightarrow \mathbb{R}$ be the FIF associated with the HIFS in (45). For any $x = \sum_{k=0}^{+\infty} x_k \beta^k \in D \subset K_p$, $x_k \in \{0, 1, \dots, p-1\}$, $k = 0, 1, 2, \dots$, one has

$$f(x) = \begin{cases} \sum_{j=0}^{+\infty} \left(\prod_{k=0}^{j-1} \alpha_{x_k} \right) \varphi_{x_j}(\sigma^{j+1}x), & x \in D, \\ 0, & \text{otherwise,} \end{cases} \quad (51)$$

where one agrees on $\prod_{k=-1}^0 \alpha_{x_k} = 1$ and the shift operator $\sigma : D \rightarrow D$ is defined by $\sigma x = (\beta^{-1}x)|_D$; that is,

$$\sigma(x_0\beta^0 + x_1\beta^1 + x_2\beta^2 + \dots) = x_1\beta^0 + x_2\beta^1 + \dots. \quad (52)$$

In a simple case, we suppose $\alpha_0 = \alpha_1 = \dots = \alpha_{p-1} = \alpha$, and, for any $x \in D$, $\varphi_0(x) = \varphi_1(x) = \dots = \varphi_{p-1}(x) = \varphi(x) \in \text{Lip}(\gamma, K_p)$, $\gamma \geq 1$. Then we have the following.

Lemma 16. Let $f : D \rightarrow \mathbb{R}$ be the FIF associated with the HIFS

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_n(x) \\ F_n(x, y) \end{pmatrix} = \begin{pmatrix} n\beta^0 + x\beta^1 \\ \alpha y + \varphi(x) \end{pmatrix}, \quad (53)$$

$$n = 0, 1, \dots, p-1,$$

where $|\alpha| < 1$, $\gamma \geq 1$, and $\varphi \in \text{Lip}(\gamma, K_p)$ with $\gamma \neq \log_p(1/|\alpha|)$; then f is continuous in D ; moreover,

$$f \in \text{Lip} \left(\min \left\{ \gamma, \log_p \frac{1}{|\alpha|} \right\}, K_p \right). \quad (54)$$

Proof. For any given $x, y \in D$, let r be a nonnegative integer such that $|x - y| = p^{-r}$. By (51), we have

$$\begin{aligned} & |f(x) - f(y)| \\ &= \left| \sum_{j=0}^{+\infty} \alpha^j [\varphi(\sigma^{j+1}x) - \varphi(\sigma^{j+1}y)] \right| \\ &\leq \sum_{j=0}^{+\infty} |\alpha|^j \cdot |\sigma^{j+1}x - \sigma^{j+1}y|^\gamma \\ &\leq \sum_{j=0}^{r-2} |\alpha|^j \left(\frac{1}{p} \right)^{(r-j-1)\gamma} + \sum_{j=r-1}^{+\infty} |\alpha|^j \cdot |\sigma^{j+1}x - \sigma^{j+1}y|^\gamma \\ &\leq \frac{p^{(1-r)\gamma} - |\alpha|^{r-1}}{1 - |\alpha| p^\gamma} + \frac{|\alpha|^{r-1}}{1 - |\alpha|} \\ &= \frac{p^{(1-r)\gamma} (1 - |\alpha|) - |\alpha|^r (p^\gamma - 1)}{(1 - |\alpha| p^\gamma) (1 - |\alpha|)}. \end{aligned} \quad (55)$$

When $\log_p(1/|\alpha|) > \gamma \geq 1$, then $0 < |\alpha| p^\gamma < 1$; we have

$$|f(x) - f(y)| \leq \frac{p^{(1-r)\gamma}}{1 - |\alpha| p^\gamma} = \frac{p^\gamma}{1 - |\alpha| p^\gamma} |x - y|^\gamma; \quad (56)$$

it follows that $f \in \text{Lip}(\gamma, K_p)$.

When $0 < \log_p(1/|\alpha|) < \gamma$, then $|\alpha| p^\gamma > 1$; we have

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{|\alpha|^r (p^\gamma - 1)}{(|\alpha| p^\gamma - 1) (1 - |\alpha|)} \\ &= \frac{p^\gamma - 1}{(|\alpha| p^\gamma - 1) (1 - |\alpha|)} |x - y|^{\log_p(1/|\alpha|)}; \end{aligned} \quad (57)$$

it follows that $f \in \text{Lip}(\log_p(1/|\alpha|), K_p)$.

We complete the proof. $\square$

Lemma 17 (see [16, Theorem 1]). If $f \in C^\lambda(K)$ ($\lambda \geq 0$), then, for any $\tau \in [0, \lambda]$, the p -type derivative of order τ of f exists, and $T_{(\cdot)^\tau} f \in C^{\lambda-\tau}(K)$.

Lemma 18 (see [15, Theorem 1]). For a local fields K , one has

$$\text{Lip}(\lambda, K) = C^\lambda(K), \quad \lambda \in (0, +\infty). \quad (58)$$

From the above three Lemmas, we can conclude the following.

Theorem 19. Let $f : D \rightarrow \mathbb{R}$ be the FIF associated with the HIFS in (53), where $|\alpha| < 1$, $\gamma \geq 1$, and $\varphi \in \text{Lip}(\gamma, K_p)$ with $\gamma \neq \log_p(1/|\alpha|)$; then f is m -order differentiable in the sense of (42) with any $m \in [0, \min\{\gamma, \log_p(1/|\alpha|)\})$.

Moreover, we get a similar conclusion in a more general case.

Theorem 20. Let $f : D \rightarrow \mathbb{R}$ be the FIF associated with the HIFS in (45), where $|\alpha_n| < 1$, $\gamma_n \geq 1$ and $\varphi_n \in \text{Lip}(\gamma_n, K_p)$ for each $n \in \{0, 1, \dots, p-1\}$. Let $\alpha = \max_{0 \leq n \leq p-1} |\alpha_n|$ and $\gamma = \min_{0 \leq n \leq p-1} \gamma_n$ with $\gamma \neq \log_p(1/|\alpha|)$. Then f is m -order differentiable in the sense of (42) with any $m \in [0, \min\{\gamma, \log_p(1/|\alpha|)\})$.

Proof. For $\varphi_n \in \text{Lip}(\gamma_n, K_p)$ and each φ_n that has a compact support D , there exists $M > 0$ independent of x , such that $\sum_{n=0}^{p-1} |\varphi_n(x)| \leq M$ for any $x \in D$.

Suppose $x, y \in D$ and $|x - y| = p^{-r}$; then

$$\begin{aligned} & |f(x) - f(y)| \\ &= \left| \sum_{j=0}^{+\infty} \left[\left(\prod_{k=0}^{j-1} \alpha_{x_k} \right) \varphi_{x_j}(\sigma^{j+1}x) - \left(\prod_{k=0}^{j-1} \alpha_{y_k} \right) \varphi_{y_j}(\sigma^{j+1}y) \right] \right| \\ &\leq \sum_{j=0}^{r-1} \left(\prod_{k=0}^{j-1} |\alpha_{x_k}| \right) |\varphi_{x_j}(\sigma^{j+1}x) - \varphi_{x_j}(\sigma^{j+1}y)| + M \sum_{j=r}^{+\infty} |\alpha|^j \\ &\leq \sum_{j=0}^{r-1} |\alpha|^j |\sigma^{j+1}x - \sigma^{j+1}y|^{\gamma_{x_j}} + M \sum_{j=r}^{+\infty} |\alpha|^j \end{aligned}$$

FIGURE 4: The graph of $W(x)$ in Example 21 with $s = 1.7$ and $s = 1.3$.

$$\begin{aligned}
 &\leq \max\{1, M\} \left(\sum_{j=0}^{r-1} \alpha^j |\sigma^{j+1}x - \sigma^{j+1}y|^y + \sum_{j=r}^{\infty} \alpha^j \right) \\
 &= \max\{1, M\} \frac{p^{(1-r)y} (1 - \alpha) - \alpha^r (p^y - 1)}{(1 - \alpha p^y)(1 - \alpha)}.
 \end{aligned} \tag{59}$$

The proof is complete. $\square$

Example 21. The Weierstrass type function on the dyadic series field K_2 is defined by [17]

$$W(x) = \sum_{j=1}^{+\infty} 2^{(s-2)j} \operatorname{Re} \chi(\beta^{-j}x), \quad x \in D, \quad 1 \leq s < 2. \tag{60}$$

Then $W(x)$ is m -order differentiable in the sense of (42) with $m < 2 - s$.

Since every $x \in D$ can be expressed as $x = \sum_{j=0}^{+\infty} x_j \beta^j$, $x_j \in \{0, 1\}$, $j = 0, 1, \dots$, it follows that

$$W(x) = \sum_{j=1}^{+\infty} 2^{(s-2)j} (-1)^{x_{j-1}}. \tag{61}$$

Thus $W(x)$ is the FIF associated with the HIFS

$$\begin{aligned}
 w_1 \left(\begin{matrix} x \\ y \end{matrix} \right) &= \left(\left(\frac{1}{2} \right)^{2-s} \beta x + \left(\frac{1}{2} \right)^{2-s} \right), \\
 w_2 \left(\begin{matrix} x \\ y \end{matrix} \right) &= \left(\left(\frac{1}{2} \right)^{2-s} \beta x + \left(\frac{1}{2} \right)^{2-s} \right),
 \end{aligned} \tag{62}$$

where $\alpha_1 = \alpha_2 = (1/2)^{2-s}$, $\varphi_1(x) = (1/2)^{2-s}$, and $\varphi_2(x) = -(1/2)^{2-s}$.

For every $\gamma > 0$, $\varphi_1, \varphi_2 \in \operatorname{Lip}(\gamma, K_2)$, by Theorem 20, we can get the result (see Figure 4).

Denote by $G(W, D)$ the graph of the Weierstrass type function $W(x) = \sum_{j=1}^{+\infty} 2^{(s-2)j} (-1)^{x_{j-1}}$ on the dyadic series field K_2 defined in Example 21 and by $G(W^{(m)}, D)$ the graph of m -order p -type derivative $W^{(m)}(x)$ of $W(x)$; then [17] gives the following elegant results:

- (1) the fractal dimensions of the graph $G(W, D)$ satisfy for a.e. $s \in (1, 2)$

$$\dim_H G(W, D) = \dim_B G(W, D) = \dim_P G(W, D) = s; \tag{63}$$

- (2) the fractal dimensions of the graph $G(W^{(m)}, D)$ satisfy for a.e. $m \in (1 - s, 2 - s)$

$$\begin{aligned}
 \dim_H G(W^{(m)}, D) &= \dim_B G(W^{(m)}, D) \\
 &= \dim_P G(W^{(m)}, D) = s + m,
 \end{aligned} \tag{64}$$

where $\dim_H E$, $\dim_B E$, and $\dim_P E$ are the Hausdorff, Box, and Packing dimension of E .

5. A Comparison between the FIF on $\mathbb{R}$ and on K_p

The structure of local fields is quite different from that of Euclidean spaces [1], so that there are essential different properties between FIF on $\mathbb{R}$ and that on K_p .

Firstly, we note that, in $\mathbb{R}$ case, the differentiability of FIF $g^\alpha(x)$ could not be guaranteed by that of $g(x)$, even $g(x)$ has more higher order smoothness. See Example 7; the function $g(x) = \cos x + 1/(2^\lambda - 1)$ is infinitely often differentiable; however, the set of nondifferential points of $W_\lambda(x)$ is dense in $[0, 2\pi]$.

Compare with the case in K_p ; the differentiability of FIF $f(x)$ has m -order smoothness depending on the α_n and γ_n of IFS. Since we use the results in Lemmas 17 and 18 which only

hold for K_p but not for $\mathbb{R}$. The relationship $\text{Lip}(\lambda, K) = C^\lambda(K)$, $\lambda \in (0, +\infty)$ on K_p helps us to get the smoothness of the Weierstrass type function in Example 21. For more examples see [1, 18].

Secondly, in the $\mathbb{R}$ case, the fractal dimensions of FIF have few research results. In the K_p case, not only may we have the p -type smoothness of FIF but also may determine the space $C^r(K_p)$ which FIF lives. Moreover, for some special IFS, not only may we determine the fractal dimensions of FIF but also may determine the fractal dimensions of the p -type derivatives of FIF, for instance, for the Weierstrass type functions in Example 21, and for more examples see [1]. However, for a general FIF, the relations between the orders of smoothness of FIF and the fractal dimensions of FIF are still open problems.

We also note that, in the $\mathbb{R}$ case, the interpolation points are $x_0 < x_1 < \dots < x_N$, and $x_0 = a$, $x_N = b$, where N is a natural number; one may choose it as one wants. However, in the K_p case, $x_n \in D$, $n \in \{0, 1, \dots, p-1\}$ with a prime p .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their gratitude to referees for their valuable comments and suggestions. This work is supported by NSFC (10571084).

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