Research Article

Dynamical Analysis in a Delayed Predator-Prey System with Stage-Structure for Both the Predator and the Prey

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A predator-prey system with two delays and stage-structure for both the predator and the prey is considered. Sufficient conditions for the local stability and the existence of periodic solutions via Hopf bifurcation with respect to both delays are obtained by analyzing the distribution of the roots of the associated characteristic equation. Specially, the direction of the Hopf bifurcation and the stability of the periodic solutions bifurcating from the Hopf bifurcation are determined by applying the normal form theory and center manifold argument. Some numerical simulations for justifying the theoretical analysis are also provided.

1. Introduction

The predator-prey systems with stage-structure have been studied by many authors [1–5]. In [2], Aiello and Freedman built a stage-structured model of single species:

\begin{align}
\frac{dx_1(t)}{dt} &= ax_m(t) - y x_1(t) - ae^{-\tau_1} x_m(t - \tau), \\
\frac{dx_m(t)}{dt} &= ae^{-\tau_1} x_m(t - \tau) - \beta x_m^2(t),
\end{align}

where $x_i(t)$ and $x_m(t)$ denote the densities of the immature population and the mature population at time $t$, respectively. However, it is well known that the predator-prey relationship often appears in population ecology. Multispecies predator-prey systems with stage structure are very important and have received much attention in recent years [6–16].

In [13], Xu investigated a predator-prey system with stage structure for the prey:

\begin{align}
\frac{dy_1(t)}{dt} &= \frac{a_2 x(t - \tau) y_2(t - \tau)}{1 + mx(t - \tau)} - r_1 y_1(t) - Dy_1(t), \\
\frac{dy_2(t)}{dt} &= Dy_1(t) - r_2 y_2(t),
\end{align}

where $x(t)$ denotes the density of the prey at time $t$. $y_1(t)$ and $y_2(t)$ denote the densities of the immature predator and the mature predator at time $t$, respectively. Xu [13] proved that system (2) is permanent under certain conditions by means of the persistence theory on infinite-dimensional systems. And sufficient conditions were derived for the local and global stability of the coexistence equilibrium of the system. In [15], F. Li and H. Li investigated a predator-prey system with stage structure for the prey:

\begin{align}
\frac{dx_1(t)}{dt} &= ax_2(t) - r_1 x_1(t) - bx_1(t), \\
\frac{dx_2(t)}{dt} &= bx_1(t) - r_2 x_2(t) - b_1 x_2^2(t) - \frac{a_1 x_2^2(t) y(t)}{1 + mx_2^2(t)}, \\
\frac{dy(t)}{dt} &= \frac{a_2 x_2^2(t - \tau) y(t - \tau)}{1 + mx_2^2(t - \tau)} - r y(t),
\end{align}

(3)
where \( x_1(t) \) and \( x_2(t) \) denote the densities of the immature prey and the mature prey at time \( t \), respectively. \( y(t) \) denotes the density of the predator at time \( t \). F. Li and H. Li [15] studied the effect of the gestation time of the predator on the dynamics of system (3).

Since both the predator and the prey have a life history that takes them through immature stage and mature stage, it is reasonable to consider the predator-prey system with stage-structure for both the predator and the prey. Starting from this point, Wang [17] proposed the following delayed system:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= r x_2(t) - a x_1(t) x_1(t - \tau_1) - (d_1 + r_1) x_1(t) - a_1 x_1(t) y_2(t), \\
\frac{dx_2(t)}{dt} &= r_1 x_1(t) - d_2 x_2(t), \\
\frac{dy_1(t)}{dt} &= a_1 x_1(t - \tau) y_2(t - \tau) - (d_3 + r_2) y_1(t), \\
\frac{dy_2(t)}{dt} &= r_2 y_1(t) - d_4 y_2(t),
\end{align*}
\]

(4)

where \( x_1(t) \) and \( x_2(t) \) denote the densities of the immature prey and the mature prey at time \( t \), respectively. \( y_1(t) \) and \( y_2(t) \) denote the densities of the immature predator and the mature predator at time \( t \), respectively. \( a \) is the interspecific competition coefficient of the immature prey, and \( a_1 \) and \( a_2 \) are the interspecific interaction coefficients between the immature prey and the mature predator. \( d_1 \) and \( d_2 \) are the death rates of the immature prey and the mature prey, respectively. \( d_3 \) and \( d_4 \) are the death rates of the immature predator and the mature predator, respectively. \( r \) is the birth rate of the immature prey, \( r_1 \) and \( r_2 \) are the transformation rates from immature individuals to mature individuals for the prey and the predator, respectively. All the parameters in system (4) are assumed to be positive constants. And \( \tau \geq 0 \) is a constant delay due to the gestation of the mature predator. Wang [17] considered the bifurcation phenomenon and the properties of periodic solutions of system (4).

The predator-prey systems with single delay have been investigated by many researchers. There are also many papers on the bifurcations of predator-prey systems with two or multiple delays [18–24]. Cui and Yan [20] investigated the stability and Hopf bifurcations of a three-species Lotka-Volterra food chain system by taking the sum of the two delays in the system as the bifurcation parameter. Gakkhar and Singh [23] investigated a modified Leslie-Gower predator-prey system with two delays and obtained the sufficient conditions for existence of Hopf bifurcation for possible combinations of the two delays. Motivated by the work above, we consider the following predator-prey system with two delays:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= r x_2(t) - a x_1(t) x_1(t - \tau_1) - (d_1 + r_1) x_1(t) - a_1 x_1(t) y_2(t), \\
\frac{dx_2(t)}{dt} &= r_1 x_1(t) - d_2 x_2(t), \\
\frac{dy_1(t)}{dt} &= a_1 x_1(t - \tau) y_2(t - \tau) - (d_3 + r_2) y_1(t), \\
\frac{dy_2(t)}{dt} &= r_2 y_1(t) - d_4 y_2(t),
\end{align*}
\]

(5)

where \( \tau_1 \geq 0 \) is feedback delay of the immature prey to the growth of the species itself and \( \tau_2 \geq 0 \) is the time delay due to the gestation of the mature predator.

The rest of this paper is organized as follows. In Section 2, sufficient conditions are established for the local stability of the positive equilibrium and the existence of Hopf bifurcation for possible combinations of the two delays in system (5). Section 3 is devoted to the properties of the Hopf bifurcation on the normal form theory and center manifold argument. Numerical simulations supporting the theoretical analysis are presented in Section 4. Finally, conclusions are given in Section 5.

2. Local Stability and Existence of Hopf Bifurcation

Considering the significance of ecology, we are interested only in positive equilibrium of system (5). It is not difficult to verify that if the condition (H):

\[
a_2 r_1 r_2 > a d_2 d_4 (d_3 + r_2) + a_2 d_2 r_2 (d_1 + r_1)
\]

(6)

holds, then system (5) has a unique positive equilibrium \( E^* (x_1^*, x_2^*, y_1^*, y_2^*) \), where

\[
\begin{align*}
x_1^* &= \frac{d_4 (d_3 + r_2)}{a_2 r_2}, \\
x_2^* &= \frac{d_4 r_1 (d_4 + r_2)}{a_2 d_2 r_2}, \\
y_1^* &= \frac{d_4 y_2^*}{r_2}, \\
y_2^* &= \frac{r r_1 - d_2 (d_1 + r_1)}{a_1 d_2} - \frac{a d_4 (d_3 + r_2)}{a_1 a_2 r_2}.
\end{align*}
\]

(7)

Let \( \bar{x}_1 = x_1 - x_1^*, \bar{x}_2 = x_2 - x_2^*, \bar{y}_1 = y_1 - y_1^* \), and \( \bar{y}_2 = y_2 - y_2^* \). Dropping the bars, system (5) can be transformed to the following form:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= a_{11} x_1(t) + a_{12} x_2(t) + a_{14} y_2(t) + b_{11} x_1(t - \tau_1) + g_1 x_1(t) y_2(t) + g_2 x_1(t) x_1(t - \tau_1), \\
\frac{dx_2(t)}{dt} &= a_{21} x_1(t) + a_{22} x_2(t), \\
\frac{dy_1(t)}{dt} &= a_{33} y_1(t) + c_{31} x_1(t - \tau_2) + c_{34} y_2(t - \tau_2) + h_1 x_1(t - \tau_2) y_2(t - \tau_2), \\
\frac{dy_2(t)}{dt} &= a_{43} y_1(t) + a_{44} y_2(t),
\end{align*}
\]

(8)
where

\[ a_{11} = -ax_1^* - a_1y_2^* - (d_1 + r_1), \]
\[ a_{12} = r, \quad a_{14} = -a_1x_1^*, \]
\[ a_{21} = r_1, \quad a_{22} = -d_2, \quad a_{33} = -(d_3 + r_2), \]
\[ a_{43} = r_2, \quad a_{44} = -d_4, \quad b_1 = -ax_1^*, \]
\[ c_{31} = a_2y_2^*, \quad c_{34} = a_2x_1^*, \]
\[ g_1 = -a_1, \quad g_2 = -a, \quad h_1 = a_2. \]

The linear system of system (8) is

\[
\frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + a_{14}y_2(t) + b_1x_1(t - \tau_1), \\
\frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t), \\
\frac{dy_1(t)}{dt} = a_{33}y_1(t) + c_{31}x_1(t - \tau_2) + c_{34}y_2(t - \tau_2), \\
\frac{dy_2(t)}{dt} = a_{43}y_1(t) + a_{44}y_2(t). 
\]

The characteristic equation of system (10) is as follows:

\[
\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0) e^{-\lambda\tau_1} + (p_2\lambda^2 + p_1\lambda + p_0) e^{-\lambda\tau_2} + (q_1\lambda + q_0) e^{-\lambda(\tau_1 + \tau_2)} = 0, 
\]

where

\[ m_3 = -(a_{11} + a_{22} + a_{33} + a_{44}), \]
\[ m_2 = a_{12}a_{21} + a_{33}a_{44} - a_{13}a_{24}, \]
\[ m_1 = (a_{12}a_{21} - a_{11}a_{22})(a_{33} + a_{44}) - (a_{11} + a_{22})(a_{33} + a_{44}), \]
\[ m_0 = (a_{11}a_{22} - a_{12}a_{21})a_{33}a_{44}, \]
\[ n_3 = -b_{11}, \quad n_2 = (a_{22} + a_{33} + a_{44})b_{11}, \]
\[ n_1 = -a_{22}b_{11}(a_{33} + a_{44}) - a_{33}a_{44}b_{11}, \]
\[ n_0 = a_{22}a_{33}a_{44}b_{11}, \quad p_2 = -a_{33}c_{34}, \]
\[ p_1 = (a_{11} + a_{22})a_{43}c_{34} - a_{13}a_{44}c_{34}, \]
\[ p_0 = (a_{12}a_{21} - a_{11}a_{22})a_{43}c_{34} + a_{14}a_{22}a_{43}c_{34}, \]
\[ q_1 = a_{43}b_{11}c_{34}, \quad q_0 = -a_{22}a_{43}b_{11}c_{34}. \]

Case 1 ($\tau_1 = \tau_2 = 0$). For $\tau_1 = \tau_2 = 0$, (11) can be rewritten in the following form:

\[
\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{13}
\]

where

\[ A_{13} = m_3 + n_3, \quad A_{12} = 2ax_1^* + a_1y_2^* + d_1 + d_2 + d_3 + d_4 + r_1 + r_2, \]
\[ A_{11} = m_1 + n_1 + p_1 + q_1, \quad A_{10} = m_0 + n_0 + p_0 + q_0. \tag{14}\]

Clearly, $A_{13} > 0$. Therefore, by the Routh-Hurwitz theorem, if the conditions $(H_{11}) A_{10} > 0$, $A_{13}A_{12} > A_{11}$ and $(H_{12}) A_{11}A_{12}A_{13} > A_{11}^2 + A_{10}A_{13}$ hold, then the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (5) without time delay is locally asymptotically stable.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). On substituting $\tau_2 = 0$, (11) becomes

\[
\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} + (B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}) e^{-\lambda\tau_1} = 0, \tag{15}
\]

where

\[ A_{23} = m_3, \quad A_{22} = m_2 + p_2, \]
\[ A_{21} = m_1 + p_1, \quad A_{20} = m_0 + p_0, \quad B_{23} = n_3, \quad B_{22} = n_2, \]
\[ B_{21} = n_1 + q_1, \quad B_{20} = n_0 + q_0. \tag{16}\]

Let $\lambda = i\omega_1 (\omega_1 > 0)$ be the root of (15). Then, we have

\[
(B_{21}\omega_1 - B_{23}\omega_1^3)\sin\tau_1\omega_1 + (B_{20} - B_{22}\omega_1^2)\cos\tau_1\omega_1 = A_{22}\omega_1^2 - \omega_1^4 - A_{20}, \tag{17}
\]

\[(B_{21}\omega_1 - B_{23}\omega_1^3)\cos\tau_1\omega_1 - (B_{20} - B_{22}\omega_1^2)\sin\tau_1\omega_1 = A_{23}\omega_1^3 - A_{21}\omega_1, \]

which follows that

\[ \omega_1^6 + e_{23}\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1 + e_{20} = 0 \tag{18} \]

with

\[ e_{23} = A_{23}^2 - B_{23}^2 - 2A_{22}, \]
\[ e_{22} = A_{22}^2 - B_{22}^2 + 2B_{21}B_{23} - 2A_{21}A_{23} + 2A_{20}, \]
\[ e_{21} = A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \]
\[ e_{20} = A_{20}^2 - B_{20}^2. \tag{19} \]

Let $\omega_1^2 = \nu_1$, then (18) becomes

\[ \nu_1^6 + e_{23}\nu_1^6 + e_{22}\nu_1^4 + e_{21}\nu_1 + e_{20} = 0. \tag{20} \]
Define
\[
f_1(v_i) = v_i^4 + e_{23}v_i^3 + e_{22}v_i^2 + e_{21}v_i + e_{20},
\]
\[
p = \frac{1}{2}e_{22} - \frac{3}{16}e_{23}, \quad q = \frac{1}{32}e_{23} - \frac{1}{8}e_{22}e_{23} + e_{21},
\]
\[
\alpha_i = \left(\frac{q}{2}\right)^3 + \left(\frac{p}{3}\right)^3, \quad \beta_i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,
\]
\[
y_1 = \frac{-q}{2} + \sqrt{\alpha_i} + \frac{q}{2} - \sqrt{\alpha_i},
\]
\[
y_2 = \beta_i\frac{-q}{2} + \sqrt{\alpha_i} + \beta_i\frac{q}{2} - \sqrt{\alpha_i},
\]
\[
y_3 = \beta_i\frac{-q}{2} + \sqrt{\alpha_i} + \beta_i\frac{q}{2} - \sqrt{\alpha_i},
\]
\[
v_{i1} = y_i - \frac{3e_{23}}{4}, \quad i = 1, 2, 3.
\]

Discussion about the roots of (20) is similar to that in [25]. Therefore, we have the following lemma.

**Lemma 1.** For (20), one has the following.
(i) If \(e_{23} < 0\), (20) has at least one positive root.
(ii) If \(e_{20} \geq 0\) and \(\alpha_i \geq 0\), (20) has positive roots if and only if \(v_{i1} > 0\) and \(f_1(v_{i1}) < 0\).
(iii) If \(e_{20} \geq 0\) and \(\alpha_i < 0\), (20) has positive roots if and only if there exists at least one \(v_{i1} \in \{v_{i1}, v_{i2}, v_{i3}\}\) such that \(v_{i1} > 0\) and \(f_1(v_{i1}) \leq 0\).

In what follows, one assumes that (H21): the coefficients in \(f_1(v_i)\) satisfy one of the following conditions in (a)-(c):
(a) \(e_{20} < 0\);
(b) \(e_{20} \geq 0\), \(\alpha_i \geq 0\), \(v_{i1} > 0\) and \(f_1(v_{i1}) < 0\);
(c) \(e_{20} \geq 0\), \(\alpha_i < 0\), and there exists at least one \(v_{i1} \in \{v_{i1}, v_{i2}, v_{i3}\}\) such that \(v_{i1} > 0\) and \(f_1(v_{i1}) \leq 0\).

If (H21) holds, one knows that (18) has at least a positive root \(\omega_{10}\) such that (15) has a pair of purely imaginary roots \(\pm i\omega_{10}\). The corresponding critical value of the delay is
\[
\tau_{1t} = \frac{1}{\omega_{10}} \arccos \left( \frac{((B_{22} - A_{22}B_{23})\omega_{10}^6}{A_{33}B_{33} + A_{32}B_{23}}\omega_{10}^4 + (A_{22}B_{23} + A_{33}B_{21})\omega_{10}^2} - A_{22}B_{22} - B_{20} \right) + (B_{22} - 2B_{21}B_{23})\omega_{10}^4 + (B_{21}^2 - 2B_{20}B_{22})\omega_{10}^2 + (B_{20}^2)^{-1} \right) - A_{22}B_{22} - B_{20} \right)
\]

\[
\times \left( (B_{22} - A_{22}B_{23})\omega_{10}^6 + (A_{22}B_{23} + A_{33}B_{21})\omega_{10}^4 + (A_{22}B_{23} + A_{33}B_{21})\omega_{10}^2 + (B_{21}^2 - 2B_{20}B_{22})\omega_{10}^4 + (B_{20}^2)^{-1} \right)
\]

Differentiating both sides of (15) regarding \(\tau_1\), one gets
\[
\left[ \frac{d\lambda}{d\tau_1} \right]^{-1} = -\frac{4\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda(\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20})}
\]

\[
+ \frac{3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21}}{\lambda(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20})} - \frac{\tau_1}{\lambda}.
\]

Thus,
\[
\Re\left[ \frac{d\lambda}{d\tau_1} \right] = \tau_1 \Rightarrow \tau_1 = \tau_{10}.
\]

\[
f_1'(v_i^*) \times (B_{23}\omega_{10}^6 + (B_{22} - 2B_{21}B_{23})\omega_{10}^4 + (B_{21}^2 - 2B_{20}B_{22})\omega_{10}^2 + (B_{20}^2)^{-1},
\]

where \(v_i^* = \omega_{10}^2\). Therefore, if the condition (H22) : \(f_1'(v_i^*) \neq 0\) holds, then \(\tau_{10} \neq \tau_1\). Namely, if the condition (H22) : \(f_1'(v_i^*) \neq 0\) holds, the transversality condition is satisfied. By the Hopf bifurcation theorem in [26], one has the following results.

**Theorem 2.** If the conditions (H21)-(H22) hold, then
(i) the positive equilibrium \(E^*(x_1^*, x_2^*, y_1^*, y_2^*)\) of system (5) is asymptotically stable for \(\tau_1 \in [0, \tau_{10}]\);
(ii) system (5) undergoes a Hopf bifurcation at the equilibrium \(E^*(x_1^*, x_2^*, y_1^*, y_2^*)\) when \(\tau_1 = \tau_{10}\), and a family of periodic solutions bifurcate from \(E^*(x_1^*, y_1^*, y_2^*)\) near \(\tau_1 = \tau_{10}\).

Case 3 (\(\tau_1 = 0, \tau_2 > 0\)). On substituting \(\tau_1 = 0\), (11) becomes
\[
\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30}
\]

\[
+ (B_{23}\lambda^2 + B_{22}\lambda + B_{20}) e^{-\lambda\tau_2} = 0,
\]

where
\[
A_{33} = m_3 + n_3, \quad A_{32} = m_2 + n_2, \quad A_{31} = m_1 + n_1, \quad A_{30} = m_0 + n_0, \quad B_{32} = p_2, \quad B_{31} = p_1 + q_1, \quad B_{30} = p_0 + q_0.
\]

Let \(\lambda = i\omega_2, (\omega_2 > 0)\) be the root of (25). Then, we have
\[
B_{33} \sin \tau_2 \omega_2 + (B_{30} - B_{32}\omega_2^2) \cos \tau_2 \omega_2
\]

\[
= A_{32}\omega_2^2 - \omega_2^4 - A_{30},
\]

\[
B_{31} \cos \tau_2 \omega_2 - (B_{30} - B_{32}\omega_2^2) \sin \tau_2 \omega_2
\]

\[
= A_{33}\omega_2^3 - A_{31}\omega_2.
\]
It follows that
\[ \omega_2^8 + e_{33} \omega_2^6 + e_{32} \omega_2^4 + e_{31} \omega_2 + e_{30} = 0, \] (28)
where
\[ e_{33} = A_{33}^2 - 2A_{32}, \]
\[ e_{32} = A_{32}^2 - B_{32}^2 + 2A_{30} - 2A_{31}A_{33}, \]
\[ e_{31} = A_{31}^2 - B_{31}^2 - 2A_{30}A_{32} + 2B_{30}B_{32}, \]
\[ e_{30} = A_{30}^2 - B_{30}^2. \]

Let \( \omega_2 = v_2 \), then (28) becomes
\[ v_2^4 + e_{33} v_2^3 + e_{32} v_2^2 + e_{31} v_2 + e_{30} = 0. \] (30)

Define
\[
\begin{align*}
    f_2(v_2) &= v_2^4 + e_{33} v_2^3 + e_{32} v_2^2 + e_{31} v_2 + e_{30}, \\
    m &= \frac{1}{2} e_{32} - \frac{3}{16} e_{33}, \\
    n &= \frac{1}{32} e_{33} - \frac{1}{8} e_{32} e_{33} + e_{31},
\end{align*}
\]
\[ \alpha_2 = \left( \frac{n}{2} \right)^{1/2} + \left( \frac{m}{3} \right)^{1/2}, \]
\[ \beta_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}, \]
\[ z_1 = \sqrt{\frac{n}{2}} + \sqrt{\alpha_2}, \quad z_2 = \sqrt{\frac{n}{2}} - \sqrt{\alpha_2}, \quad z_3 = \sqrt{\frac{n}{2}} + \sqrt{\beta_2}, \quad z_4 = \sqrt{\frac{n}{2}} - \sqrt{\beta_2}, \]
\[ v_{2i} = z_i - \frac{3e_{33}}{4}, \quad i = 1, 2, 3. \] (31)

According to Lemma 1, we know that if \((H_{32})\), the coefficients in \(f_2(v_2)\) satisfy one of the following conditions in \((a')-(c')\):

(a') \( e_{30} < 0 \);
(b') \( e_{30} \geq 0, \alpha_2 \geq 0, v_{21} > 0 \), and \( f_2(v_{21}) < 0 \);
(c') \( e_{30} \geq 0, \alpha_2 < 0 \), and there exists at least one \( v_{2*} \in \{v_{21}, v_{22}, v_{23}\} \) such that \( v_{2*} > 0 \) and \( f_2(v_{2*}) \leq 0 \).

Then, (28) has at least a positive root \( \omega_{20} \) such that (25) has a pair of purely imaginary roots \( \pm i \omega_{20} \). The corresponding critical value of the delay is
\[ \tau_{2k} = \frac{1}{\omega_{20}} < \arccos \left( \frac{B_{32} \omega_{20}^6 + (A_{33} B_{31} - A_{32} B_{32} - B_{30}) \omega_{20}^4}{(A_{30} B_{32} + A_{32} B_{30} - A_{31} B_{31}) \omega_{20}^2} \right). \]

Similarly as in Case 2, we can conclude that if the condition \((H_{32})\) \( f_2'(v_2^*) \neq 0 \), \( (v_2^* = \omega_{20}^*) \), then Re[\( dA/d\tau \)] \( \tau_{2k} \neq 0 \). That is, if the condition \((H_{32})\) \( f_2'(v_2^*) \neq 0 \) holds, the transversality condition is satisfied. According to the Hopf bifurcation theorem in [26], we obtain the following results.

**Theorem 3.** If the conditions \((H_{31})-(H_{32})\) hold, then

(i) the positive equilibrium \( E^* (x_1^*, x_2^*, y_1^*, y_2^*) \) of system (5) is asymptotically stable for \( \tau_2 \in (0, \tau_{20}) \);

(ii) system (5) undergoes a Hopf bifurcation at the positive equilibrium \( E^* (x_1^*, x_2^*, y_1^*, y_2^*) \) when \( \tau_2 = \tau_{20} \), and a family of periodic solutions bifurcate from \( E^* (x_1^*, x_2^*, y_1^*, y_2^*) \) near \( \tau_2 = \tau_{20} \).

Case 4 \( (\tau_1 = \tau_2 = \tau > 0) \). For \( \tau_1 = \tau_2 = \tau > 0 \), (II) can be transformed into the following form:

\[ \lambda^4 + A_{43} \lambda^3 + A_{42} \lambda^2 + A_{41} \lambda + A_{40} = \left( B_{43} \lambda^3 + B_{42} \lambda^2 + B_{41} \lambda + B_{40} \right) e^{-\lambda \tau} \] (33)
\[ + (C_{41} \lambda + C_{40}) e^{-2\lambda \tau} = 0, \]

where
\[ A_{43} = m_3, \quad A_{42} = m_2, \quad A_{41} = m_1, \quad A_{40} = m_0, \]
\[ B_{43} = n_3, \quad B_{42} = n_2 + p_2, \]
\[ B_{41} = n_1 + p_1, \]
\[ C_{41} = q_1, \]
\[ C_{40} = q_0. \] (34)

Multiplying by \( e^{\lambda \tau} \), (33) becomes

\[ \left( \lambda^4 + A_{43} \lambda^3 + A_{42} \lambda^2 + A_{41} \lambda + A_{40} \right) e^{\lambda \tau} \]
\[ + (C_{41} \lambda + C_{40}) e^{-\lambda \tau} + B_{43} \lambda^3 + B_{42} \lambda^2 \]
\[ + B_{41} \lambda + B_{40} = 0. \] (35)

Let \( \lambda = i \omega \) \( (\omega > 0) \) be the root of (35). Then, we can get

\[ \omega^4 - A_{43} \omega^2 + A_{40} + C_{40} \sin \omega \]
\[ + \left( A_{43} \omega^3 - A_{41} \omega + C_{41} \omega \right) \cos \omega = B_{42} \omega^2 - B_{40}, \]
\[ \omega^4 - A_{43} \omega^2 + A_{40} - C_{40} \sin \omega \]
\[ - \left( A_{43} \omega^3 - A_{41} \omega - C_{41} \omega \right) \cos \omega = B_{42} \omega^2 - B_{40}. \] (36)
Then, we obtain
\[
\begin{align*}
\sin \tau \omega &= \frac{A_7 \omega^7 + A_5 \omega^5 + A_3 \omega^3 + A_1 \omega}{\omega^8 + B_6 \omega^6 + B_4 \omega^4 + B_2 \omega^2 + B_0}, \\
\cos \tau \omega &= \frac{A_8 \omega^6 + A_4 \omega^4 + A_2 \omega^2 + A_0}{\omega^8 + B_6 \omega^6 + B_4 \omega^4 + B_2 \omega^2 + B_0},
\end{align*}
\]
where
\[
\begin{align*}
A_0 &= (C_{40} - A_{40}) B_{40}, \\
A_1 &= (A_{41} + C_{41}) B_{40} - (A_{40} + C_{40}) B_{41}, \\
A_2 &= A_{40} B_{42} + A_{42} B_{40} + B_{41} C_{41} - A_{41} B_{41} - B_{42} C_{40}, \\
A_3 &= A_{40} B_{43} + A_{43} B_{41} + B_{42} C_{42} \\
&\quad - A_{41} B_{42} - A_{42} B_{40} - B_{43} C_{41} - B_{40}, \\
A_5 &= A_{43} B_{42} - A_{42} B_{43} - B_{41}, \\
A_6 &= B_{42} - A_{43} B_{43}, \\
A_7 &= B_{43}, \\
B_0 &= A_2^2 - C_4^2, \\
B_2 &= A_1^2 - C_4^2 - 2 A_0 A_4, \\
B_4 &= A_2^2 + 2 A_4 - 2 A_1 A_4, \\
B_6 &= A_3^2 - 2 A_2^2.
\end{align*}
\]
It is well known that \(\sin^2 \tau \omega + \cos^2 \tau \omega = 1\). Therefore, we have
\[
\begin{align*}
\omega^{16} + e_{47} \omega^{14} + e_{46} \omega^{12} + e_{45} \omega^{10} + e_{44} \omega^8 \\
+ e_{43} \omega^6 + e_{42} \omega^4 + e_{41} \omega^2 + e_0 = 0,
\end{align*}
\]
where
\[
\begin{align*}
e_{47} &= 2 B_6 - A_7^2, \\
e_{46} &= B_6^2 - A_7^2 + 2 B_4 - 2 A_5 A_7, \\
e_{45} &= 2 B_2 + 2 B_4 B_0 - A_5^2 - 2 A_3 A_7 - 2 A_4 A_6, \\
e_{44} &= B_2^2 + 2 B_4 B_0 - A_5^2 - 2 A_1 A_7 - 2 A_3 A_5 - 2 A_2 A_6, \\
e_{43} &= 2 B_0 B_6 + 2 B_2 B_4 - 2 A_1 A_5 - 2 A_0 A_6 - 2 A_2 A_4 - A_3^2, \\
e_{42} &= B_2^2 + B_4 B_0 - 2 A_1 A_3 - 2 A_0 A_4, \\
e_{41} &= 2 B_0 B_2 - 2 A_0 A_2 - A_1^2, \\
e_0 &= B_0^2 - A_0^2.
\end{align*}
\]
Let \(\omega^2 = \nu\), then (39) becomes
\[
\begin{align*}
\nu^8 + e_{47} \nu^6 + e_{46} \nu^4 + e_{45} \nu^2 + e_{44} \nu^4 + e_{43} \nu^6 \\
+ e_{42} \nu^2 + e_{41} \nu + e_0 = 0.
\end{align*}
\]
If all the parameters of system (5) are given, it is easy to calculate the roots of (41). Thus, we suppose the following:
\[(H_{41})\] Equation (41) has at least one positive real root.

Without loss of generality, we assume that (41) has eight positive real roots, which are denoted as \(\nu_1, \nu_2, \nu_3, \ldots, \nu_8\). Then (39) has eight positive roots \(\omega_k = \sqrt{\nu_k}, k = 1, 2, 3, \ldots, 8\). And for every fixed \(\omega_k, k = 1, 2, 3, \ldots, 8\), the corresponding critical value of time delay \(\tau_k^{(j)}\) is
\[
\tau_k^{(j)} = \frac{1}{\omega_k^2} \arccos \left( \frac{A_4 \omega_k^6 + A_2 \omega_k^4 + A_0^2}{\omega_k^8 + B_6 \omega_k^6 + B_4 \omega_k^4 + B_2 \omega_k^2 + B_0^2} + \frac{2 j \pi}{\omega_k} \right),
\]

\(k = 1, 2, 3, \ldots, 8; j = 0, 1, 2, \ldots\)

(42)

Let
\[
\tau_0 = \min \{\tau_k^{(0)}\}, \quad \omega_0 = \omega_{k_1}, \quad k = 1, 2, 3, \ldots, 8.
\]

Next, we verify the transversality condition. Differentiating both sides of (35) with respect to \(\tau\), we get
\[
\begin{align*}
\left[ \frac{d\lambda}{d\tau} \right]^{-1} &= -(\left( 4 \lambda^3 + 3 A_{43} \lambda^2 + 2 A_{42} \lambda + A_{41} \right) e^{\lambda \tau} \\
&\quad + C_{41} e^{-\lambda \tau} + 3 B_{43} \lambda^2 + 2 B_{42} \lambda + B_{41}) \\
&\times \left( e^{\lambda \tau} - (C_{41} \lambda + C_{40}) e^{-\lambda \tau} \right) - \frac{\tau}{\lambda},
\end{align*}
\]

Thus,
\[
\text{Re} \left[ \frac{d\lambda}{d\tau} \right]^{-1} = -\frac{P_{41} Q_{41} + P_{42} Q_{42}}{Q_{41}^2 + Q_{42}^2},
\]

where
\[
\begin{align*}
P_{41} &= (A_{41} + C_{41} - 3 A_{43} \omega_0^2) \cos \tau_0 \omega_0 \\
&\quad - (2 A_{42} \omega_0 - 4 \omega_0^3) \sin \tau_0 \omega_0 - 3 B_{43} \omega_0^2 + B_{41}, \\
P_{42} &= (A_{41} + C_{41} - 3 A_{43} \omega_0^2) \sin \tau_0 \omega_0 \\
&\quad + (2 A_{42} \omega_0 - 4 \omega_0^3) \cos \tau_0 \omega_0 + 2 B_{42} \omega_0, \\
Q_{41} &= (A_{43} \omega_0^5 - A_{41} \omega_0^3 - C_{41} \omega_0) \cos \tau_0 \omega_0 \\
&\quad - \omega_0^5 - A_{42} \omega_0^3 + A_{40} \omega_0 + C_{40} \omega_0 \sin \tau_0 \omega_0, \\
Q_{42} &= (A_{43} \omega_0^5 - A_{41} \omega_0^3 + C_{41} \omega_0) \sin \tau_0 \omega_0 \\
&\quad + (\omega_0^5 - A_{42} \omega_0^3 + 3 A_{40} \omega_0 - C_{40} \omega_0) \cos \tau_0 \omega_0.
\end{align*}
\]

Obviously, if the condition \((H_{41})\) : \(P_{41} Q_{41} + P_{42} Q_{42} \neq 0\) holds, then \(\text{Re}[d\lambda/d\tau]^{-1} = 0\). Namely, if the condition \((H_{42})\) : \(P_{41} Q_{41} + P_{42} Q_{42} \neq 0\) holds, the transversality condition is satisfied. In conclusion, we have the following results.

**Theorem 4.** If the conditions \((H_{41})-(H_{42})\) hold, then

(i) the positive equilibrium \(E^* (x_1^*, x_2^*, y_1^*, y_2^*)\) of system (5) is asymptotically stable for \(\tau \in [0, \tau_0];\)**
(ii) system (5) undergoes a Hopf bifurcation at the equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ when $\tau = \tau_0$, and a family of periodic solutions bifurcate from $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ near $\tau = \tau_0$.

Case 5 ($r_1 > 0, r_2 > 0$ and $r_1 \in (0, r_{10})$). We consider system (5) under Case 2. That is, $r_1$ is in its stable interval and $r_2$ is considered as a parameter. Let $\lambda = i\omega_2' (\omega_2' > 0)$ be the root of (II). Then, we can get

\[
\Delta_{51} \sin \tau_i \omega_2' + \Delta_{52} \cos \tau_i \omega_2' = \Delta_{53},
\]
\[
\Delta_{51} \cos \tau_i \omega_2' - \Delta_{52} \sin \tau_i \omega_2' = \Delta_{54},
\]
where

\[
\Delta_{51} = q_1 \omega_2' \cos \tau_i \omega_2' - q_0 \sin \tau_i \omega_2' + p_1 \omega_2',
\]
\[
\Delta_{52} = q_1 \omega_2' \sin \tau_i \omega_2' + q_0 \cos \tau_i \omega_2' - p_2 (\omega_2')^2 + p_0,
\]
\[
\Delta_{53} = (n_2 (\omega_2')^2 - n_0) \cos \tau_i \omega_2' + (n_3 (\omega_2')^3 - n_1 \omega_2'),
\]
\[
\times \sin \tau_i \omega_2' - (\omega_2')^4 + m_2 (\omega_2')^2 - m_0,
\]
\[
\Delta_{54} = (n_0 - n_2 (\omega_2')^2) \sin \tau_i \omega_2' + (n_3 (\omega_2')^3 - n_1 \omega_2'),
\]
\[
\times \cos \tau_i \omega_2' + m_3 (\omega_2')^3 - m_1 \omega_2'.
\]

It follows that

\[
\epsilon_0 (\omega_2') + \epsilon_1 (\omega_2') \cos \tau_i \omega_2' + \epsilon_2 (\omega_2') \sin \tau_i \omega_2' = 0
\]
with

\[
c_0 (\omega_2') = (\omega_2')^8 + (m_2^2 + n_2^3 - 2m_2) (\omega_2')^6
\]
\[
+ (m_2^2 + n_2^3 - 2p_2^2 + 2m_0 - 2m_1 m_3 - 2n_1 n_3) (\omega_2')^4
\]
\[
+ (m_1^2 + n_1^3 - p_1^2 - q_0^2 - 2m_0 m_2
\]
\[
- 2n_0 n_2 + 2p_0 p_2) (\omega_2')^2
\]
\[
+ m_0^2 + n_0^3 - q_0^2 - q_0^2,
\]
\[
c_1 (\omega_2') = 2 (m_3 n_3 - n_2) (\omega_2')^6
\]
\[
+ 2 (m_3 n_3 + n_0 - m_3 n_3 - m_1 n_3) (\omega_2')^4
\]
\[
+ 2 (m_3 n_2 - m_0 n_2 - m_2 n_0 - p_1 q_1 + p_2 q_0) (\omega_2')^2
\]
\[
+ 2 (m_0 q_2 - p_0 q_0),
\]
\[
c_2 (\omega_2') = -2n_0 (\omega_2')^7
\]
\[
+ 2 (m_3 n_3 - m_3 n_3 + n_1) (\omega_2')^5
\]
\[
+ 2 (m_3 n_3 + m_3 n_3 - m_1 n_3)
\]
\[
- m_3 n_1 + p_2 q_1) (\omega_2')^3
\]
\[
+ 2 (m_3 n_1 - m_3 n_1 + p_1 q_1 - p_1 q_0) \omega_2'.
\]

Suppose that $(H_3)$: Equation (49) has at least finite positive real roots. If the condition $(H_3)$ holds, we denote the positive roots of (49) as $\omega_2', \omega_3', \ldots, \omega_k$. Then, for every fixed $\omega_2'$ ($i = 1, 2, 3, \ldots, k$), the corresponding critical value of time delay $\{\tau_2^{(j)} \mid j = 0, 1, 2, \ldots\}$ is

\[
\tau_2^{(j)} = \frac{1}{\omega_2'} \arccos \left(\frac{\Delta_{51} \Delta_{54} + \Delta_{52} \Delta_{53}}{\Delta_{51}^2 + \Delta_{52}^2} + 2 \frac{j\pi}{\omega_2'}, \right)
\]
\[
i = 1, 2, 3, \ldots, k, \quad j = 0, 1, 2, \ldots.
\]

Let

\[
\tau_2^* = \min \left\{ \tau_2^{(j)} \mid i = 1, 2, 3, \ldots, k \right\}, \quad \omega_2^* = \omega_2'|_{\tau_2 = \tau_2^*}.
\]

Differentiating (II) regarding $\tau_2$, we obtain

\[
\left[ \frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{g_0 (\lambda) + g_1 (\lambda) e^{-\lambda \tau_1} + g_2 (\lambda) e^{-\lambda \tau_2} + g_3 (\lambda) e^{-\lambda (\tau_1 + \tau_2)}}{h_2 (\lambda) e^{-\lambda \tau_2} + h_3 (\lambda) e^{-\lambda (\tau_1 + \tau_2)}}
\]
\[
- \frac{\tau_2}{\lambda}
\]

with

\[
g_0 (\lambda) = 4\lambda^3 + 3m_3 \lambda^2 + 2m_2 \lambda + m_1,
\]
\[
g_1 (\lambda) = -\tau_1 n_3 \lambda^3 + (3n_1 - \tau_1 n_2) \lambda^2
\]
\[
+ (2n_2 - \tau_1 n_1) \lambda + n_1 - \tau_1 n_0,
\]
\[
g_2 (\lambda) = 2p_2 \lambda + p_1,
\]
\[
g_3 (\lambda) = q_1,
\]
\[
h_2 (\lambda) = p_2 \lambda^3 + p_1 \lambda^2 + p_0 \lambda,
\]
\[
h_3 (\lambda) = q_1 \lambda^2 + q_0 \lambda.
\]

Hence,

\[
\text{Re} \left[ \frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{P_{51} Q_{53} - P_{52} Q_{52}}{Q_{51}^2 + Q_{52}^2},
\]

where

\[
P_{51} = (2p_2 \omega_2^* - q_1 \sin \tau_2 \omega_2^*) \sin \tau_2 \omega_2^*
\]
\[
+ (p_1 + q_1 \cos \tau_2 \omega_2^*) \cos \tau_2 \omega_2^*
\]
\[
+ \left( \tau_1 n_3 (\omega_2^*)^3 + (2n_2 - \tau_1 n_1) \omega_2^* \right) \sin \tau_2 \omega_2^*
\]
\[
+ \left( (\tau_1 n_2 - 3n_3) (\omega_2^*)^2 + n_1 - \tau_1 n_0 \right) \cos \tau_2 \omega_2^*
\]
\[
- 3m_3 (\omega_2^*)^2 + m_1,
\]
\[ P_{52} = (2\omega_2^* - q_1 \sin \tau_1 \omega_2^*) \cos \tau_2^* \omega_2^* \]
\[ - (p_1 + q_1 \cos \tau_1 \omega_2^*) \sin \tau_1 \omega_2^* \]
\[ + (r_1 n_5 \omega_2^*)^3 + (2n_2 - r_1 n_1 \omega_2^*) \cos \tau_1 \omega_2^* \]
\[ - \left[ (r_1 n_2 - 3n_4) \omega_2^* \cos \tau_1 \omega_2^* + n_1 - r_1 n_0 \right] \sin \tau_1 \omega_2^* \]
\[ - 4(\omega_2^*)^3 + 2m_2 \omega_2^*, \]
\[ Q_{51} = (p_2 \omega_2^*)^3 - p_0 \omega_2^* - q_1 \omega_2^* \sin \tau_1 \omega_2^* \]
\[ - q_0 \omega_2^* \cos \tau_1 \omega_2^* \]
\[ + (p_1(\omega_2^*)^2 + q_1(\omega_2^*)^2) \cos \tau_1 \omega_2^* \]
\[ - q_0 \omega_2^* \sin \tau_1 \omega_2^*, \]
\[ Q_{52} = (p_2 \omega_2^*)^3 - p_0 \omega_2^* - q_1 \omega_2^* \sin \tau_1 \omega_2^* \]
\[ - q_0 \omega_2^* \cos \tau_1 \omega_2^* \]
\[ + (p_1(\omega_2^*)^2 + q_1(\omega_2^*)^2) \cos \tau_1 \omega_2^* \]
\[ - q_0 \omega_2^* \sin \tau_1 \omega_2^*. \]

where
\[ L_\mu \phi = (\tau_2^* + \mu) \left( A' \phi(0) + B' \phi \left( -\frac{\tau_1^*}{\tau_2^*} \right) + C' \phi(-1) \right), \]
\[ F(\mu, \phi) = (\tau_2^* + \mu) \]
\[ \begin{pmatrix} g_1 x_1(t) y_2(t) + g_2 x_1(t) x_1(t - \tau_1) \\ h_1 x_1(t - \tau_2) y_2(t - \tau_2) \end{pmatrix} \]
\[ (58) \]

with
\[ \phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C([−1, 0], R^4), \]
\[ A' = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \]
\[ B' = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ C' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_{51} & 0 & 0 & c_{54} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]
\[ (59) \]

Using the Riesz representation theorem, there exists a 4×4 matrix function \( \eta(\theta, \mu) : [-1, 0] \rightarrow R^4 \) whose elements are of bounded variation, such that
\[ L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C([−1, 0], R_4). \]
\[ (60) \]

In fact, we can choose
\[ \eta(\theta, \mu) = \begin{cases} \left( \tau_2^* + \mu \right) \left( A' + B' + C' \right), & \theta = 0, \\ \left( \tau_2^* + \mu \right) \left( B' + C' \right), & \theta \in \left( -\frac{\tau_1^*}{\tau_2^*}, 0 \right), \\ \left( \tau_2^* + \mu \right) C', & \theta \in \left( -1, -\frac{\tau_1^*}{\tau_2^*} \right), \\ 0, & \theta = -1. \end{cases} \]
\[ (61) \]

For \( \phi \in C([−1, 0], R^4) \), we define
\[ A(\mu) \phi = \int_{-1}^{0} \frac{d\phi(\theta)}{d\theta}, \quad -1 \leq \theta < 0, \]
\[ \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \quad \theta = 0, \]
\[ R(\mu) \phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \]
\[ (62) \]
Then system (57) can be transformed into the following operator equation:

\[
\dot{u}(t) = A(\mu)u_t + R(\mu)u_t,
\]

where \(u_t = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta), u_3(t+\theta), u_4(t+\theta))\).

The adjoint operator \(A^*\) of \(A\) is defined by

\[
A^*(\phi) = \begin{cases}
-\frac{d\phi(s)}{ds}, & 0 < s \leq 1, \\
\int_{-1}^{0} d\eta^T(s,\mu)\phi(-s), & s = 0,
\end{cases}
\]

associated with a bilinear inner product:

\[
\langle \phi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta) d\eta(\theta) \phi(\xi) d\xi,
\]

where \(\eta(\theta) = \eta(\theta,0)\).

By the discussion above, we know that \(\pm i\tau^* \omega^*_2\) are eigenvalues of \(A(0)\) and they are also eigenvalues of \(A^*(0)\). We assume that \(q(\theta) = (1, q_2, q_3, q_4)^T e^{i\tau^* \omega^*_2 \theta}\) are the eigenvectors of \(A(0)\) belonging to the eigenvalue \(+i\tau^* \omega^*_2\) and \(q^*(s) = \)

---

**Figure 1:** The track of the states \(x_1, x_2, y_1, y_2\) for \(\tau_1 = 4.2500 < \tau_{10} = 4.5061\).

**Figure 2:** The phase plot of the states \(x_1, x_2, y_2\) for \(\tau_1 = 4.2500 < \tau_{10} = 4.5061\).
$D(1, q_2^*, q_3^*, q_4^*) e^{i\tau_2 \omega_2^*}$ are the eigenvectors of $A^*(0)$ belonging to the eigenvalue $-i\tau_2^* \omega_2^*$.

By a simple computation, we can get

$$
q_2 = \frac{a_{21}}{i \omega_2^* - a_{22}},
q_3 = \frac{c_{34} (i \omega_2^* - a_{11}) + a_{14} c_{31} - b_1 c_{34} e^{-i\tau_1^* \omega_1^*}}{a_{14} (i \omega_2^* - a_{33}) e^{i\tau_2^* \omega_2^*}} - \frac{a_{12} a_{21} c_{34}}{a_{14} (i \omega_2^* - a_{33}) e^{i\tau_2^* \omega_2^*}},
q_4 = \left( \frac{i \omega_2^* - a_{11}}{i \omega_2^* - a_{22}} - a_{12} a_{21}
\right)
- b_{11} (i \omega_2^* - a_{22}) e^{-i\tau_1^* \omega_1^*}) \times (a_{14} (i \omega_2^* - a_{22}))^{-1},
q_4^* = \frac{-a_{12}}{i \omega_2^* + a_{22}},
q_4^* = -\left( (i \omega_2^* + a_{11}) (i \omega_2^* + a_{22}) - a_{12} a_{21} + b_{11}
\times (i \omega_2^* + a_{22}) e^{i\tau_1^* \omega_1^*}) \times (c_{31} (i \omega_2^* + a_{33}) e^{i\tau_2^* \omega_2^*})^{-1},
q_4^* = \frac{c_{34} (i \omega_2^* + a_{11}) - a_{14} c_{31} - b_1 c_{34} e^{i\tau_1^* \omega_1^*}}{c_{31} (i \omega_2^* + a_{44})}
- \frac{a_{12} a_{21} c_{34}}{c_{31} (i \omega_2^* + a_{22}) (i \omega_2^* + a_{44})}.
$$

From (65), we can get

$$
\langle q^* (s), q (\theta) \rangle = \bar{q}^* (0) q (0)
- \int_{\theta-1}^{\theta} \int_{0}^{\theta} \bar{q}^* (\xi - \theta) d\eta (\theta) q (\xi) d\xi
= \bar{D} (1, \bar{q}^*_2, \bar{q}^*_3, \bar{q}^*_4) (1, q_2, q_3, q_4)^T
- \bar{D} \int_{\theta-1}^{\theta} \int_{0}^{\theta} (1, \bar{q}^*_2, \bar{q}^*_3, \bar{q}^*_4)
\times e^{i\tau_1^* \omega_1^* (\theta - \xi)} d\eta (\theta)
\times (1, q_2, q_3, q_4)^T e^{i\tau_2^* \omega_2^* d\xi}
= \bar{D} \left[ 1 + q_2 \bar{q}^*_2 + q_3 \bar{q}^*_3 + q_4 \bar{q}^*_4 - (1, \bar{q}^*_2, \bar{q}^*_3, \bar{q}^*_4)^T
\int_{-1}^{0} e^{i\tau_1^* \omega_1^* (\theta - \xi)} d\eta (\theta) (1, q_2, q_3, q_4)^T \right]
\times \left[ 1 + q_2 \bar{q}^*_2 + q_3 \bar{q}^*_3 + q_4 \bar{q}^*_4 + b_{11} \tau_1^* e^{-i\tau_1^* \omega_1^*}
+ \bar{q}^*_3 \tau_2^* e^{-i\tau_2^* \omega_2^*} (c_{31} + c_{44} q_4) \right].
$$

Normalizing $q$ and $q^*$ by the condition $\langle q^*, q \rangle = 1$ and $\langle q^*, q \rangle$, one can get

$$
\bar{D} = \left[ 1 + q_2 \bar{q}^*_2 + q_3 \bar{q}^*_3 + q_4 \bar{q}^*_4 + b_{11} \tau_1^* e^{-i\tau_1^* \omega_1^*}
+ \bar{q}^*_3 \tau_2^* e^{-i\tau_2^* \omega_2^*} (c_{31} + c_{44} q_4) \right]^{-1}.
$$

Next, we can get the coefficients determining the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions by the algorithms given in [26]:

$$
g_{20} = 2 \tau_2^* \bar{D} \left[ g_1 q^{(4)} (0) + g_2 q^{(4)} \left( -\frac{\tau_1^*}{\tau_2^*} \right) \right]
+ h_1 \bar{q}^*_3 (q^{(1)} (1) - q^{(4)} (1)),
$$

$$
g_{31} = \tau_2^* \bar{D} \left[ g_1 (\bar{q}^*_2 (0) + q^{(4)} (0))
+ g_2 \left( \bar{q}^*_1 \left( -\frac{\tau_1^*}{\tau_2^*} \right) + q^{(4)} \left( -\frac{\tau_1^*}{\tau_2^*} \right) \right)
+ h_1 \bar{q}^*_3 (q^{(1)} (1) - q^{(4)} (1))\right],
$$

$$
g_{02} = 2 \tau_2^* \bar{D} \left[ g_1 q^{(4)} (0) + g_2 q^{(4)} \left( -\frac{\tau_1^*}{\tau_2^*} \right) \right]
+ h_1 \bar{q}^*_3 (q^{(1)} (1) - q^{(4)} (1)),
$$

$$
g_{21} = 2 \tau_2^* \bar{D} \left[ g_1 (W_1^{(1)} (0) q^{(4)} (0) + \frac{1}{2} W_2^{(1)} (0) \bar{q}^{(4)} (0)
+ W_1^{(4)} (0) + \frac{1}{2} W_2^{(4)} (0) \right)
+ g_2 \left( W_1^{(1)} (0) q^{(1)} \left( -\frac{\tau_1^*}{\tau_2^*} \right) + \frac{1}{2} W_2^{(1)} (0) \bar{q}^{(1)} \right)
\times \left( -\frac{\tau_1^*}{\tau_2^*} \right) + W_1^{(4)} \left( -\frac{\tau_1^*}{\tau_2^*} \right).
with

\[
W_{20}(\theta) = \frac{i q_{20} q_{0}(0)}{\tau_2^* \omega_2^*} e^{i \tau_2^* \omega_2^* \theta} + \frac{i \overline{q}_{20} \overline{q}(0)}{3 \tau_2^* \omega_2^*} e^{-i \tau_2^* \omega_2^* \theta} + E_1 e^{2i \tau_2^* \omega_2^* \theta},
\]

\[
W_{11}(	heta) = -\frac{i q_{11} q_{0}(0)}{\tau_1^* \omega_1^*} e^{i \tau_1^* \omega_1^* \theta} + \frac{i \overline{q}_{11} \overline{q}(0)}{\tau_1^* \omega_1^*} e^{-i \tau_1^* \omega_1^* \theta} + E_2,
\]

(70)

where \( E_1 \) and \( E_2 \) can be determined by the following equations, respectively:

\[
\begin{align*}
+ \frac{1}{2} W_{20}^{(i)} \left( \frac{\tau_1^*}{\tau_2^*} \right) \\
+ h_1 \overline{q}_{3}^{(i)} \left[ W_{11}^{(i)} (-1) q^{(4)} (-1) \\
+ \frac{1}{2} W_{20}^{(i)} (-1) \overline{q}^{(4)} (-1) \\
+ W_{11}^{(4)} (-1) q^{(1)} (-1) \\
+ \frac{1}{2} W_{20}^{(4)} (-1) \overline{q}^{(1)} (-1) \right]
\end{align*}
\]

(69)
Therefore, we can calculate the following values:

\[
C_1(0) = \frac{i}{2\tau_2 \omega_2^*} \left( g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{21}|^2}{3} \right) + \frac{g_{21}}{2},
\]

\[
\beta_2 = 2 \text{ Re } \{ C_1(0) \},
\]

\[
T_2 = -\frac{\text{ Im } \{ C_1(0) \} + \mu_2 \text{ Im } \{ \lambda' (\tau_2^2) \}}{\tau_2^2 \omega_2^*},
\]

(72)

By the results of Hassard et al. [26], the following results are obtained.
Theorem 6. For system (5),

(i) the direction of the Hopf bifurcation is determined by \( \mu_2 \): if \( \mu_2 > 0 \), the Hopf bifurcation is supercritical; if \( \mu_2 < 0 \), the Hopf bifurcation is subcritical;

(ii) the stability of the bifurcating periodic solutions is determined by \( \beta_2 \): if \( \beta_2 < 0 \), the bifurcating periodic solutions are stable; if \( \beta_2 > 0 \), the bifurcating periodic solutions are unstable;

(iii) the period of the bifurcating periodic solution is determined by \( \tau_2 \): if \( \tau_2 > 0 \), the period of the bifurcating periodic solutions increases; if \( \tau_2 < 0 \), the period of the bifurcating periodic solutions decreases.

4. Numerical Simulation

In this section, we present some numerical simulations to support the theoretical analysis in Sections 2 and 3. We consider the following system by taking the same coefficients as in [17]:

\[
\frac{dx_1(t)}{dt} = 5x_2(t) - x_1(t)x_1(t - \tau_1) - 2x_1(t) - x_1(t)y_2(t),
\]

\[
\frac{dx_2(t)}{dt} = x_1(t) - x_2(t),
\]

\[
\frac{dy_1(t)}{dt} = 2x_1(t - \tau_2)y_2(t - \tau_2) - 2.2y_1(t),
\]

\[
\frac{dy_2(t)}{dt} = 1.2y_1(t) - y_2(t),
\]

from which we can get \( a_0r_1r_2 = 12, ad_2d_4(d_3 + r_2) + a_0d_2r_2(d_1 + r_1) = 7 \). Obviously, \( a_0r_1r_2 > ad_2d_4(d_3 + r_2) + a_0d_2r_2(d_1 + r_1) \). Therefore, the condition \( (H) \) is satisfied. Further, we get the unique positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) of system (73) by MATLAB software package. Then, we have \( A_{10} = 4.5833, A_{11} = 7.5163, A_{13}A_{12} = 233.1899, A_{11}A_{12}A_{13} = 1752.7 \), and

\[
A_{11}^2 + A_{10}^2A_{13}^2 = 525.5744.
\]

Obviously, \( A_{10} > 0, A_{13}A_{12} > A_{11} \), and \( A_{11}A_{12}A_{13} > A_{11}^2 + A_{10}A_{13}^2 \). Namely, the conditions \( (H_{11}) \) and \( (H_{13}) \) hold.

For \( \tau_1 > 0, \tau_2 = 0 \). From (18), we have

\[
\omega^4 + 45.3999\omega^3 + 284.0324\omega^2 - 163.6041\omega + 21.0076 = 0.
\]

Then, (20) becomes

\[
\omega^4 + 45.3999\omega^3 + 284.0324\omega^2 - 163.6041\omega + 21.0076 = 0
\]

and it has two positive real roots: \( \nu_{11} = 0.3397, \nu_{12} = 0.2003 \). Thus, the condition \( (H_{21}) \) is satisfied. Further, we have \( \omega_{10} = 0.4475, \tau_{10} = 4.5061 \). Then, we get \( f_1'(\nu_{1*}) = -44.3242 \neq 0 \). Therefore, the condition \( (H_{22}) \) holds. From Theorem 2, we know that the positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) is asymptotically stable for \( \tau_1 \in (0, 4.5061) \), which can be shown in Figures 1, 2, and 3. The positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) is unstable when \( \tau_1 > 4.5061 \) and system (73) undergoes a Hopf bifurcation at \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \), and a family of periodic solutions bifurcate from \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \). This property can be illustrated by Figures 4–6. Similarly, we have \( \omega_{20} = 0.1517, \tau_{20} = 9.9183 \) for \( \tau_1 = 0, \tau_2 > 0 \). For \( \tau_2 = 7.7500 \in [0, \tau_{20}) \), the positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) is asymptotically stable from Theorem 3 and this property can be shown in Figures 7, 8, and 9. If \( \tau_2 = 10.7500 > \tau_{20} = 9.9183 \), the positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) is unstable and a Hopf bifurcation occurs, and the corresponding waveforms and phase plots are shown in Figures 10, 11, and 12.

For \( \tau_1 = \tau_2 = \tau > 0 \), we can get \( \omega_0 = 0.3412, \tau_0 = 3.1043 \). From Theorem 4, we know that the positive equilibrium \( E^*(0.9167, 0.9167, 1.7361, 2.0833) \) is asymptotically stable for \( \tau \in (0, \tau_0) \), which can be illustrated by Figures 13, 14, and 15. As can be seen from Figure 5 that when \( \tau = 2.7500 \in (0, \tau_0) \), the positive equilibrium
Let $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable. However, if $r = 3.2500 > r_0 = 3.1043$, then the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ becomes unstable and a family of bifurcating periodic solutions occur, which is illustrated by Figures 16, 17, and 18.

Lastly, regarding $r_2$ as a parameter and letting $r_1 = 2.5 \in (0, r_{10})$, we can obtain that $\omega_2^* = 0.2253$. Further, we have $r_2^* = 6.6494$ and $\lambda'(r_2^*) = 0.0466 + 0.0314i$. Letting $r_2 = 5.5500 \in (0, r_2^*)$, we can know that the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable from Theorem 5. When $r_2 =$
Figure 13: The track of the states $x_1, x_2, y_1, y_2$ for $\tau = 2.7500 < \tau_0 = 3.1043$.

Figure 14: The phase plot of the states $x_1, x_2, y_2$ for $\tau = 2.7500 < \tau_0 = 3.1043$.

Figure 15: The phase plot of the states $x_2, y_1, y_2$ for $\tau = 2.7500 < \tau_0 = 3.1043$.

7.5000 > $\tau^*_2 = 6.6494$, then the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ becomes unstable and a Hopf bifurcation occurs. The corresponding waveforms and phase plots are illustrated in Figures 19, 20, 21, 22, 23, and 24.

In addition, for system (73), we have $C_1(0) = -31.3276 + 4.4079i$ by a simple computation. Further, from (72), we have $\mu_2 = 672.2661 > 0$, $\beta_2 = -62.6552 < 0$, and $T_2 = -17.0330 < 0$. Therefore, by Theorem 6, we can know that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the bifurcated periodic solutions decrease. Since the bifurcating periodic solutions of system (73) are stable, the predator and the prey species in system (73) can coexist in an oscillatory mode from the view of biology.

5. Conclusions

In this paper, a delayed predator-prey system with stage-structure for both the predator and the prey population is investigated. The sufficient conditions for the local stability of the positive equilibrium and the existence of Hopf bifurcation for the possible combinations of two delays are obtained.
The main results are given in Theorems 2–5. The results show that the two time delays can play a complicated role in system (5). Furthermore, the explicit formulae determining the direction of the bifurcation and the stability of the bifurcating periodic solutions are established when $\tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$ by using the normal form theory and center manifold theorem. The main results are given in Theorem 6. If the bifurcating periodic solutions are stable, the predator and the prey species can coexist in an oscillatory mode from the view of biology. Compared with literature [17], we
Figure 19: The track of the states $x_1(t), x_2(t), y_1(t), y_2(t)$ for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

Figure 20: The phase plot of the states $x_1(t), x_2(t), y_1(t), y_2(t)$ for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

Figure 21: The phase plot of the states $x_1(t), y_1(t), y_2(t)$ for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

consider not only the time delay due to the gestation of the mature predator but also the negative feedback delay of the immature prey. Wang [17] obtained that the species in system (5) with only the time delay due to the gestation of the mature predator could coexist under some conditions. However, we get that the species could also coexist with some available time delays of the immature prey and the mature predator under certain conditions. This is valuable from the view of ecology. Some numerical simulations are also included to support the obtained theoretical results.
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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