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Research Article

Dynamical Analysis in a Delayed Predator-Prey System with Stage-Structure for Both the Predator and the Prey

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A predator-prey system with two delays and stage-structure for both the predator and the prey is considered. Sufficient conditions for the local stability and the existence of periodic solutions via Hopf bifurcation with respect to both delays are obtained by analyzing the distribution of the roots of the associated characteristic equation. Specially, the direction of the Hopf bifurcation and the stability of the periodic solutions bifurcating from the Hopf bifurcation are determined by applying the normal form theory and center manifold argument. Some numerical simulations for justifying the theoretical analysis are also provided.

1. Introduction

The predator-prey systems with stage-structure have been studied by many authors [1–5]. In [2], Aiello and Freedman built a stage-structured model of single species:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \frac{dx_m(t)}{dt} &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t), \end{aligned} \tag{1}$$

where $x_i(t)$ and $x_m(t)$ denote the densities of the immature population and the mature population at time t, respectively. However, it is well known that the predator-prey relationship often appears in population ecology. Multispecies predator-prey systems with stage structure are very important and have received much attention in recent years [6–16].

In [13], Xu investigated a predator-prey system with stage structure for the predator:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(r - ax(t) - \frac{a_1 y_2(t)}{1+mx(t)}\right), \\ \frac{dy_1(t)}{dt} &= \frac{a_2 x(t-\tau) y_2(t-\tau)}{1+mx(t-\tau)} - r_1 y_1(t) - Dy_1(t), \\ \frac{dy_2(t)}{dt} &= Dy_1(t) - r_2 y_2(t), \end{aligned} \tag{2}$$

where $x(t)$ denotes the density of the prey at time t. $y_1(t)$ and $y_2(t)$ denote the densities of the immature predator and the mature predator at time t, respectively. Xu [13] proved that system (2) is permanent under certain conditions by means of the persistence theory on infinite-dimensional systems. And sufficient conditions were derived for the local and global stability of the coexistence equilibrium of the system. In [15], F. Li and H. Li investigated a predator-prey system with stage structure for the prey:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= ax_2(t) - r_1 x_1(t) - bx_1(t), \\ \frac{dx_2(t)}{dt} &= bx_1(t) - r_2 x_2(t) - b_1 x_2^2(t) - \frac{a_1 x_2^2(t) y(t)}{1+mx_2^2(t)}, \\ \frac{dy(t)}{dt} &= \frac{a_2 x_2^2(t-\tau) y(t-\tau)}{1+mx_2^2(t-\tau)} - ry(t), \end{aligned} \tag{3}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature prey and the mature prey at time t, respectively. $y(t)$ denotes the density of the predator at time t. F. Li and H. Li [15] studied the effect of the gestation time of the predator on the dynamics of system (3).

Since both the predator and the prey have a life history that takes them through immature stage and mature stage, it is reasonable to consider the predator-prey system with stage-structure for both the predator and the prey. Starting from this point, Wang [17] proposed the following delayed system:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= rx_2(t) - ax_1^2(t) - (d_1 + r_1)x_1(t) - a_1x_1(t)y_2(t),\\ \frac{dx_2(t)}{dt} &= r_1x_1(t) - d_2x_2(t),\\ \frac{dy_1(t)}{dt} &= a_2x_1(t-\tau)y_2(t-\tau) - (d_3 + r_2)y_1(t),\\ \frac{dy_2(t)}{dt} &= r_2y_1(t) - d_4y_2(t),\end{aligned} \tag{4}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature prey and the mature prey at time t, respectively. $y_1(t)$ and $y_2(t)$ denote the densities of the immature predator and the mature predator at time t, respectively. a is the intraspecific competition coefficient of the immature prey. a_1 and a_2 are the interspecific interaction coefficients between the immature prey and the mature predator. d_1 and d_2 are the death rates of the immature prey and the mature prey, respectively. d_3 and d_4 are the death rates of the immature predator and the mature predator, respectively. r is the birth rate of the immature prey. r_1 and r_2 are the transformation rates from immature individuals to mature individuals for the prey and the predator, respectively. All the parameters in system (4) are assumed to be positive constants. And $\tau \geq 0$ is a constant delay due to the gestation of the mature predator. Wang [17] considered the bifurcation phenomenon and the properties of periodic solutions of system (4).

The predator-prey systems with single delay have been investigated by many researchers. There are also many papers on the bifurcations of predator-prey systems with two or multiple delays [18–24]. Cui and Yan [20] investigated the stability and Hopf bifurcations of a three-species Lotka-Volterra food chain system by taking the sum of the two delays in the system as the bifurcation parameter. Gakkhar and Singh [23] investigated a modified Leslie-Gower predator-prey system with two delays and obtained the sufficient conditions for existence of Hopf bifurcation for possible combinations of the two delays. Motivated by the work above, we consider the following predator-prey system with two delays:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= rx_2(t) - ax_1(t)x_1(t-\tau_1)\\ &\quad - (d_1 + r_1)x_1(t) - a_1x_1(t)y_2(t),\\ \frac{dx_2(t)}{dt} &= r_1x_1(t) - d_2x_2(t),\\ \frac{dy_1(t)}{dt} &= a_2x_1(t-\tau_2)y_2(t-\tau_2) - (d_3 + r_2)y_1(t),\\ \frac{dy_2(t)}{dt} &= r_2y_1(t) - d_4y_2(t),\end{aligned} \tag{5}$$

where $\tau_1 \geq 0$ is feedback delay of the immature prey to the growth of the species itself and $\tau_2 \geq 0$ is the time delay due to the gestation of the mature predator.

The rest of this paper is organized as follows. In Section 2, sufficient conditions are established for the local stability of the positive equilibrium and the existence of Hopf bifurcation for possible combinations of the two delays in system (5). Section 3 is devoted to the properties of the Hopf bifurcation on the normal form theory and center manifold argument. Numerical simulations supporting the theoretical analysis are presented in Section 4. Finally, conclusions are given in Section 5.

2. Local Stability and Existence of Hopf Bifurcation

Considering the significance of ecology, we are interested only in positive equilibrium of system (5). It is not difficult to verify that if the condition (H):

$$a_2rr_1r_2 > ad_2d_4(d_3 + r_2) + a_2d_2r_2(d_1 + r_1) \tag{6}$$

holds, then system (5) has a unique positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$, where

$$\begin{aligned}x_1^* &= \frac{d_4(d_3 + r_2)}{a_2r_2}, \qquad x_2^* = \frac{d_4r_1(d_3 + r_2)}{a_2d_2r_2},\\ y_1^* &= \frac{d_4}{r_2}y_2^*, \qquad y_2^* = \frac{rr_1 - d_2(d_1 + r_1)}{a_1d_2} - \frac{ad_4(d_3 + r_2)}{a_1a_2r_2}.\end{aligned} \tag{7}$$

Let $\overline{x}_1 = x_1 - x_1^*$, $\overline{x}_2 = x_2 - x_2^*$, $\overline{y}_1 = y_1 - y_1^*$, and $\overline{y}_2 = y_2 - y_2^*$. Dropping the bars, system (5) can be transformed to the following form:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= a_{11}x_1(t) + a_{12}x_2(t) + a_{14}y_2(t) + b_{11}x_1(t-\tau_1)\\ &\quad + g_1x_1(t)y_2(t) + g_2x_1(t)x_1(t-\tau_1),\\ \frac{dx_2(t)}{dt} &= a_{21}x_1(t) + a_{22}x_2(t),\\ \frac{dy_1(t)}{dt} &= a_{33}y_1(t) + c_{31}x_1(t-\tau_2) + c_{34}y_2(t-\tau_2)\\ &\quad + h_1x_1(t-\tau_2)y_2(t-\tau_2),\\ \frac{dy_2(t)}{dt} &= a_{43}y_1(t) + a_{44}y_2(t),\end{aligned} \tag{8}$$

where

$$\begin{gathered} a_{11} = -ax_1^* - a_1 y_2^* - (d_1 + r_1), \\ a_{12} = r, \qquad a_{14} = -a_1 x_1^*, \\ a_{21} = r_1, \qquad a_{22} = -d_2, \qquad a_{33} = -(d_3 + r_2), \\ a_{43} = r_2, \qquad a_{44} = -d_4, \qquad b_{11} = -ax_1^*, \\ c_{31} = a_2 y_2^*, \qquad c_{34} = a_2 x_1^*, \\ g_1 = -a_1, \qquad g_2 = -a, \qquad h_1 = a_2. \end{gathered} \tag{9}$$

The linear system of system (8) is

$$\begin{gathered} \frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + a_{14}y_2(t) + b_{11}x_1(t-\tau_1), \\ \frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t), \\ \frac{dy_1(t)}{dt} = a_{33}y_1(t) + c_{31}x_1(t-\tau_2) + c_{34}y_2(t-\tau_2), \\ \frac{dy_2(t)}{dt} = a_{43}y_1(t) + a_{44}y_2(t). \end{gathered} \tag{10}$$

The characteristic equation of system (10) is as follows:

$$\begin{aligned} &\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 \\ &\quad + \left(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau_1} \\ &\quad + \left(p_2\lambda^2 + p_1\lambda + p_0\right)e^{-\lambda\tau_2} \\ &\quad + \left(q_1\lambda + q_0\right)e^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \tag{11}$$

where

$$\begin{gathered} m_3 = -(a_{11} + a_{22} + a_{33} + a_{44}), \\ m_2 = a_{11}a_{22} + a_{33}a_{44} - a_{12}a_{21} \\ \quad + (a_{11} + a_{22})(a_{33} + a_{44}), \\ m_1 = (a_{12}a_{21} - a_{11}a_{22})(a_{33} + a_{44}) \\ \quad - (a_{11} + a_{22})a_{33}a_{44}, \\ m_0 = (a_{11}a_{22} - a_{12}a_{21})a_{33}a_{44}, \\ n_3 = -b_{11}, \qquad n_2 = (a_{22} + a_{33} + a_{44})b_{11}, \\ n_1 = -a_{22}b_{11}(a_{33} + a_{44}) - a_{33}a_{44}b_{11}, \\ n_0 = a_{22}a_{33}a_{44}b_{11}, \qquad p_2 = -a_{43}c_{34}, \\ p_1 = (a_{11} + a_{22})a_{43}c_{34} - a_{14}a_{43}c_{31}, \\ p_0 = (a_{12}a_{21} - a_{11}a_{22})a_{43}c_{34} + a_{14}a_{22}a_{43}c_{31}, \\ q_1 = a_{43}b_{11}c_{34}, \qquad q_0 = -a_{22}a_{43}b_{11}c_{34}. \end{gathered} \tag{12}$$

Case 1 ($\tau_1 = \tau_2 = 0$). For $\tau_1 = \tau_2 = 0$, (11) can be rewritten in the following form:

$$\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{13}$$

where

$$\begin{aligned} A_{13} &= m_3 + n_3 \\ &= 2ax_1^* + a_1y_2^* + d_1 + d_2 + d_3 + d_4 + r_1 + r_2, \\ A_{12} &= m_2 + n_2 + p_2, \\ A_{11} &= m_1 + n_1 + p_1 + q_1, \qquad A_{10} = m_0 + n_0 + p_0 + q_0. \end{aligned} \tag{14}$$

Clearly, $A_{13} > 0$. Therefore, by the Routh-Hurwitz theorem, if the conditions (H_{11}) $A_{10} > 0$, $A_{13}A_{12} > A_{11}$ and (H_{12}) $A_{11}A_{12}A_{13} > A_{11}^2 + A_{10}A_{13}^2$ hold, then the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (5) without time delay is locally asymptotically stable.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). On substituting $\tau_2 = 0$, (11) becomes

$$\begin{aligned} &\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} \\ &\quad + \left(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)e^{-\lambda\tau_1} = 0, \end{aligned} \tag{15}$$

where

$$\begin{gathered} A_{23} = m_3, \qquad A_{22} = m_2 + p_2, \\ A_{21} = m_1 + p_1, \qquad A_{20} = m_0 + p_0, \\ B_{23} = n_3, \qquad B_{22} = n_2, \\ B_{21} = n_1 + q_1, \qquad B_{20} = n_0 + q_0. \end{gathered} \tag{16}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be the root of (15). Then, we have

$$\begin{aligned} &\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\sin\tau_1\omega_1 + \left(B_{20} - B_{22}\omega_1^2\right)\cos\tau_1\omega_1 \\ &\quad = A_{22}\omega_1^2 - \omega_1^4 - A_{20}, \\ &\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\cos\tau_1\omega_1 - \left(B_{20} - B_{22}\omega_1^2\right)\sin\tau_1\omega_1 \\ &\quad = A_{23}\omega_1^3 - A_{21}\omega_1, \end{aligned} \tag{17}$$

which follows that

$$\omega_1^8 + e_{23}\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1 + e_{20} = 0 \tag{18}$$

with

$$\begin{gathered} e_{23} = A_{23}^2 - B_{23}^2 - 2A_{22}, \\ e_{22} = A_{22}^2 - B_{22}^2 + 2B_{21}B_{23} - 2A_{21}A_{23} + 2A_{20}, \\ e_{21} = A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \\ e_{20} = A_{20}^2 - B_{20}^2. \end{gathered} \tag{19}$$

Let $\omega_1^2 = v_1$, then (18) becomes

$$v_1^4 + e_{23}v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{20}$$

Define

$$\begin{aligned}
&f_1(v_1) = v_1^4 + e_{23}v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20},\\
&p = \frac{1}{2}e_{22} - \frac{3}{16}e_{23}^2, \qquad q = \frac{1}{32}e_{23}^3 - \frac{1}{8}e_{22}e_{23} + e_{21},\\
&\alpha_1 = \left(\frac{q}{2}\right)^3 + \left(\frac{p}{3}\right)^3, \qquad \beta_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\\
&y_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}},\\
&y_2 = \beta_1\sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \beta_1^2\sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}},\\
&y_3 = \beta_1^2\sqrt[3]{-\frac{q}{2} + \sqrt{\alpha_1}} + \beta_1\sqrt[3]{-\frac{q}{2} - \sqrt{\alpha_1}},\\
&v_{1i} = y_i - \frac{3e_{23}}{4}, \quad i = 1, 2, 3.
\end{aligned} \tag{21}$$

Discussion about the roots of (20) is similar to that in [25]. Therefore, we have the following lemma.

Lemma 1. *For* (20), *one has the following.*

(i) *If* $e_{20} < 0$, (20) *has at least one positive root.*

(ii) *If* $e_{20} \geq 0$ *and* $\alpha_1 \geq 0$, (20) *has positive roots if and only if* $v_{11} > 0$ *and* $f_1(v_{11}) < 0$.

(iii) *If* $e_{20} \geq 0$ *and* $\alpha_1 < 0$, (20) *has positive roots if and only if there exists at least one* $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ *such that* $v_{1*} > 0$ *and* $f_1(v_{1*}) \leq 0$.

In what follows, one assumes that (H_{21}): *the coefficients in* $f_1(v_1)$ *satisfy one of the following conditions in* (a)–(c):

(a) $e_{20} < 0$;

(b) $e_{20} \geq 0$, $\alpha_1 \geq 0$, $v_{11} > 0$ *and* $f_1(v_{11}) < 0$;

(c) $e_{20} \geq 0$, $\alpha_1 < 0$, *and there exists at least one* $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ *such that* $v_{1*} > 0$ *and* $f_1(v_{1*}) \leq 0$.

If (H_{21}) *holds, one knows that* (18) *has at least a positive root* ω_{10} *such that* (15) *has a pair of purely imaginary roots* $\pm i\omega_{10}$. *The corresponding critical value of the delay is*

$$\begin{aligned}
\tau_{1k} = {} & \frac{1}{\omega_{10}} \\
& \times \arccos\Big\{\Big((B_{22} - A_{23}B_{23})\omega_{10}^6 \\
& \quad + (A_{21}B_{23} + A_{23}B_{21} \\
& \quad - A_{22}B_{22} - B_{20})\omega_{10}^4\Big) \\
& \times \Big(B_{23}^2\omega_{10}^6 + \left(B_{22}^2 - 2B_{21}B_{23}\right)\omega_{10}^4 \\
& \quad + \left(B_{21}^2 - 2B_{20}B_{22}\right)\omega_{10}^2 + B_{20}^2\Big)^{-1} \\
& + \left((A_{20}B_{22} + A_{22}B_{20} - A_{21}B_{21})\,\omega_{10}^2 \right. \\
& \quad \left. - A_{20}B_{20}\right) \\
& \times \left(B_{23}^2\omega_{10}^6 + \left(B_{22}^2 - 2B_{21}B_{23}\right)\omega_{10}^4 \right. \\
& \quad \left. + \left(B_{21}^2 - 2B_{20}B_{22}\right)\omega_{10}^2 + B_{20}^2\right)^{-1}\Big\} \\
& + \frac{2k\pi}{\omega_{10}}, \quad k = 0, 1, 2, \ldots
\end{aligned} \tag{22}$$

Differentiating both sides of (15) *regarding* τ_1, *one gets*

$$\begin{aligned}
\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = {} & -\frac{4\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda\left(\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20}\right)} \\
& + \frac{3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21}}{\lambda\left(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)} - \frac{\tau_1}{\lambda}.
\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}
&\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} \\
&\quad = f_1'(v_1^*) \times \Big(B_{23}^2\omega_{10}^6 + \left(B_{22}^2 - 2B_{21}B_{23}\right)\omega_{10}^4 \\
&\qquad + \left(B_{21}^2 - 2B_{20}B_{22}\right)\omega_{10}^2 + B_{20}^2\Big)^{-1},
\end{aligned} \tag{24}$$

where $v_1^* = \omega_{10}^2$. *Therefore, if the condition* (H_{22}) : $f_1'(v_1^*) \neq 0$ *holds, then* $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}} \neq 0$. *Namely, if the condition* (H_{22}) : $f_1'(v_1^*) \neq 0$ *holds, the transversality condition is satisfied. By the Hopf bifurcation theorem in [26], one has the following results.*

Theorem 2. *If the conditions* (H_{21})-(H_{22}) *hold, then*

(i) *the positive equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *of system* (5) *is asymptotically stable for* $\tau_1 \in [0, \tau_{10})$;

(ii) *system* (5) *undergoes a Hopf bifurcation at the equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *when* $\tau_1 = \tau_{10}$, *and a family of periodic solutions bifurcate from* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *near* $\tau_1 = \tau_{10}$.

Case 3 $(\tau_1 = 0, \tau_2 > 0)$. On substituting $\tau_1 = 0$, (11) becomes

$$\begin{aligned}
&\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} \\
&\quad + \left(B_{32}\lambda^2 + B_{31}\lambda + B_{30}\right)e^{-\lambda\tau_2} = 0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&A_{33} = m_3 + n_3, \qquad A_{32} = m_2 + n_2,\\
&A_{31} = m_1 + n_1, \qquad A_{30} = m_0 + n_0,\\
&B_{32} = p_2, \qquad B_{31} = p_1 + q_1, \qquad B_{30} = p_0 + q_0.
\end{aligned} \tag{26}$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be the root of (25). Then, we have

$$\begin{aligned}
&B_{31}\sin\tau_2\omega_2 + \left(B_{30} - B_{32}\omega_2^2\right)\cos\tau_2\omega_2 \\
&\quad = A_{32}\omega_2^2 - \omega_2^4 - A_{30},\\
&B_{31}\cos\tau_2\omega_2 - \left(B_{30} - B_{32}\omega_2^2\right)\sin\tau_2\omega_2 \\
&\quad = A_{33}\omega_2^3 - A_{31}\omega_2.
\end{aligned} \tag{27}$$

It follows that

$$\omega_2^8 + e_{33}\omega_2^6 + e_{32}\omega_2^4 + e_{31}\omega_2 + e_{30} = 0, \tag{28}$$

where

$$\begin{aligned} e_{33} &= A_{33}^2 - 2A_{32}, \\ e_{32} &= A_{32}^2 - B_{32}^2 + 2A_{30} - 2A_{31}A_{33}, \\ e_{31} &= A_{31}^2 - B_{31}^2 - 2A_{30}A_{32} + 2B_{30}B_{32}, \\ e_{30} &= A_{30}^2 - B_{30}^2. \end{aligned} \tag{29}$$

Let $\omega_2^2 = v_2$, then (28) becomes

$$v_2^4 + e_{33}v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30} = 0. \tag{30}$$

Define

$$\begin{aligned} f_2(v_2) &= v_2^4 + e_{33}v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30}, \\ m &= \frac{1}{2}e_{32} - \frac{3}{16}e_{33}^2, \qquad n = \frac{1}{32}e_{33}^3 - \frac{1}{8}e_{32}e_{33} + e_{31}, \\ \alpha_2 &= \left(\frac{n}{2}\right)^3 + \left(\frac{m}{3}\right)^3, \qquad \beta_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \\ z_1 &= \sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ z_2 &= \beta_2\sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \beta_2^2\sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ z_3 &= \beta_2^2\sqrt[3]{-\frac{n}{2} + \sqrt{\alpha_2}} + \beta_2\sqrt[3]{-\frac{n}{2} - \sqrt{\alpha_2}}, \\ v_{2i} &= z_i - \frac{3e_{33}}{4}, \quad i = 1, 2, 3. \end{aligned} \tag{31}$$

According to Lemma 1, we know that if (H_{31}) the coefficients in $f_2(v_2)$ satisfy one of the following conditions in (a′)–(c′):

(a′) $e_{30} < 0$;

(b′) $e_{30} \geq 0$, $\alpha_2 \geq 0$, $v_{21} > 0$, and $f_2(v_{21}) < 0$;

(c′) $e_{30} \geq 0$, $\alpha_2 < 0$, and there exists at least one $v_{2*} \in \{v_{21}, v_{22}, v_{23}\}$ such that $v_{2*} > 0$ and $f_2(v_{2*}) \leq 0$.

Then, (28) has at least a positive roots ω_{20} such that (25) has a pair of purely imaginary roots $\pm i\omega_{20}$. The corresponding critical value of the delay is

$$\begin{aligned} \tau_{2k} = \frac{1}{\omega_{20}} < \arccos\Big(&\Big(B_{32}\omega_{20}^6 + \left(A_{33}B_{31} - A_{32}B_{32} - B_{30}\right)\omega_{20}^4 \\ &+ \left(A_{30}B_{32} + A_{32}B_{30} - A_{31}B_{31}\right)\omega_{20}^2\Big) \\ &\times \Big(B_{32}^2\omega_{20}^4 + \Big(B_{31}^2 - 2B_{30}B_{32}\Big) \\ &\times \omega_{20}^2 + B_{30}^2\Big)^{-1}\Big) \\ &+ \frac{2k\pi}{\omega_{20}}, \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{32}$$

Similarly as in Case 2, we can conclude that if the condition $(H_{32})\, f_2'(v_2^*) \neq 0$, $(v_2^* = \omega_{20}^2)$, then $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}} \neq 0$. That is, if the condition $(H_{32})\, f_2'(v_2^*) \neq 0$ holds, the transversality condition is satisfied. According to the Hopf bifurcation theorem in [26], we obtain the following results.

Theorem 3. *If the conditions (H_{31})-(H_{32}) hold, then*

(i) *the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (5) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$;*

(ii) *system (5) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ when $\tau_2 = \tau_{20}$, and a family of periodic solutions bifurcate from $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ near $\tau_2 = \tau_{20}$.*

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). For $\tau_1 = \tau_2 = \tau > 0$, (11) can be transformed into the following form:

$$\begin{aligned} &\lambda^4 + A_{43}\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40} \\ &\quad + \left(B_{43}\lambda^3 + B_{42}\lambda^2 + B_{41}\lambda + B_{40}\right)e^{-\lambda\tau} \\ &\quad + \left(C_{41}\lambda + C_{40}\right)e^{-2\lambda\tau} = 0, \end{aligned} \tag{33}$$

where

$$\begin{aligned} &A_{43} = m_3, \qquad A_{42} = m_2, \qquad A_{41} = m_1, \qquad A_{40} = m_0, \\ &B_{43} = n_3, \qquad B_{42} = n_2 + p_2, \\ &B_{41} = n_1 + p_1, \qquad B_{40} = n_0 + p_0, \\ &C_{41} = q_1, \qquad C_{40} = q_0. \end{aligned} \tag{34}$$

Multiplying by $e^{\lambda\tau}$, (33) becomes

$$\begin{aligned} &\left(\lambda^4 + A_{43}\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40}\right)e^{\lambda\tau} \\ &\quad + \left(C_{41}\lambda + C_{40}\right)e^{-\lambda\tau} + B_{43}\lambda^3 + B_{42}\lambda^2 \\ &\quad + B_{41}\lambda + B_{40} = 0. \end{aligned} \tag{35}$$

Let $\lambda = i\omega$ $(\omega > 0)$ be the root of (35). Then, we can get

$$\begin{aligned} &\left(\omega^4 - A_{42}\omega^2 + A_{40} + C_{40}\right)\cos\tau\omega \\ &\quad + \left(A_{43}\omega^3 - A_{41}\omega + C_{41}\omega\right)\sin\tau\omega = B_{42}\omega^2 - B_{40}, \\ &\left(\omega^4 - A_{42}\omega^2 + A_{40} - C_{40}\right)\sin\tau\omega \\ &\quad - \left(A_{43}\omega^3 - A_{41}\omega - C_{41}\omega\right)\cos\tau\omega \\ &= B_{43}\omega^3 - B_{41}\omega. \end{aligned} \tag{36}$$

Then, we obtain

$$\sin\tau\omega = \frac{A_7\omega^7 + A_5\omega^5 + A_3\omega^3 + A_1\omega}{\omega^8 + B_6\omega^6 + B_4\omega^4 + B_2\omega^2 + B_0},$$
$$\cos\tau\omega = \frac{A_6\omega^6 + A_4\omega^4 + A_2\omega^2 + A_0}{\omega^8 + B_6\omega^6 + B_4\omega^4 + B_2\omega^2 + B_0}, \tag{37}$$

where

$$\begin{aligned}
A_0 &= (C_{40} - A_{40})B_{40},\\
A_1 &= (A_{41} + C_{41})B_{40} - (A_{40} + C_{40})B_{41},\\
A_2 &= A_{40}B_{42} + A_{42}B_{40} + B_{41}C_{41} - A_{41}B_{41} - B_{42}C_{40},\\
A_3 &= A_{40}B_{43} + A_{42}B_{41} + B_{43}C_{40}\\
&\quad - A_{41}B_{42} - A_{43}B_{40} - B_{42}C_{41},\\
A_4 &= A_{41}B_{43} + A_{43}B_{41} - A_{42}B_{42} - B_{43}C_{41} - B_{40},\\
A_5 &= A_{43}B_{42} - A_{42}B_{43} - B_{41},\\
A_6 &= B_{42} - A_{43}B_{43}, \qquad A_7 = B_{43},\\
B_0 &= A_{40}^2 - C_{40}^2, \qquad B_2 = A_{41}^2 - C_{41}^2 - 2A_{40}A_{42},\\
B_4 &= A_{42}^2 + 2A_{40} - 2A_{41}A_{43}, \qquad B_6 = A_{43}^2 - 2A_{42}.
\end{aligned} \tag{38}$$

It is well known that $\sin^2\tau\omega + \cos^2\tau\omega = 1$. Therefore, we have

$$\begin{aligned}
&\omega^{16} + e_{47}\omega^{14} + e_{46}\omega^{12} + e_{45}\omega^{10} + e_{44}\omega^8\\
&\quad + e_{43}\omega^6 + e_{42}\omega^4 + e_{41}\omega^2 + e_{40} = 0,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
e_{47} &= 2B_6 - A_7^2, \qquad e_{46} = B_6^2 - A_6^2 + 2B_4 - 2A_5A_7,\\
e_{45} &= 2B_2 + 2B_4B_6 - A_5^2 - 2A_3A_7 - 2A_4A_6,\\
e_{44} &= B_4^2 + 2B_0 + 2B_2B_6 - A_4^2 - 2A_1A_7 - 2A_3A_5 - 2A_2A_6,\\
e_{43} &= 2B_0B_6 + 2B_2B_4 - 2A_1A_5 - 2A_0A_6 - 2A_2A_4 - A_3^2,\\
e_{42} &= B_2^2 - A_2^2 + 2B_0B_4 - 2A_1A_3 - 2A_0A_4,\\
e_{42} &= 2B_0B_2 - 2A_0A_2 - A_1^2, \qquad e_{40} = B_0^2 - A_0^2.
\end{aligned} \tag{40}$$

Let $\omega^2 = v$, then (39) becomes

$$\begin{aligned}
&v^8 + e_{47}v^7 + e_{46}v^6 + e_{45}v^5 + e_{44}v^4 + e_{43}v^3\\
&\quad + e_{42}v^2 + e_{41}v + e_{40} = 0.
\end{aligned} \tag{41}$$

If all the parameters of system (5) are given, it is easy to calculate the roots of (41). Thus, we suppose the following:

(H_{41}) Equation (41) has at least one positive real root.

Without loss of generality, we assume that (41) has eight positive real roots, which are denoted as $v_1, v_2, v_3, \ldots, v_8$. Then (39) has eight positive roots $\omega_k = \sqrt{v_k}$, $k = 1, 2, 3, \ldots, 8$. And for every fixed ω_k, $k = 1, 2, 3, \ldots, 8$, the corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\frac{A_6\omega_k^6 + A_4\omega_k^4 + A_2\omega_k^2 + A_0}{\omega_k^8 + B_6\omega_k^6 + B_4\omega_k^4 + B_2\omega_k^2 + B_0} + \frac{2j\pi}{\omega_k},$$
$$k = 1, 2, 3, \ldots, 8; \quad j = 0, 1, 2, \ldots. \tag{42}$$

Let

$$\tau_0 = \min\left\{\tau_k^{(0)}\right\}, \quad \omega_0 = \omega_k|_{\tau=\tau_0}, \quad k = 1, 2, 3, \ldots, 8. \tag{43}$$

Next, we verify the transversality condition. Differentiating both sides of (35) with respect to τ, we get

$$\begin{aligned}
\left[\frac{d\lambda}{d\tau}\right]^{-1} = &-\Big(\Big(\left(4\lambda^3 + 3A_{43}\lambda^2 + 2A_{42}\lambda + A_{41}\right)e^{\lambda\tau}\\
&+ C_{41}e^{-\lambda\tau} + 3B_{43}\lambda^2 + 2B_{42}\lambda + B_{41}\Big)\\
&\times\Big(\lambda\Big(\left(\lambda^4 + A_{43}\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40}\right)\\
&\times e^{\lambda\tau} - (C_{41}\lambda + C_{40})e^{-\lambda\tau}\Big)\Big)^{-1}\Big) - \frac{\tau}{\lambda}.
\end{aligned} \tag{44}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = -\frac{P_{41}Q_{41} + P_{42}Q_{42}}{Q_{41}^2 + Q_{42}^2}, \tag{45}$$

where

$$\begin{aligned}
P_{41} &= \left(A_{41} + C_{41} - 3A_{43}\omega_0^2\right)\cos\tau_0\omega_0\\
&\quad - \left(2A_{42}\omega_0 - 4\omega_0^3\right)\sin\tau_0\omega_0 - 3B_{43}\omega_0^2 + B_{41},\\
P_{42} &= \left(A_{41} - C_{41} - 3A_{43}\omega_0^2\right)\sin\tau_0\omega_0\\
&\quad + \left(2A_{42}\omega_0 - 4\omega_0^3\right)\cos\tau_0\omega_0 + 2B_{42}\omega_0,\\
Q_{41} &= \left(A_{43}\omega_0^4 - A_{41}\omega_0^2 - C_{41}\omega_0^2\right)\cos\tau_0\omega_0\\
&\quad - \left(\omega_0^5 - A_{42}\omega_0^3 + A_{40}\omega_0 + C_{40}\omega_0\right)\sin\tau_0\omega_0,\\
Q_{42} &= \left(A_{43}\omega_0^4 - A_{41}\omega_0^2 + C_{41}\omega_0^2\right)\sin\tau_0\omega_0\\
&\quad + \left(\omega_0^5 - A_{42}\omega_0^3 + A_{40}\omega_0 - C_{40}\omega_0\right)\cos\tau_0\omega_0.
\end{aligned} \tag{46}$$

Obviously, if the condition (H_{42}) : $P_{41}Q_{41} + P_{42}Q_{42} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. Namely, if the condition (H_{42}) : $P_{41}Q_{41} + P_{42}Q_{42} \neq 0$ holds, the transversality condition is satisfied. In conclusion, we have the following results.

Theorem 4. *If the conditions (H_{41})-(H_{42}) hold, then*

(i) *the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (5) is asymptotically stable for $\tau \in [0, \tau_0)$;*

(ii) *system* (5) *undergoes a Hopf bifurcation at the equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *when* $\tau = \tau_0$, *and a family of periodic solutions bifurcate from* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *near* $\tau = \tau_0$.

Case 5 ($\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$). We consider system (5) under Case 2. That is, τ_1 is in its stable interval and τ_2 is considered as a parameter. Let $\lambda = i\omega_2'$ ($\omega_2' > 0$) be the root of (11). Then, we can get

$$\begin{aligned} \Delta_{51} \sin \tau_2 \omega_2' + \Delta_{52} \cos \tau_2 \omega_2' &= \Delta_{53}, \\ \Delta_{51} \cos \tau_2 \omega_2' - \Delta_{52} \sin \tau_2 \omega_2' &= \Delta_{54}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \Delta_{51} &= q_1 \omega_2' \cos \tau_1 \omega_2' - q_0 \sin \tau_1 \omega_2' + p_1 \omega_2', \\ \Delta_{52} &= q_1 \omega_2' \sin \tau_1 \omega_2' + q_0 \cos \tau_1 \omega_2' - p_2 (\omega_2')^2 + p_0, \\ \Delta_{53} &= \left(n_2 (\omega_2')^2 - n_0\right) \cos \tau_1 \omega_2' + \left(n_3 (\omega_2')^3 - n_1 \omega_2'\right) \\ &\quad \times \sin \tau_1 \omega_2' - (\omega_2')^4 + m_2 (\omega_2')^2 - m_0, \\ \Delta_{54} &= \left(n_0 - n_2 (\omega_2')^2\right) \sin \tau_1 \omega_2' + \left(n_3 (\omega_2')^3 - n_1 \omega_2'\right) \\ &\quad \times \cos \tau_1 \omega_2' + m_3 (\omega_2')^3 - m_1 \omega_2'. \end{aligned} \tag{48}$$

It follows that

$$c_0(\omega_2') + c_1(\omega_2') \cos \tau_1 \omega_2' + c_2(\omega_2') \sin \tau_1 \omega_2' = 0 \tag{49}$$

with

$$\begin{aligned} c_0(\omega_2') &= (\omega_2')^8 + \left(m_3^2 + n_3^2 - 2m_2\right)(\omega_2')^6 \\ &\quad + \left(m_2^2 + n_2^2 - p_2^2 + 2m_0 - 2m_1 m_3 - 2n_1 n_3\right)(\omega_2')^4 \\ &\quad + \left(m_1^2 + n_1^2 - p_1^2 - q_1^2 - 2m_0 m_2 \right. \\ &\qquad \left. - 2n_0 n_2 + 2p_0 p_2\right)(\omega_2')^2 \\ &\quad + m_0^2 + n_0^2 - p_0^2 - q_0^2, \\ c_1(\omega_2') &= 2(m_3 n_3 - n_2)(\omega_2')^6 \\ &\quad + 2(m_2 n_2 + n_0 - m_3 n_1 - m_1 n_3)(\omega_2')^4 \\ &\quad + 2(m_1 n_1 - m_0 n_2 - m_2 n_0 - p_1 q_1 + p_2 q_0)(\omega_2')^2 \\ &\quad + 2(m_0 n_0 - p_0 q_0), \\ c_2(\omega_2') &= -2n_3 (\omega_2')^7 \\ &\quad + 2(m_2 n_3 - m_3 n_2 + n_1)(\omega_2')^5 \\ &\quad + 2(m_3 n_0 + m_1 n_2 - m_0 n_3 \\ &\qquad - m_2 n_1 + p_2 q_1)(\omega_2')^3 \\ &\quad + 2(m_0 n_1 - m_1 n_0 + p_1 q_0 - p_0 q_1)\omega_2'. \end{aligned} \tag{50}$$

Suppose that (H_{51}): Equation (49) has at least finite positive real roots. If the condition (H_{51}) holds, we denote the positive roots of (49) as $\omega_{21}', \omega_{22}', \ldots, \omega_{2k}'$. Then, for every fixed ω_{2i}' ($i = 1, 2, 3, \ldots, k$), the corresponding critical value of time delay $\{\tau_{2i}'^{(j)} \mid j = 0, 1, 2, \ldots\}$ is

$$\begin{aligned} \tau_{2i}'^{(j)} &= \frac{1}{\omega_{2i}'} \arccos \frac{\Delta_{51}\Delta_{54} + \Delta_{52}\Delta_{53}}{\Delta_{51}^2 + \Delta_{52}^2} + \frac{2j\pi}{\omega_{2i}'}, \\ &\quad i = 1, 2, 3, \ldots, k, \quad j = 0, 1, 2, \ldots. \end{aligned} \tag{51}$$

Let

$$\tau_2^* = \min\left\{\tau_{2i}'^{(0)} \mid i = 1, 2, 3, \ldots, k,\right\}, \qquad \omega_2^* = \omega_{2i}'|_{\tau_2 = \tau_2^*}. \tag{52}$$

Differentiating (11) regarding τ_2, we obtain

$$\begin{aligned} &\left[\frac{d\lambda}{d\tau_2}\right]^{-1} \\ &= \frac{g_0(\lambda) + g_1(\lambda) e^{-\lambda\tau_1} + g_2(\lambda) e^{-\lambda\tau_2} + g_3(\lambda) e^{-\lambda(\tau_1+\tau_2)}}{h_2(\lambda) e^{-\lambda\tau_2} + h_3(\lambda) e^{-\lambda(\tau_1+\tau_2)}} \\ &\quad - \frac{\tau_2}{\lambda} \end{aligned} \tag{53}$$

with

$$\begin{aligned} g_0(\lambda) &= 4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1, \\ g_1(\lambda) &= -\tau_1 n_3 \lambda^3 + (3n_3 - \tau_1 n_2)\lambda^2 \\ &\quad + (2n_2 - \tau_1 n_1)\lambda + n_1 - \tau_1 n_0, \\ g_2(\lambda) &= 2p_2\lambda + p_1, \qquad g_3(\lambda) = q_1, \\ h_2(\lambda) &= p_2\lambda^3 + p_1\lambda^2 + p_0\lambda, \qquad h_3(\lambda) = q_1\lambda^2 + q_0\lambda. \end{aligned} \tag{54}$$

Hence,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_2^*} = -\frac{P_{51}Q_{51} - P_{52}Q_{52}}{Q_{51}^2 + Q_{52}^2}, \tag{55}$$

where

$$\begin{aligned} P_{51} &= (2p_2\omega_2^* - q_1 \sin \tau_1 \omega_2^*) \sin \tau_2^* \omega_2^* \\ &\quad + (p_1 + q_1 \cos \tau_1 \omega_2^*) \cos \tau_2^* \omega_2^* \\ &\quad + \left(\tau_1 n_3 (\omega_2^*)^3 + (2n_2 - \tau_1 n_1)\omega_2^*\right) \sin \tau_1 \omega_2^* \\ &\quad + \left((\tau_1 n_2 - 3n_3)(\omega_2^*)^2 + n_1 - \tau_1 n_0\right) \cos \tau_1 \omega_2^* \\ &\quad - 3m_3 (\omega_2^*)^2 + m_1, \end{aligned}$$

$$\begin{aligned}
P_{52} &= \left(2p_2\omega_2^* - q_1 \sin\tau_1\omega_2^*\right)\cos\tau_2^*\omega_2^* \\
&\quad - \left(p_1 + q_1\cos\tau_1\omega_2^*\right)\sin\tau_2^*\omega_2^* \\
&\quad + \left(\tau_1 n_3 (\omega_2^*)^3 + (2n_2 - \tau_1 n_1)\omega_2^*\right)\cos\tau_1\omega_2^* \\
&\quad - \left((\tau_1 n_2 - 3n_3)(\omega_2^*)^2 + n_1 - \tau_1 n_0\right)\sin\tau_1\omega_2^* \\
&\quad - 4(\omega_2^*)^3 + 2m_2\omega_2^*, \\
Q_{51} &= \left(p_2(\omega_2^*)^3 - p_0\omega_2^* - q_1(\omega_2^*)^2\sin\tau_1\omega_2^*\right. \\
&\quad \left. - q_0\omega_2^*\cos\tau_1\omega_2^*\right)\sin\tau_2^*\omega_2^* \\
&\quad + \left(p_1(\omega_2^*)^2 + q_1(\omega_2^*)^2\cos\tau_1\omega_2^*\right. \\
&\quad \left. - q_0\omega_2^*\sin\tau_1\omega_2^*\right)\cos\tau_2^*\omega_2^*, \\
Q_{52} &= \left(p_2(\omega_2^*)^3 - p_0\omega_2^* - q_1(\omega_2^*)^2\sin\tau_1\omega_2^*\right. \\
&\quad \left. - q_0\omega_2^*\cos\tau_1\omega_2^*\right)\cos\tau_2^*\omega_2^* \\
&\quad - \left(p_1(\omega_2^*)^2 + q_1(\omega_2^*)^2\cos\tau_1\omega_2^*\right. \\
&\quad \left. - q_0\omega_2^*\sin\tau_1\omega_2^*\right)\sin\tau_2^*\omega_2^*.
\end{aligned} \tag{56}$$

Clearly, if the condition $(H_{52})P_{51}Q_{51} - P_{52}Q_{52} \neq 0$ holds, then $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_2^*} \neq 0$. Thus, we have the following results.

Theorem 5. *If the conditions* (H_{51})-(H_{52}) *hold and* $\tau_1 \in (0, \tau_{10})$, *then*

(i) *the positive equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *of system* (5) *is asymptotically stable for* $\tau_2 \in [0, \tau_2^*)$;

(ii) *system* (5) *undergoes a Hopf bifurcation at the equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *when* $\tau_2 = \tau_2^*$, *and a family of periodic solutions bifurcate from* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *near* $\tau_2 = \tau_2^*$.

3. Stability of the Bifurcating Periodic Solutions

In this section, we will determine the direction of Hopf bifurcation and stability of the bifurcated periodic solutions of system (5) with respect to τ_2 for $\tau_1 \in (0, \tau_{10})$ by using the normal form method and center manifold theorem introduced by Hassard et al. in [25]. Without loss of generality, we assume that $\tau_1^* < \tau_2^*$, where $\tau_1^* \in (0, \tau_{10})$.

Let $\tau_2 = \tau_2^* + \mu$, $\mu \in R$. Then, μ is the Hopf bifurcation value of system (5). Rescaling the time $t \to t/\tau_2$, then system (5) can be written as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{57}$$

where

$$\begin{aligned}
L_\mu\phi &= (\tau_2^* + \mu)\left(A'\phi(0) + B'\phi\left(-\frac{\tau_1^*}{\tau_2^*}\right) + C'\phi(-1)\right), \\
F(\mu, \phi) &= (\tau_2^* + \mu) \\
&\quad \times \begin{pmatrix} g_1 x_1(t) y_2(t) + g_2 x_1(t) x_1(t-\tau_1) \\ 0 \\ h_1 x_1(t-\tau_2) y_2(t-\tau_2) \\ 0 \end{pmatrix}
\end{aligned} \tag{58}$$

with

$$\begin{aligned}
\phi(\theta) &= \left(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)\right)^T \in C\left([-1, 0], R^4\right), \\
A' &= \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}, \\
B' &= \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
C' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_{31} & 0 & 0 & c_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{59}$$

Using the Riesz representation theorem, there exists a 4×4 matrix function $\eta(\theta, \mu) : [-1, 0] \to R^4$ whose elements are of bounded variation, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C([-1, 0], R_4). \tag{60}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_2^* + \mu)\left(A' + B' + C'\right), & \theta = 0, \\ (\tau_2^* + \mu)\left(B' + C'\right), & \theta \in \left[-\dfrac{\tau_1^*}{\tau_2^*}, 0\right), \\ (\tau_2^* + \mu)C', & \theta \in \left(-1, -\dfrac{\tau_1^*}{\tau_2^*}\right), \\ 0, & \theta = -1. \end{cases} \tag{61}$$

For $\phi \in C([-1, 0], R^4)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases} \tag{62}$$

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

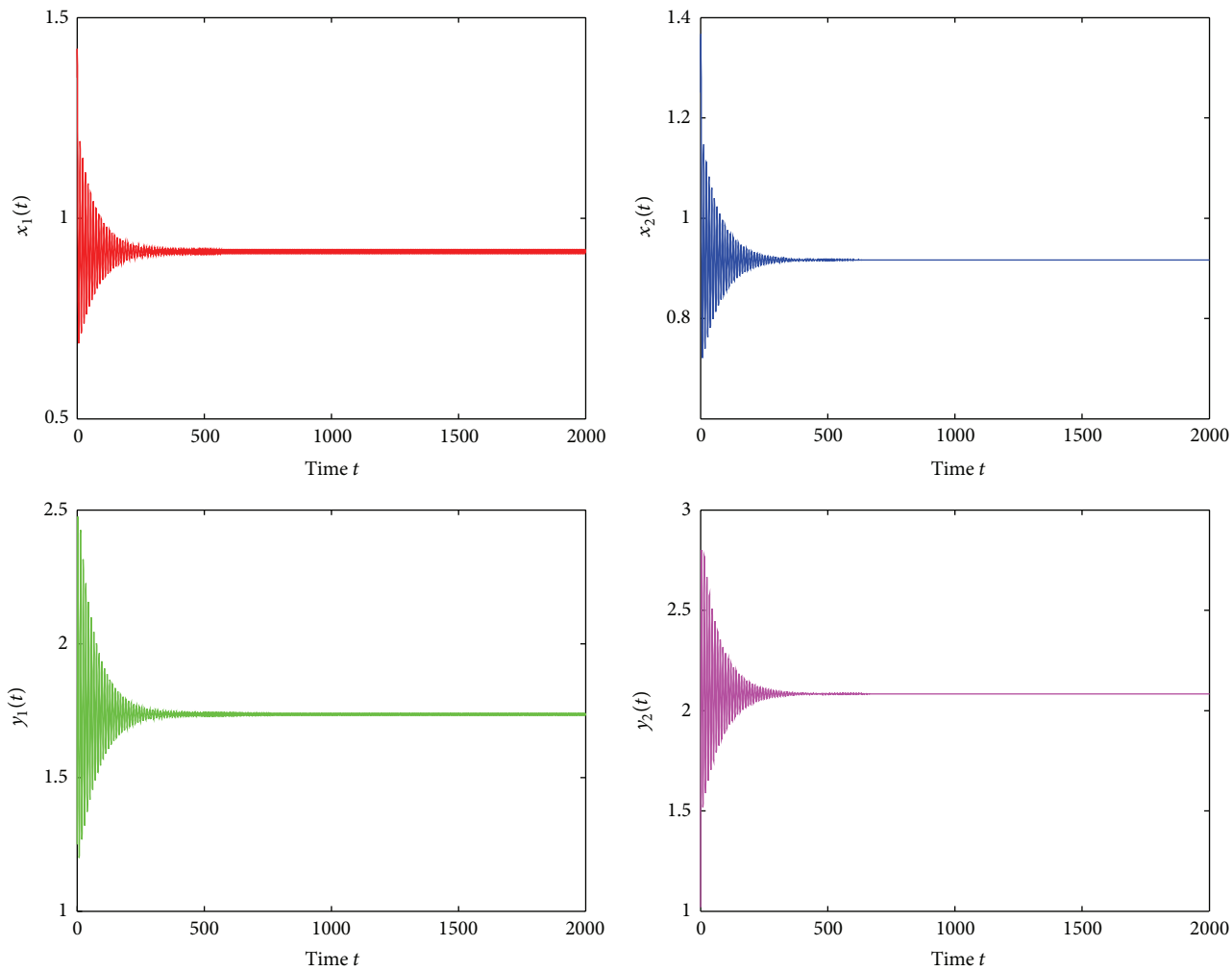

FIGURE 1: The track of the states x_1, x_2, y_1, y_2 for $\tau_1 = 4.2500 < \tau_{10} = 4.5061$.

Then system (57) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu) u_t + R(\mu) u_t, \tag{63}$$

where $u_t = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta), u_3(t+\theta), u_4(t+\theta))$.

The adjoint operator A^* of A is defined by

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,\mu)\varphi(-s), & s = 0, \end{cases} \tag{64}$$

associated with a bilinear inner product:

$$\begin{aligned} \langle \varphi(s), \phi(\theta) \rangle &= \overline{\varphi}(0)\phi(0) \\ &\quad - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \overline{\varphi}(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi, \end{aligned} \tag{65}$$

where $\eta(\theta) = \eta(\theta, 0)$.

By the discussion above, we know that $\pm i\tau_2^*\omega_2^*$ are eigenvalues of $A(0)$ and they are also eigenvalues of $A^*(0)$. We assume that $q(\theta) = (1, q_2, q_3, q_4)^T e^{i\tau_2^*\omega_2^*\theta}$ are the eigenvectors of $A(0)$ belonging to the eigenvalue $+i\tau_2^*\omega_2^*$ and $q^*(s) =$

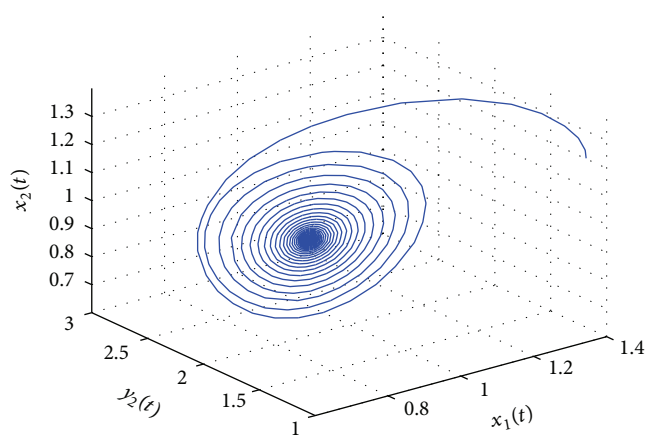

FIGURE 2: The phase plot of the states x_1, x_2, y_2 for $\tau_1 = 4.2500 < \tau_{10} = 4.5061$.

$D(1, q_2^*, q_3^*, q_4^*)e^{i\tau_2^*\omega_2^* s}$ are the eigenvectors of $A^*(0)$ belonging to the eigenvalue $-i\tau_2^*\omega_2^*$.

By a simple computation, we can get

$$\begin{aligned}
q_2 &= \frac{a_{21}}{i\omega_2^* - a_{22}},\\
q_3 &= \frac{c_{34}\left(i\omega_2^* - a_{11}\right) + a_{14}c_{31} - b_{11}c_{34}e^{-i\tau_1^*\omega_2^*}}{a_{14}\left(i\omega_2^* - a_{33}\right)e^{i\tau_2^*\omega_2^*}}\\
&\quad - \frac{a_{12}a_{21}c_{34}}{a_{14}\left(i\omega_2^* - a_{22}\right)\left(i\omega_2^* - a_{33}\right)e^{i\tau_2^*\omega_2^*}},\\
q_4 &= \Big(\left(i\omega_2^* - a_{11}\right)\left(i\omega_2^* - a_{22}\right) - a_{12}a_{21}\\
&\quad - b_{11}\left(i\omega_2^* - a_{22}\right)e^{-i\tau_1^*\omega_2^*}\Big) \times \left(a_{14}\left(i\omega_2^* - a_{22}\right)\right)^{-1},\\
q_2^* &= -\frac{a_{12}}{i\omega_2^* + a_{22}},\\
q_3^* &= -\Big(\left(i\omega_2^* + a_{11}\right)\left(i\omega_2^* + a_{22}\right) - a_{12}a_{21} + b_{11}\\
&\quad \times \left(i\omega_2^* + a_{22}\right)e^{i\tau_1^*\omega_2^*}\Big) \times \left(c_{31}\left(i\omega_2^* + a_{33}\right)e^{i\tau_2^*\omega_2^*}\right)^{-1},\\
q_4^* &= \frac{c_{34}\left(i\omega_2^* + a_{11}\right) - a_{14}c_{31} - b_{11}c_{34}e^{i\tau_1^*\omega_2^*}}{c_{31}\left(i\omega_2^* + a_{44}\right)}\\
&\quad - \frac{a_{12}a_{21}c_{34}}{c_{31}\left(i\omega_2^* + a_{22}\right)\left(i\omega_2^* + a_{44}\right)}.
\end{aligned} \tag{66}$$

From (65), we can get

$$\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \overline{q}^*(0)\,q(0)\\
&\quad - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{q}^*(\xi-\theta)\,d\eta(\theta)\,q(\xi)\,d\xi\\
&= \overline{D}\left(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*\right)\left(1, q_2, q_3, q_4\right)^T\\
&\quad - \overline{D}\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\left(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*\right)\\
&\qquad \times e^{i\tau_2^*\omega_2^*(\theta-\xi)}d\eta(\theta)\\
&\qquad \times \left(1, q_2, q_3, q_4\right)^T e^{i\tau_2^*\omega_2^*\xi}d\xi\\
&= \overline{D}\Big[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* - \left(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*\right)\\
&\qquad \times \int_{-1}^{0}\theta e^{i\tau_2^*\omega_2^*\theta}d\eta(\theta)\left(1, q_2, q_3, q_4\right)^T\Big]\\
&= \overline{D}\Big[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + b_{11}\tau_1^*e^{-i\tau_1^*\omega_2^*}\\
&\qquad + \overline{q}_3^*\tau_2^*e^{-i\tau_2^*\omega_2^*}\left(c_{31} + c_{34}q_4\right)\Big].
\end{aligned} \tag{67}$$

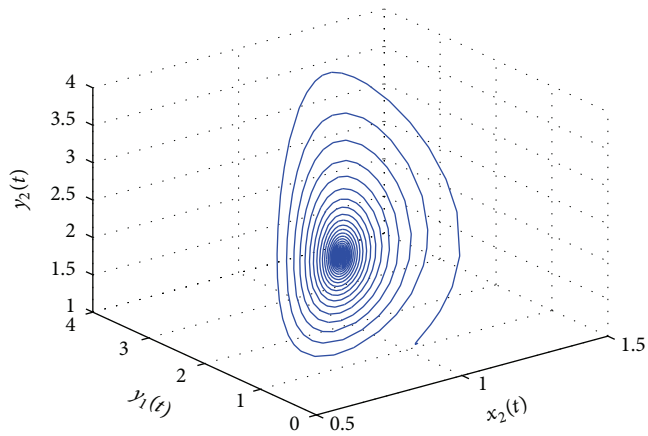

Figure 3: The phase plot of the states x_2, y_1, y_2 for $\tau_1 = 4.2500 < \tau_{10} = 4.5061$.

Normalizing q and q^* by the condition $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle$, one can get

$$\begin{aligned}
\overline{D} = \Big[&1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + b_{11}\tau_1^*e^{-i\tau_1^*\omega_2^*}\\
&+ \overline{q}_3^*\tau_2^*e^{-i\tau_2^*\omega_2^*}\left(c_{31} + c_{34}q_4\right)\Big]^{-1}.
\end{aligned} \tag{68}$$

Next, we can get the coefficients determining the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions by the algorithms given in [26]:

$$\begin{aligned}
g_{20} &= 2\tau_2^*\overline{D}\Big[g_1q^{(4)}(0) + g_2q^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\\
&\qquad + h_1\overline{q}_3^*q^{(1)}(-1)\,q^{(4)}(-1)\Big],\\
g_{11} &= \tau_2^*\overline{D}\Big[g_1\left(\overline{q}^{(4)}(0) + q^{(4)}(0)\right)\\
&\qquad + g_2\left(\overline{q}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) + q^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\right)\\
&\qquad + h_1\overline{q}_3^*\Big(q^{(1)}(-1)\,\overline{q}^{(4)}(-1)\\
&\qquad\qquad + \overline{q}^{(1)}(-1)\,q^{(4)}(-1)\Big)\Big],\\
g_{02} &= 2\tau_2^*\overline{D}\Big[g_1\overline{q}^{(4)}(0) + g_2\overline{q}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\\
&\qquad + h_1\overline{q}_3^*\overline{q}^{(1)}(-1)\,\overline{q}^{(4)}(-1)\Big],\\
g_{21} &= 2\tau_2^*\overline{D}\Big[g_1\Big(W_{11}^{(1)}(0)\,q^{(4)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}^{(4)}(0)\\
&\qquad + W_{11}^{(4)}(0) + \frac{1}{2}W_{20}^{(4)}(0)\Big)\\
&\qquad + g_2\Big(W_{11}^{(1)}(0)\,q^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}^{(1)}\\
&\qquad \times\left(-\frac{\tau_1^*}{\tau_2^*}\right) + W_{11}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)
\end{aligned}$$

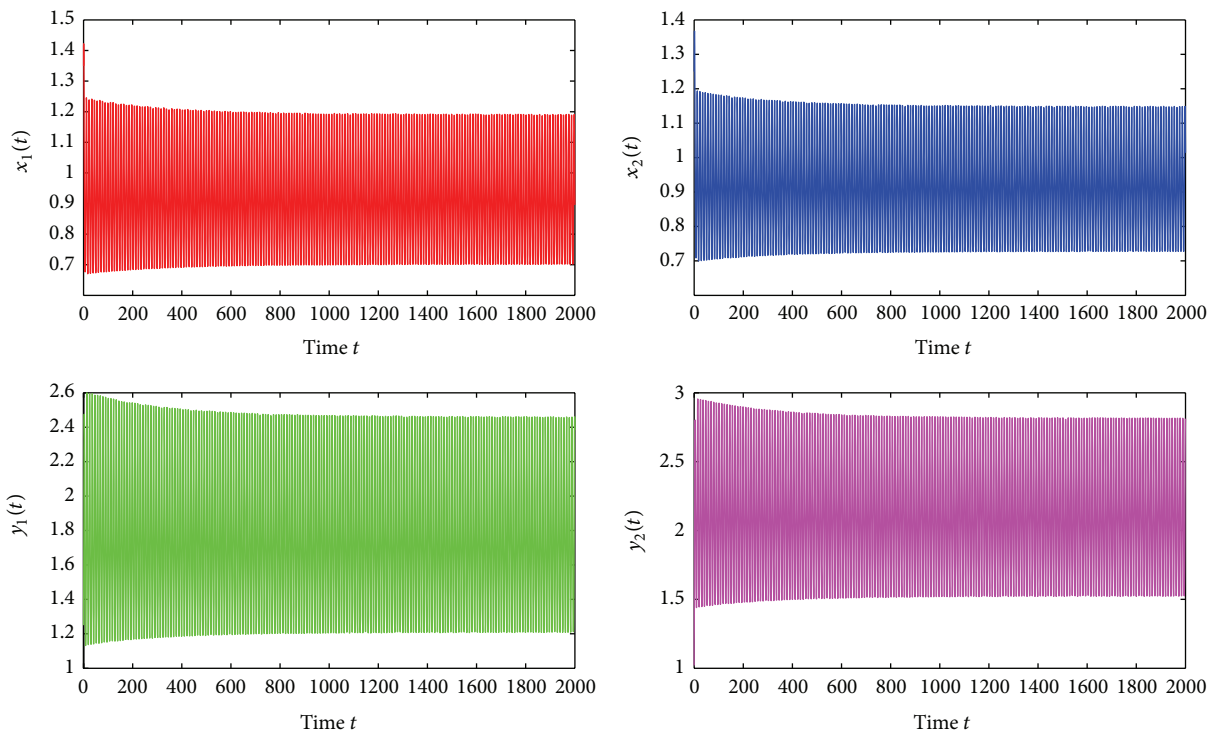

FIGURE 4: The track of the states x_1, x_2, y_1, y_2 for $\tau_1 = 4.71500 > \tau_{10} = 4.5061$.

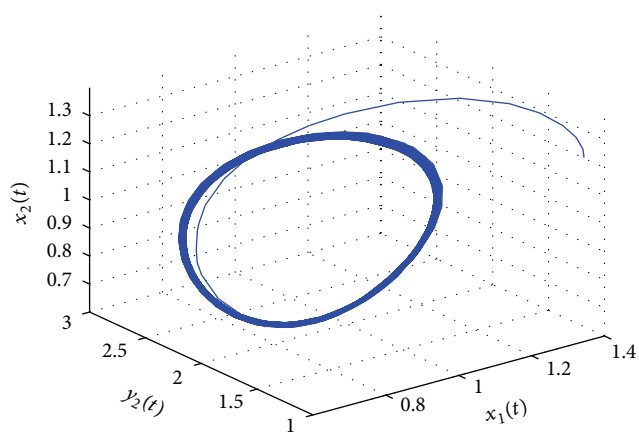

FIGURE 5: The phase plot of the states x_1, x_2, y_2 for $\tau_1 = 4.71500 > \tau_{10} = 4.5061$.

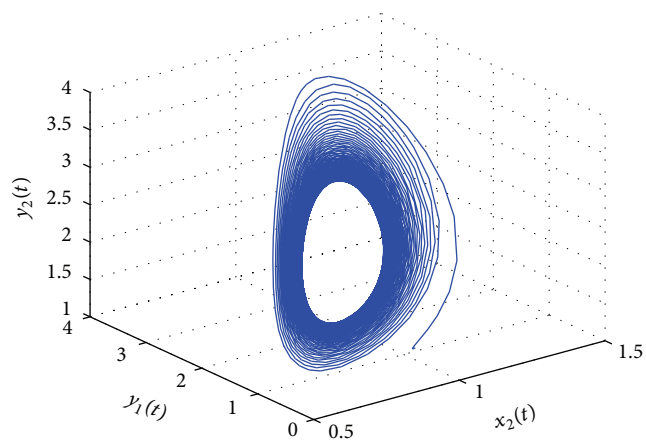

FIGURE 6: The phase plot of the states x_2, y_1, y_2 for $\tau_1 = 4.71500 > \tau_{10} = 4.5061$.

$$+\frac{1}{2}W_{20}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\Bigg)$$
$$+h_1\overline{q}_3^*\Bigg(W_{11}^{(1)}(-1)\,q^{(4)}(-1)$$
$$+\frac{1}{2}W_{20}^{(1)}(-1)\,\overline{q}^{(4)}(-1)$$
$$+W_{11}^{(4)}(-1)\,q^{(1)}(-1)$$
$$+\frac{1}{2}W_{20}^{(4)}(-1)\,\overline{q}^{(1)}(-1)\Bigg)\Bigg] \tag{69}$$

with

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\tau_2^*\omega_2^*}e^{i\tau_2^*\omega_2^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_2^*\omega_2^*}e^{-i\tau_2^*\omega_2^*\theta} + E_1e^{2i\tau_2^*\omega_2^*\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}q(0)}{\tau_2^*\omega_2^*}e^{i\tau_2^*\omega_2^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_2^*\omega_2^*}e^{-i\tau_2^*\omega_2^*\theta} + E_2, \tag{70}$$

where E_1 and E_2 can be determined by the following equations, respectively:

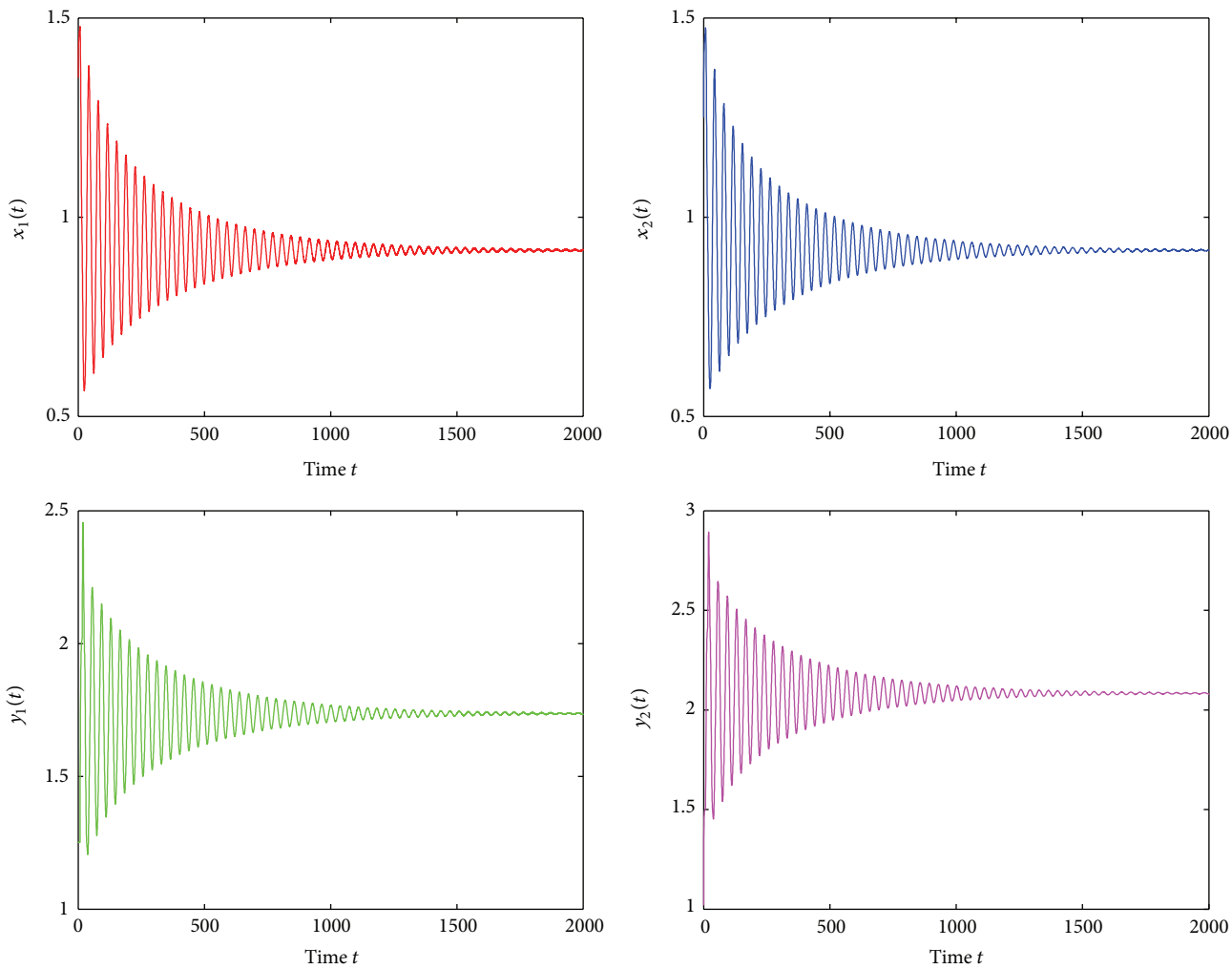

FIGURE 7: The track of the states x_1, x_2, y_1, y_2 for $\tau_2 = 7.7500 < \tau_{20} = 9.9183$.

$$E_1 = 2\begin{pmatrix} 2i\omega_2^* - a_{11} - b_{11}e^{-2i\tau_1^*\omega_2^*} & -a_{12} & 0 & -a_{14} \\ -a_{21} & 2i\omega_2^* - a_{22} & 0 & 0 \\ -c_{31}e^{-2i\tau_2^*\omega_2^*} & 0 & 2i\omega_2^* - a_{33} & -c_{34}e^{-2i\tau_2^*\omega_2^*} \\ 0 & 0 & -a_{43} & 2i\omega_2^* - a_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} g_1 q^{(4)}(0) + g_2 q^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \\ 0 \\ h_1 q^{(1)}(-1)\, q^{(4)}(-1) \\ 0 \end{pmatrix},$$

$$E_2 = -\begin{pmatrix} a_{11} + b_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ c_{31} & 0 & a_{33} & c_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} g_1\left(\overline{q}^{(4)}(0) + q^{(4)}(0)\right) + g_2\left(\overline{q}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) + q^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\right) \\ 0 \\ h_1\left(q^{(1)}(-1)\,\overline{q}^{(4)}(-1) + \overline{q}^{(1)}(-1)\, q^{(4)}(-1)\right) \\ 0 \end{pmatrix}. \tag{71}$$

Therefore, we can calculate the following values:

$$C_1(0) = \frac{i}{2\tau_2^*\omega_2^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_2^*)\}},$$

$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_2^*)\}}{\tau_2^*\omega_2^*}. \tag{72}$$

By the results of Hassard et al. [26], the following results are obtained.

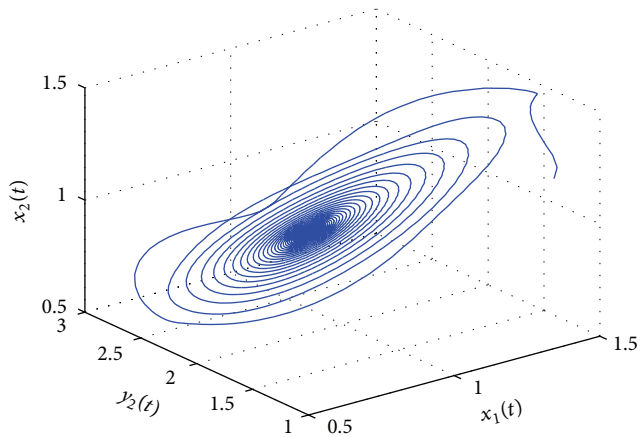

Figure 8: The phase plot of the states x_1, x_2, y_2 for $\tau_2 = 7.7500 < \tau_{20} = 9.9183$.

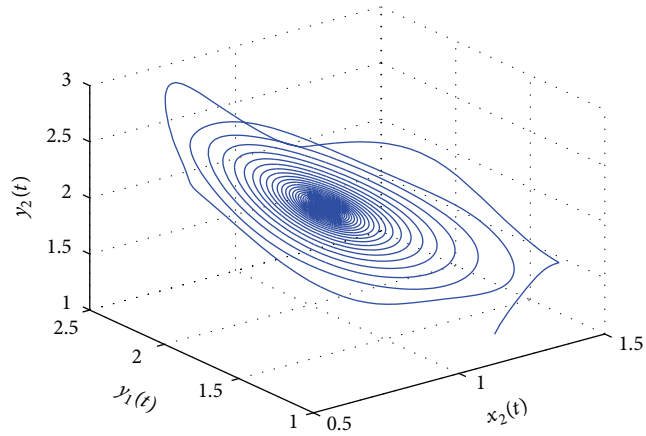

Figure 9: The phase plot of the states x_2, y_1, y_2 for $\tau_2 = 7.7500 < \tau_{20} = 9.9183$.

Theorem 6. *For system* (5),

(i) *the direction of the Hopf bifurcation is determined by μ_2: if $\mu_2 > 0$, the Hopf bifurcation is supercritical; if $\mu_2 < 0$, the Hopf bifurcation is subcritical;*

(ii) *the stability of the bifurcating periodic solutions is determined by β_2: if $\beta_2 < 0$, the bifurcating periodic solutions are stable; if $\beta_2 > 0$, the bifurcating periodic solutions are unstable;*

(iii) *the period of the bifurcating periodic solution is determined by T_2: if $T_2 > 0$, the period of the bifurcating periodic solutions increases; if $T_2 < 0$, the period of the bifurcating periodic solutions decreases.*

4. Numerical Simulation

In this section, we present some numerical simulations to support the theoretical analysis in Sections 2 and 3. We consider the following system by taking the same coefficients as in [17]:

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= 5x_2(t) - x_1(t)x_1(t-\tau_1) - 2x_1(t) - x_1(t)y_2(t),\\
\frac{dx_2(t)}{dt} &= x_1(t) - x_2(t),\\
\frac{dy_1(t)}{dt} &= 2x_1(t-\tau_2)y_2(t-\tau_2) - 2.2y_1(t),\\
\frac{dy_2(t)}{dt} &= 1.2y_1(t) - y_2(t),
\end{aligned} \tag{73}$$

from which we can get $a_2rr_1r_2 = 12$, $ad_2d_4(d_3 + r_2) + a_2d_2r_2(d_1 + r_1) = 7$. Obviously, $a_2rr_1r_2 > ad_2d_4(d_3 + r_2) + a_2d_2r_2(d_1 + r_1)$. Therefore, the condition (H) is satisfied. Further, we get the unique positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ of system (73) by MATLAB software package. Then, we have $A_{10} = 4.5833$, $A_{11} = 7.5163$, $A_{13}A_{12} = 233.1899$, $A_{11}A_{12}A_{13} = 1752.7$, and $A_{11}^2 + A_{10}A_{13}^2 = 525.5744$. Obviously, $A_{10} = 4.5833 > 0$, $A_{13}A_{12} > A_{11}$, and $A_{11}A_{12}A_{13} > A_{11}^2 + A_{10}A_{13}^2$. Namely, the conditions (H_{11}) and (H_{12}) hold.

For $\tau_1 > 0, \tau_2 = 0$. From (18), we have

$$\omega_1^8 + 45.3999\omega_1^6 + 284.0324\omega_1^4 - 163.6041\omega_1^2 + 21.0076 = 0. \tag{74}$$

Then, (20) becomes

$$\begin{aligned}
&v_1^4 + 45.3999v_1^3 + 284.0324v_1^2\\
&\quad - 163.6041v_1 + 21.0076 = 0
\end{aligned} \tag{75}$$

and it has two positive real roots: $v_{11} = 0.3397$, $v_{12} = 0.2003$. Thus, the condition (H_{21}) is satisfied. Further, we have $\omega_{10} = 0.4475$, $\tau_{10} = 4.5061$. Then, we get $f_1'(v_{1*}) = -44.3242 \neq 0$. Therefore, the condition (H_{22}) holds. From Theorem 2, we know that the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable for $\tau_1 \in [0, 4.5061)$, which can be shown in Figures 1, 2, and 3. The positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is unstable when $\tau_1 > 4.5061$ and system (73) undergoes a Hopf bifurcation at $E^*(0.9167, 0.9167, 1.7361, 2.0833)$, and a family of periodic solutions bifurcate from $E^*(0.9167, 0.9167, 1.7361, 2.0833)$. This property can be illustrated by Figures 4–6. Similarly, we have $\omega_{20} = 0.1517$, $\tau_{20} = 9.9183$ for $\tau_1 = 0, \tau_2 > 0$. For $\tau_2 = 7.7500 \in [0, \tau_{20})$, the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable from Theorem 3 and this property can be shown in Figures 7, 8, and 9. If $\tau_2 = 10.7500 > \tau_{20} = 9.9183$, the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is unstable and a Hopf bifurcation occurs, and the corresponding waveforms and phase plots are shown in Figures 10, 11, and 12.

For $\tau_1 = \tau_2 = \tau > 0$, we can get $\omega_0 = 0.3412, \tau_0 = 3.1043$. From Theorem 4, we know that the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable for $\tau \in [0, \tau_0)$, which can be illustrated by Figures 13, 14, and 15. As can be seen from Figure 5 that when $\tau = 2.7500 \in [0, \tau_0)$, the positive equilibrium

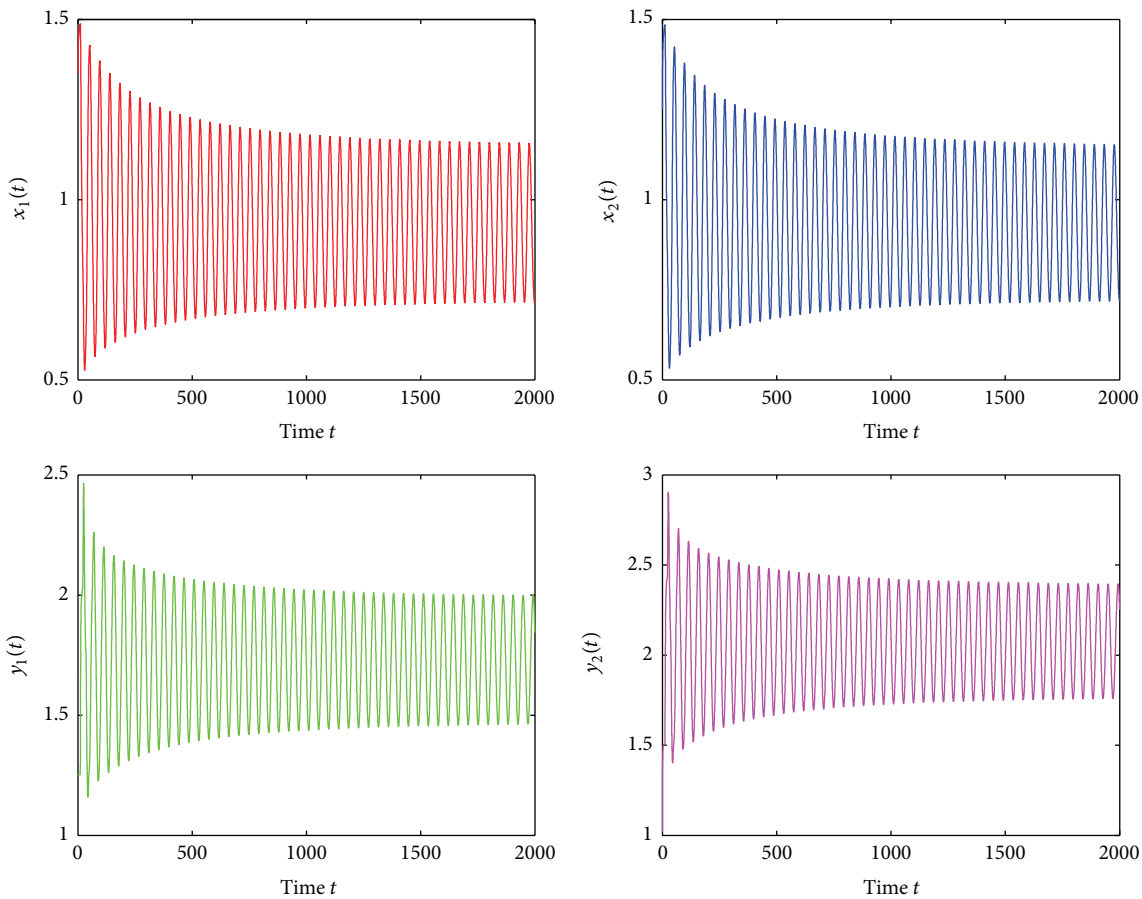

FIGURE 10: The track of the states x_1, x_2, y_1, y_2 for $\tau_2 = 10.7500 > \tau_{20} = 9.9183$.

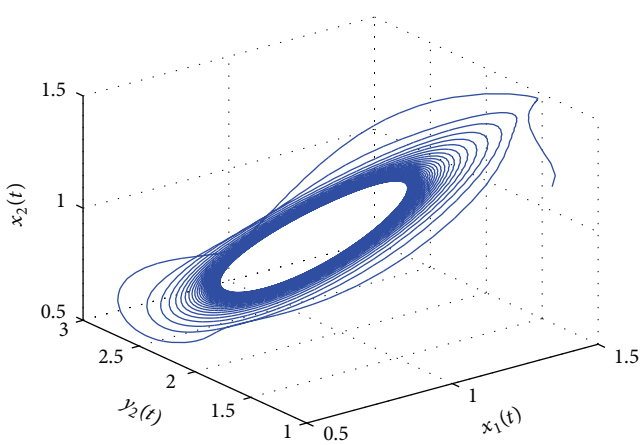

FIGURE 11: The phase plot of the states x_1, x_2, y_2 for $\tau_2 = 10.7500 > \tau_{20} = 9.9183$.

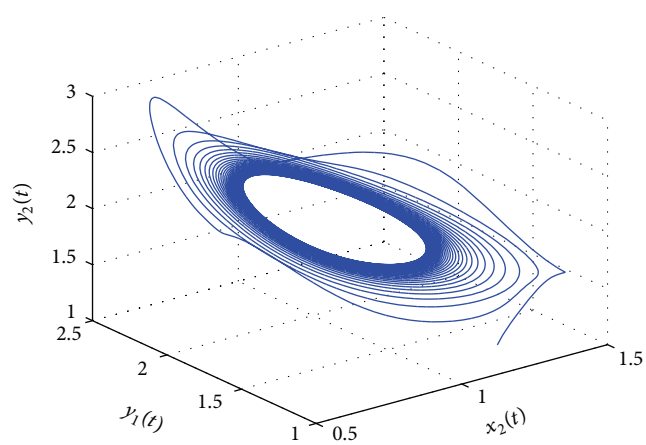

FIGURE 12: The phase plot of the states x_2, y_1, y_2 for $\tau_2 = 10.7500 > \tau_{20} = 9.9183$.

$E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable. However, if $\tau = 3.2500 > \tau_0 = 3.1043$, then the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ becomes unstable and a family of bifurcating periodic solutions occur, which is illustrated by Figures 16, 17, and 18.

Lastly, regarding τ_2 as a parameter and letting $\tau_1 = 2.5 \in (0, \tau_{10})$, we can obtain that $\omega_2^* = 0.2253$. Further, we have $\tau_2^* = 6.6494$ and $\lambda'(\tau_2^*) = 0.0466 + 0.0314i$. Letting $\tau_2 = 5.5500 \in [0, \tau_1^*)$, we can know that the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ is asymptotically stable from Theorem 5. When $\tau_2 =$

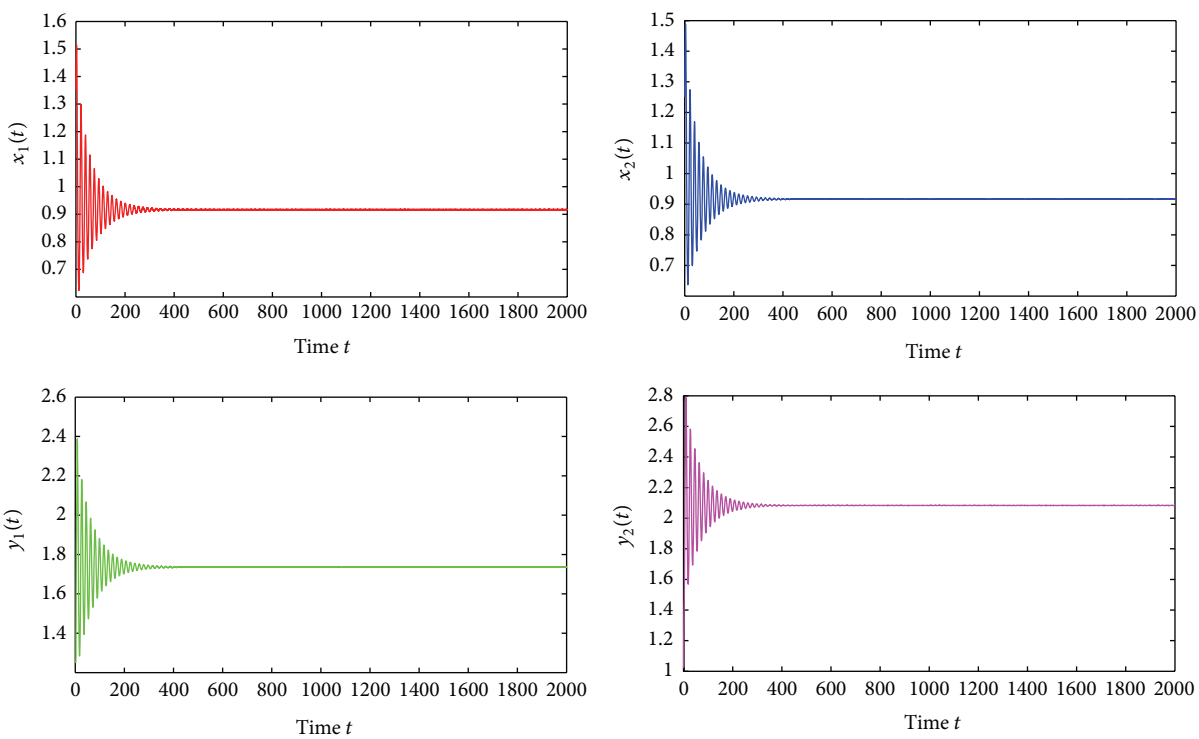

FIGURE 13: The track of the states x_1, x_2, y_1, y_2 for $\tau = 2.7500 < \tau_0 = 3.1043$.

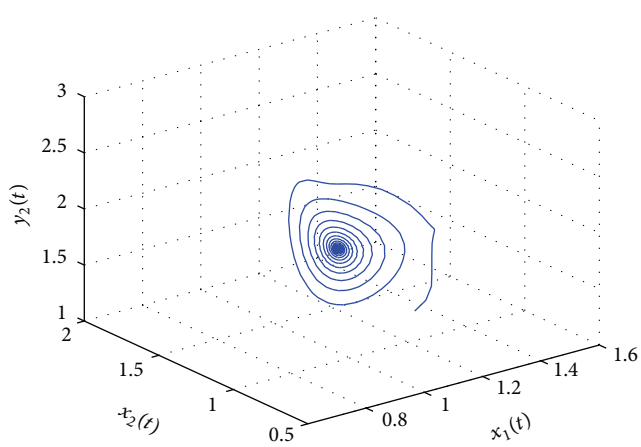

FIGURE 14: The phase plot of the states x_1, x_2, y_2 for $\tau = 2.7500 < \tau_0 = 3.1043$.

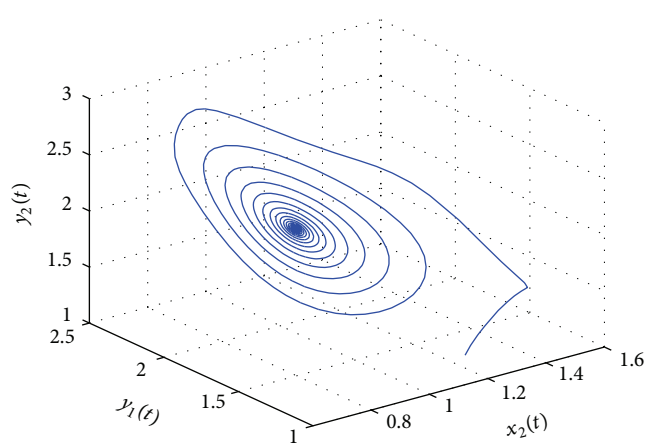

FIGURE 15: The phase plot of the states x_2, y_1, y_2 for $\tau = 2.7500 < \tau_0 = 3.1043$.

$7.5000 > \tau_2^* = 6.6494$, then the positive equilibrium $E^*(0.9167, 0.9167, 1.7361, 2.0833)$ becomes unstable and a Hopf bifurcation occurs. The corresponding waveforms and phase plots are illustrated in Figures 19, 20, 21, 22, 23, and 24.

In addition, for system (73), we have $C_1(0) = -31.3276 + 4.4079i$ by a simple computation. Further, from (72), we have $\mu_2 = 672.2661 > 0$, $\beta_2 = -62.6552 < 0$, and $T_2 = -17.0330 < 0$. Therefore, by Theorem 6, we can know that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the bifurcated periodic solutions decrease. Since the bifurcating periodic solutions of system (73) are stable, the predator and the prey species in system (73) can coexist in an oscillatory mode from the view of biology.

5. Conclusions

In this paper, a delayed predator-prey system with stage-structure for both the predator and the prey population is investigated. The sufficient conditions for the local stability of the positive equilibrium and the existence of Hopf bifurcation for the possible combinations of two delays are obtained.

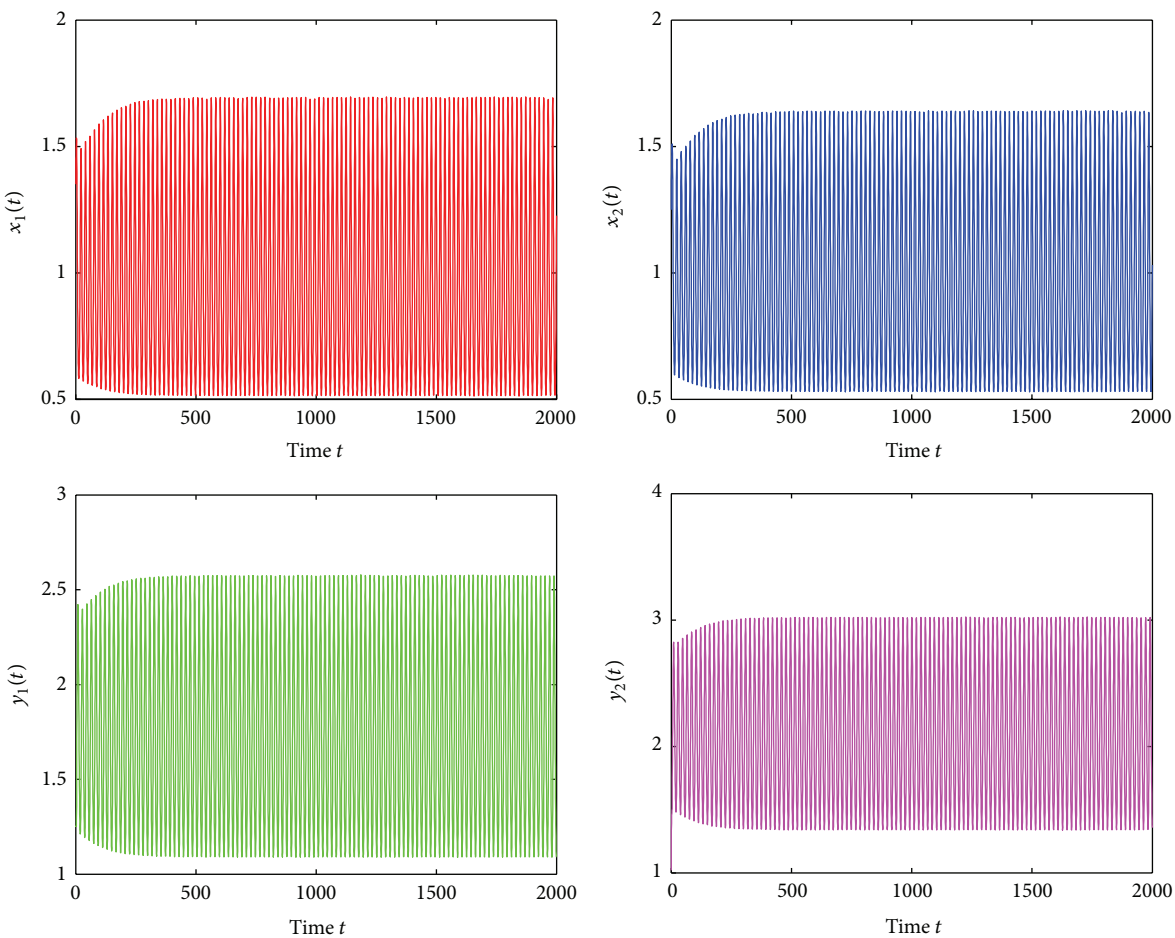

FIGURE 16: The track of the states x_1, x_2, y_1, y_2 for $\tau = 3.2500 > \tau_0 = 3.1043$.

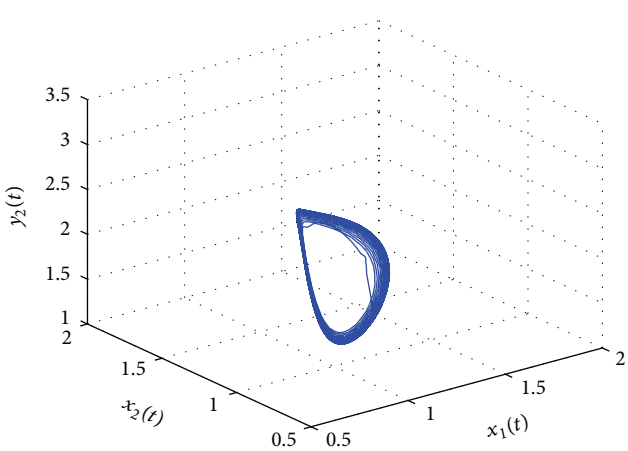

FIGURE 17: The phase plot of the states x_1, x_2, y_2 for $\tau = 3.2500 > \tau_0 = 3.1043$.

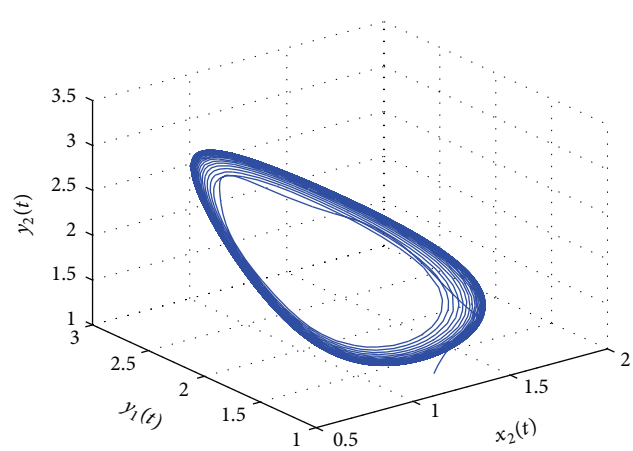

FIGURE 18: The phase plot of the states x_2, y_1, y_2 for $\tau = 3.2500 > \tau_0 = 3.1043$.

The main results are given in Theorems 2–5. The results show that the two time delays can play a complicated role in system (5). Furthermore, the explicit formulae determining the direction of the bifurcation and the stability of the bifurcating periodic solutions are established when $\tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$ by using the normal form theory and center manifold theorem. The main results are given in Theorem 6. If the bifurcating periodic solutions are stable, the predator and the prey species can coexist in an oscillatory mode from the view of biology. Compared with literature [17], we

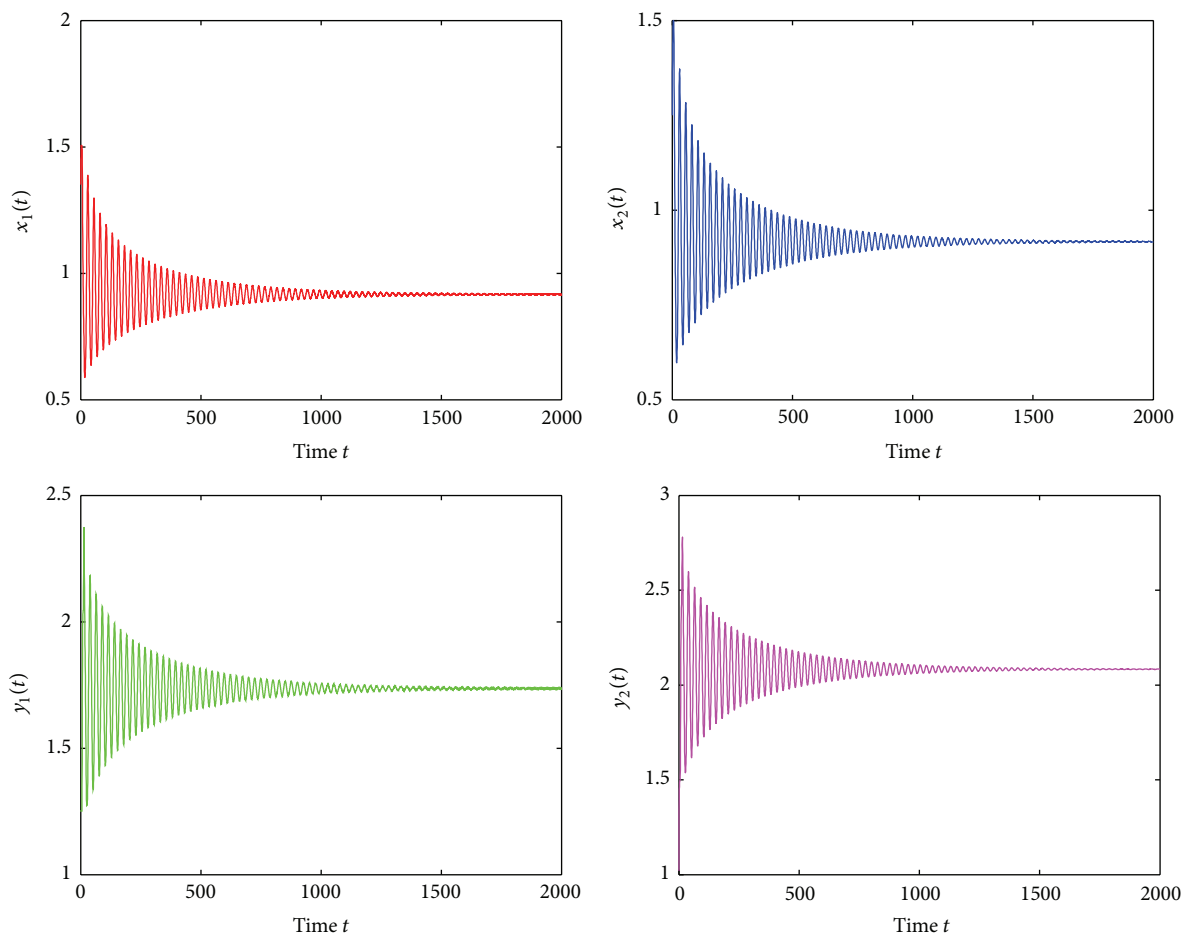

FIGURE 19: The track of the states x_1, x_2, y_1, y_2 for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

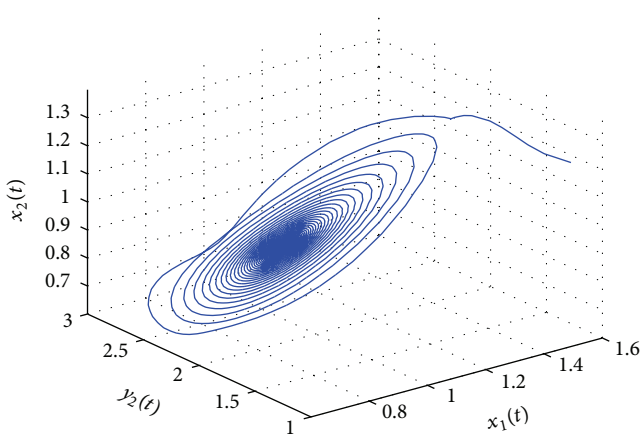

FIGURE 20: The phase plot of the states x_1, x_2, y_2 for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

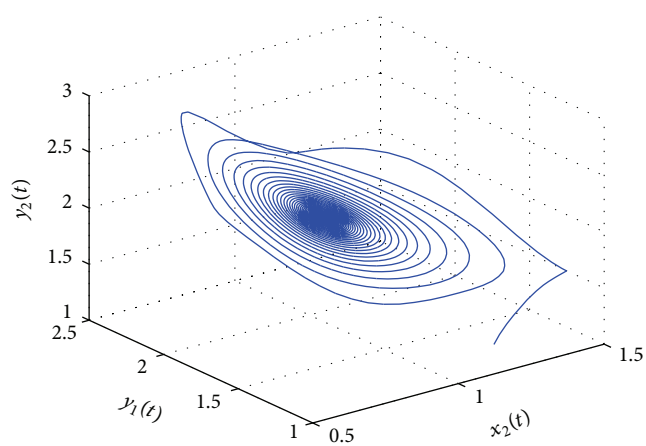

FIGURE 21: The phase plot of the states x_2, y_1, y_2 for $\tau_2 = 5.5500 < \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

consider not only the time delay due to the gestation of the mature predator but also the negative feedback delay of the immature prey. Wang [17] obtained that the species in system (5) with only the time delay due to the gestation of the mature predator could coexist under some conditions. However, we get that the species could also coexist with some available time delays of the immature prey and the mature predator under certain conditions. This is valuable from the view of ecology. Some numerical simulations are also included to support the obtained theoretical results.

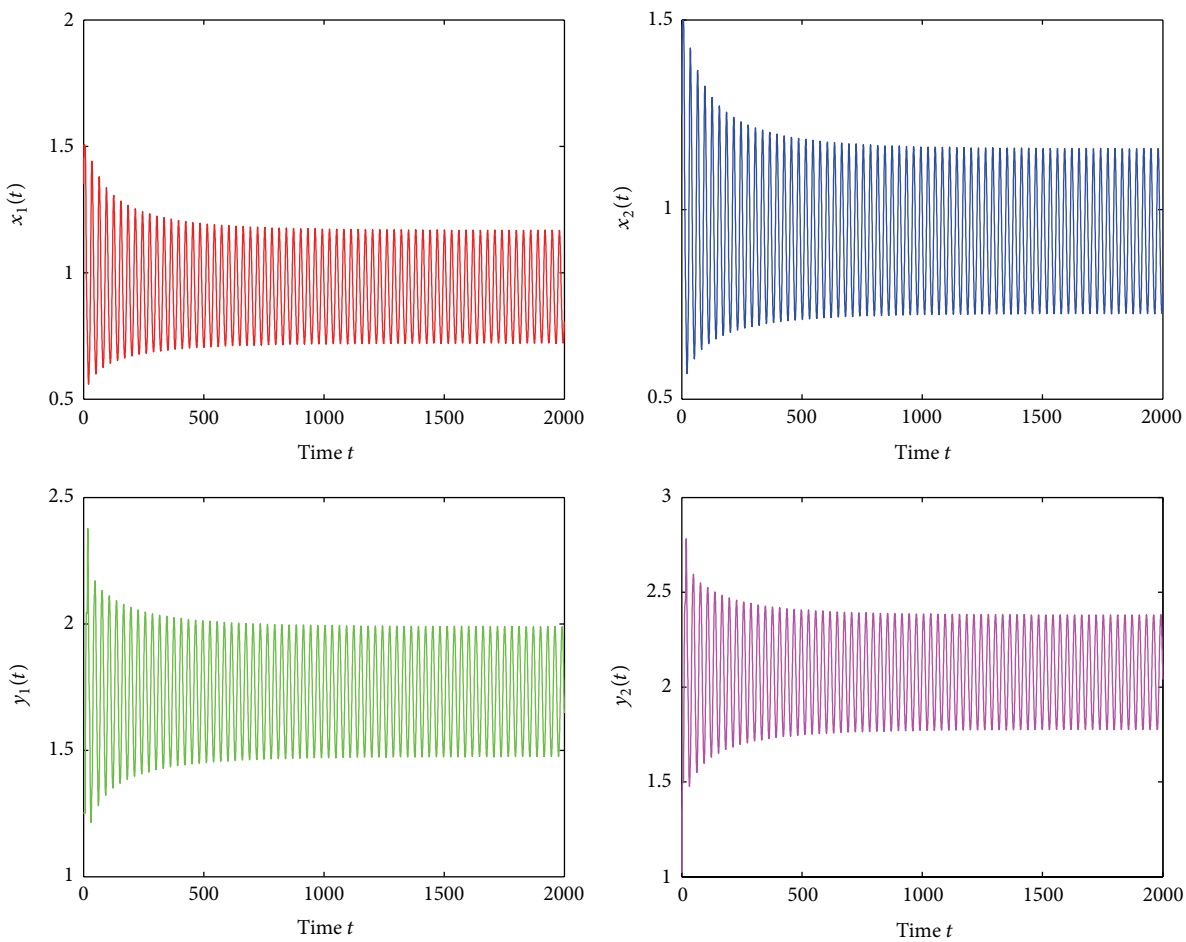

FIGURE 22: The track of the states x_1, x_2, y_1, y_2 for $\tau_2 = 7.5000 > \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

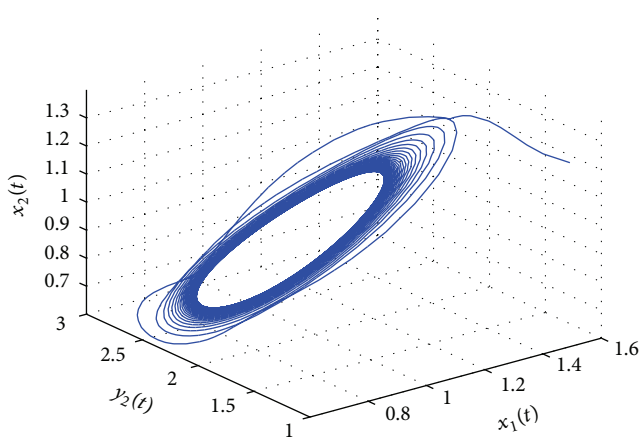

FIGURE 23: The phase plot of the states x_1, x_2, y_2 for $\tau_2 = 7.5000 > \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

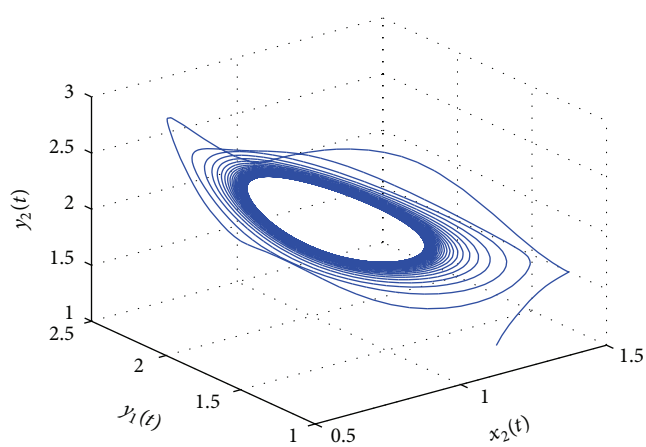

FIGURE 24: The phase plot of the states x_2, y_1, y_2 for $\tau_2 = 7.5000 > \tau_2^* = 6.6494$ and $\tau_1^* = 2.5000$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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