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*Research Article*

# Discrete Weighted Pseudo Asymptotic Periodicity of Second Order Difference Equations

**Zhinan Xia**

*Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, Zhejiang 310023, China*

Correspondence should be addressed to Zhinan Xia; xiazn299@zjut.edu.cn

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We define the concept of discrete weighted pseudo-$\mathcal{S}$-asymptotically periodic function and prove some basic results including composition theorem. We investigate the existence, and uniqueness of discrete weighted pseudo-$\mathcal{S}$-asymptotically periodic solution to nonautonomous semilinear difference equations. Furthermore, an application to scalar second order difference equations is given. The working tools are based on the exponential dichotomy theory and fixed point theorem.

## 1. Introduction

The concept of $\mathcal{S}$-asymptotic periodicity [1, 2] introduced by Henríquez et al. is natural generalization of asymptotic periodicity [3]. Since then, many contributions on the existence of $\mathcal{S}$-asymptotically periodic solutions for differential equations have been made; one can see [4–10] for more details. Further Pierri and Rolnik [11] introduced the notion of pseudo-$\mathcal{S}$-asymptotically periodic function, explored its properties, and investigated pseudo-$\mathcal{S}$-asymptotic periodicity of neutral differential equations with finite delay in Banach space. With the help of weighted function, weighted pseudo-$\mathcal{S}$-asymptotically periodic function is introduced and the applications to fractional integrodifferential equations are investigated in [12].

All the above-mentioned concepts are introduced in the continuous case, but it is rarely for the discrete type. For (discrete) $\mathcal{S}$-asymptotic periodicity, the subject has been studied in the recent paper [13], where the authors discussed the existence of (discrete) $\mathcal{S}$-asymptotically periodic solutions of semilinear difference equations with infinite delay. Motivated by the above literatures, it is natural to consider the discrete version of weighted pseudo-$\mathcal{S}$-asymptotically periodic function, which we will discuss in the present paper.

The rapid development of the theory of difference equations has been strongly promoted by the large number of applications in physics, engineering, biology, and other subjects. The asymptotic behaviour of solutions of difference equations is at present an active of research. Many researchers have made important contributions to these topics, for example, almost periodicity [14, 15], asymptotic almost periodicity [16, 17], almost automorphy [18–20], $l^p$-boundedness [21], stability [22, 23], and $\mathcal{S}$-asymptotic periodicity [13]. However, to the best of our knowledge, (discrete) weighted pseudo-$\mathcal{S}$-asymptotic periodicity of difference equations is quite new and an untreated topic. This is one of the key motivations of this study.

The principal aim of this paper is to introduce the concept of discrete weighted pseudo-$\mathcal{S}$-asymptotically periodic function, which is much more general to generalize discrete $\mathcal{S}$-asymptotically periodic function and explore its properties and applications in difference equations. The paper is organized as follows. In Section 2, first, some notations and preliminary results are presented. Next, we propose a new class of functions called discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic function, explore its properties, and establish the composition theorem. Section 3 is devoted to the existence and uniqueness of discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic solution of nonautonomous semilinear difference equations. In Section 4, discrete weighted pseudo-$\mathcal{S}$-asymptotic $\omega$-periodicity of the scalar second order difference equations is investigated.

## 2. Preliminaries and Basic Results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ stand for the set of natural numbers, integers, nonnegative integers, real numbers, nonnegative real numbers, and complex numbers, respectively. Let $A$ be bounded linear operator; $\sigma_p(A)$ denotes the point spectrum of $A$.

In order to facilitate the discussion below, we further introduce the following notations.

(i) $S^1 = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$.

(ii) $l^\infty(\mathbb{Z}, X) = \{x : \mathbb{Z} \to X : \|x\|_d := \sup_{n\in\mathbb{Z}}\|x(n)\| < \infty\}$.

(iii) $C_0(\mathbb{Z}, X) = \{x \in l^\infty(\mathbb{Z}, X) : \lim_{|n|\to\infty}\|x(n)\| = 0\}$.

(iv) $C_\omega(\mathbb{Z}, X) = \{x \in l^\infty(\mathbb{Z}, X) : x$ is $\omega$-priodic$\}$, where $\omega \in \mathbb{Z}^+ \setminus \{0\}$.

(v) $L(X, Y)$ is the Banach space of bounded linear operators from $X$ to $Y$ endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.

(vi) $\mathcal{UC}(\mathbb{Z} \times X, X)$ is the set of all functions $f : \mathbb{Z} \times X \to X$ satisfying that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$\|f(k, x) - f(k, y)\| \le \varepsilon \tag{1}$$

for all $k \in \mathbb{Z}$ and $x, y \in X$ with $\| x - y \| \le \delta$.

First, we recall the so-called Matkowski's fixed point theorem [24] and exponential dichotomy [25] which will be used in the sequel.

**Theorem 1** (Matkowski's fixed point theorem [24]). *Let $(X, d)$ be a complete metric space and let $\mathcal{F} : X \to X$ be a map such that*

$$d(\mathcal{F}x, \mathcal{F}y) \le \Phi(d(x, y)), \quad \forall x, y \in X, \tag{2}$$

*where $\Phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty}\Phi^n(t) = 0$ for all $t > 0$. Then $\mathcal{F}$ has a unique fixed point $z \in X$.*

Given a sequence $\{A(n)\}_{n\in\mathbb{Z}} \subset L(X)$ of invertible operators, define

$$\mathcal{A}(m, n) = \begin{cases} A(m-1)\cdots A(n), & \text{if } m > n, \\ \mathrm{Id}, & \text{if } m = n, \\ A^{-1}(m)\cdots A^{-1}(n-1), & \text{if } m < n, \end{cases} \tag{3}$$

where Id is the identity operator in $X$.

For the first order difference equation

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}. \tag{4}$$

*Definition 2* (see [25]). Equation (4) is said to have an exponential dichotomy if there exist projections $P(n) \in L(X)$ for all $n \in \mathbb{Z}$ and positive constants $\eta, \nu, \alpha, \beta$ such that

(i) $P(m)\mathcal{A}(m, n) = \mathcal{A}(m, n)P(n)$, $m, n \in \mathbb{N}$,

(ii) $\|\mathcal{A}(m, n)P(n)\| \le \eta e^{-\alpha(m-n)}$, $m \ge n$,

(iii) $\|\mathcal{A}(m, n)Q(n)\| \le \nu e^{-\beta(n-m)}$, $n \ge m$,

where $Q(n) = \mathrm{Id} - P(n)$ is the complementary projection of $P(n)$.

Next, we propose a new class of functions called discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic function and explore its properties including composition result.

Let $U$ denote the collection of functions (weights) $\rho : \mathbb{Z} \to (0, +\infty)$. For $\rho \in U$ and $n \in \mathbb{Z}^+$, set

$$\mu(n, \rho) := \sum_{k=-n}^{n} \rho(k). \tag{5}$$

Denote

$$U_\infty := \left\{\rho \in U : \lim_{n\to\infty}\mu(n, \rho) = \infty\right\},$$
$$U_B := \left\{\rho \in U_\infty : 0 < \inf_{k\in\mathbb{Z}}\rho(k) \le \sup_{k\in\mathbb{Z}}\rho(k) < \infty\right\}. \tag{6}$$

*Definition 3.* Let $\rho_1, \rho_2 \in U_\infty$. $\rho_1$ is said to be equivalent to $\rho_2$ (i.e., $\rho_1 \sim \rho_2$) if $(\rho_1/\rho_2) \in U_B$.

It is trivial to show that "$\sim$" is a binary equivalence relation on $U_\infty$. The equivalence class of a given weight $\rho \in U_\infty$ is denoted by $\mathrm{cl}(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho\in U_\infty}\mathrm{cl}(\rho)$.

Let $\rho \in U_\infty, m \in \mathbb{Z}$; define $\rho_m$ by $\rho_m(n) = \rho(n + m)$ for $n \in \mathbb{Z}$ and

$$U_T = \{\rho \in U_\infty : \rho \sim \rho_m \text{ for each } m \in \mathbb{Z}\}. \tag{7}$$

*Definition 4.* A function $f \in l^\infty(\mathbb{Z}, X)$ is called discrete asymptotically $\omega$-periodic if there exist $g \in C_\omega(\mathbb{Z}, X), \varphi \in C_0(\mathbb{Z}, X)$ such that $f = g + \varphi$. The collection of those functions is denoted by $AP_\omega(\mathbb{Z}, X)$.

*Definition 5.* A function $f \in l^\infty(\mathbb{Z}, X)$ is called discrete $\mathcal{S}$-asymptotically $\omega$-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty}(f(n + \omega) - f(n)) = 0$. The collection of those functions is denoted by $\mathcal{S}AP_\omega(\mathbb{Z}, X)$.

*Definition 6.* A function $f \in l^\infty(\mathbb{Z}, X)$ is called discrete pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{k=-n}^{n}\|f(k + \omega) - f(k)\| = 0. \tag{8}$$

Denote by $P\mathcal{S}AP_\omega(\mathbb{Z}, X)$ the set of such functions.

*Definition 7.* Let $\rho \in U_\infty$. A function $f \in l^\infty(\mathbb{Z}, X)$ is called discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic if there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty}\frac{1}{\mu(n, \rho)}\sum_{k=-n}^{n}\rho(k)\|f(k + \omega) - f(k)\| = 0. \tag{9}$$

Denote by $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ the set of such functions.

Next, we will show some properties of $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ including composition theorem.

**Lemma 8.** *Let $\rho \in U_\infty$; then the following properties hold:*

(i) $f \pm g \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ *if* $f, g \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$;

(ii) $\lambda f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ *if* $\lambda \in \mathbb{Z}$, $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$;

(iii) $AP_\omega(\mathbb{Z}, X) \subset \mathcal{S}AP_\omega(\mathbb{Z}, X) \subset P\mathcal{S}AP_\omega(\mathbb{Z}, X) \subset WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$;

(iv) $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ *is a Banach space when endowed with the sup norm*

$$\|f\|_d := \sup_{n\in\mathbb{Z}} \|f(n)\|. \tag{10}$$

*Proof.* The proof is straightforward, so the details are omitted here. □

**Lemma 9.** *Assume that $\rho_1, \rho_2 \in U_\infty$. If $\rho_1 \sim \rho_2$, then*

(i) $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1) = WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2)$;

(ii) $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1 + \rho_2) = WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1) = WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2)$;

(iii) $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1/\rho_2) = WP\mathcal{S}AP_\omega(\mathbb{Z}, X, cl(1)) = P\mathcal{S}AP_\omega(\mathbb{Z}, X)$;

(iv) *if $\rho \in U_B$, then $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho) = P\mathcal{S}AP_\omega(\mathbb{Z}, X)$.*

*Proof.* (i) Since $\rho_1 \sim \rho_2$, there exist $K_1 > 0$ and $K_2 > 0$ such that $K_1\rho_2 \le \rho_1 \le K_2\rho_2$; then

$$K_1\mu(n, \rho_2) \le \mu(n, \rho_1) \le K_2\mu(n, \rho_2). \tag{11}$$

Let $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2)$; then

$$\begin{aligned}
&\frac{1}{\mu(n,\rho_1)}\sum_{k=-n}^{n}\rho_1(k)\|f(k+\omega)-f(k)\| \\
&= \frac{1}{\mu(n,\rho_1)}\sum_{k=-n}^{n}\frac{\rho_1(k)}{\rho_2(k)}\rho_2(k)\|f(k+\omega)-f(k)\| \\
&\le \frac{K_2}{\mu(n,\rho_1)}\sum_{k=-n}^{n}\rho_2(k)\|f(k+\omega)-f(k)\| \\
&\le \frac{K_2}{K_1\mu(n,\rho_2)}\sum_{k=-n}^{n}\rho_2(k)\|f(k+\omega)-f(k)\|.
\end{aligned} \tag{12}$$

The fact that $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2)$ implies that

$$\lim_{n\to\infty}\frac{1}{\mu(n,\rho_2)}\sum_{k=-n}^{n}\rho_2(k)\|f(k+\omega)-f(k)\| = 0; \tag{13}$$

then

$$\lim_{n\to\infty}\frac{1}{\mu(n,\rho_1)}\sum_{k=-n}^{n}\rho_1(k)\|f(k+\omega)-f(k)\| = 0; \tag{14}$$

that is, $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1)$; hence

$$WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2) \subset WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1). \tag{15}$$

Proceeding in a similar manner, we have $WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_1) \subset WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho_2)$. Hence (i) holds.

From the proof of (i), it is not difficult to see that (ii), (iii), and (iv) hold, so the details are omitted here. The proof is completed. □

**Lemma 10.** *Let $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$, $\rho \in U_T$; then $f(\cdot + m) \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$ for all $m \in \mathbb{Z}$.*

*Proof.* Without loss of generality, we may assume that $m \in \mathbb{Z}^+$,

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+m+\omega)-f(k+m)\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k=-n+m}^{n+m}\rho_{-m}(k)\|f(k+\omega)-f(k)\| \\
&\le \frac{\mu(n+m,\rho)}{\mu(n,\rho)}\frac{1}{\mu(n+m,\rho)} \\
&\quad\times\sum_{k=-n-m}^{n+m}\frac{\rho_{-m}(k)}{\rho(k)}\rho(k)\|f(k+\omega)-f(k)\|,
\end{aligned} \tag{16}$$

since $\rho \in U_T$ implies that there exists $\kappa > 0$ such that $\rho_m(k)/\rho(k) \le \kappa$, $\rho_{-m}(k)/\rho(k) \le \kappa$. For $n > m$

$$\begin{aligned}
\mu(n+m,\rho) &= \sum_{k=-n-m}^{n+m}\rho(k) = \sum_{k=-n-m}^{n-m}\rho(k) + \sum_{k=n-m}^{n+m}\rho(k) \\
&\le \sum_{k=-n-m}^{n-m}\rho(k) + \sum_{k=-n+m}^{n+m}\rho(k) \\
&= \sum_{k=-n}^{n}\rho_{-m}(k) + \sum_{k=-n}^{n}\rho_m(k) \le 2\kappa\mu(n,\rho);
\end{aligned} \tag{17}$$

then

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+m+\omega)-f(k+m)\| \\
&\le \frac{2\kappa^2}{\mu(n+m,\rho)}\sum_{k=-n-m}^{n+m}\rho(k)\|f(k+\omega)-f(k)\|.
\end{aligned} \tag{18}$$

Note that $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$, $\rho \in U_T$; therefore $f(\cdot+m) \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$. □

We will establish composition theorem for discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic function.

**Lemma 11.** *Let $f \in l^\infty(\mathbb{Z}, X)$; then $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$, $\rho \in U_\infty$, if and only if, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k\in E_f(n,\varepsilon)}\rho(k) = 0, \tag{19}$$

*where $E_f(n,\varepsilon) = \{k \in [-n,n] \cap \mathbb{Z} : \|f(k+\omega) - f(k)\| \ge \varepsilon\}$.*

*Proof.* Sufficiency: it is clear that $M = \sup_{n\in\mathbb{Z}}\|f(n)\| < \infty$ and $\forall\varepsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$\frac{1}{\mu(n,\rho)}\sum_{k\in E_f(n,\varepsilon)}\rho(k) < \frac{\varepsilon}{2M}, \quad n > N. \tag{20}$$

Then, for $n > N$,

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+\omega)-f(k)\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k\in E_f(n,\varepsilon)}\rho(k)\|f(k+\omega)-f(k)\| \\
&\quad + \frac{1}{\mu(n,\rho)}\sum_{k\in([-n,n]\cap\mathbb{Z})\setminus E_f(n,\varepsilon)}\rho(k)\|f(k+\omega)-f(k)\| \\
&\le 2M\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in E_f(n,\varepsilon)}\rho(k) \\
&\quad + \varepsilon\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in([-n,n]\cap\mathbb{Z})\setminus E_f(n,\varepsilon)}\rho(k) \le 2\varepsilon,
\end{aligned} \tag{21}$$

so

$$\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+\omega)-f(k)\| = 0. \tag{22}$$

That is, $f \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$.

Necessity: suppose the contrary that there exists $\varepsilon_0 > 0$, such that $(1/\mu(n,\rho))\sum_{k\in E_f(n,\varepsilon)}\rho(k)$ does not converge to 0 as $n \to \infty$. That is, there exists $\delta > 0$, such that, for each $m \in \mathbb{Z}$,

$$\frac{1}{\mu(n_m,\rho)}\sum_{k\in E_f(n_m,\varepsilon_0)}\rho(k) \ge \delta, \quad \text{for some } n_m > m. \tag{23}$$

Then for $n_m > m$

$$\begin{aligned}
&\frac{1}{\mu(n_m,\rho)}\sum_{k=-n_m}^{n_m}\rho(k)\|f(k+\omega)-f(k)\| \\
&= \frac{1}{\mu(n_m,\rho)}\sum_{k\in E_f(n_m,\varepsilon_0)}\rho(k)\|f(k+\omega)-f(k)\| \\
&\quad + \frac{1}{\mu(n_m,\rho)}\sum_{k\in([-n_m,n_m]\cap\mathbb{Z})\setminus E_f(n_m,\varepsilon_0)}\rho(k)\|f(k+\omega)-f(k)\| \\
&\ge \frac{1}{\mu(n_m,\rho)}\sum_{k\in E_f(n_m,\varepsilon_0)}\rho(k)\|f(k+\omega)-f(k)\| \\
&\ge \frac{\varepsilon_0}{\mu(n_m,\rho)}\sum_{k\in E_f(n_m,\varepsilon_0)}\rho(k) \ge \delta\varepsilon_0,
\end{aligned} \tag{24}$$

which contradicts the fact that

$$\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+\omega)-f(k)\| = 0. \tag{25}$$

Thus (19) holds. □

**Theorem 12.** *Let $\rho \in U_\infty$. Assume that $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}\times X,X,\rho) \cap \mathcal{UC}(\mathbb{Z}\times X,X)$; then $\psi(\cdot) = f(\cdot,u(\cdot)) \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ if $u \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$.*

*Proof.* Since $f \in \mathcal{UC}(\mathbb{Z}\times X,X)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(k,u(k+\omega)) - f(k,u(k))\| \le L_f\varepsilon \tag{26}$$

for all $k \in \mathbb{Z}$ and $\| u(k+\omega) - u(k) \| \le \delta$. For the above $\varepsilon > 0$, since $f \in WP\mathcal{S}AP_\omega(\mathbb{Z}\times X,X,\rho)$, there exists $N \in \mathbb{N}$ such that, for $n > N$,

$$\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k+\omega,x)-f(k,x)\| \le \varepsilon \quad \text{for each } x \in X. \tag{27}$$

Denote

$$E_u(n,\delta) = \{k \in [-n,n]\cap\mathbb{Z} : \|u(k+\omega)-u(k)\| \ge \delta\}; \tag{28}$$

then $\lim_{n\to\infty}(1/\mu(n,\rho))\sum_{k\in E_u(n,\delta)}\rho(k) = 0$ by Lemma 11. So

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\|f(k,u(k+\omega)) - f(k,u(k))\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k\in E_u(n,\delta)}\rho(k)\|f(k,u(k+\omega)) - f(k,u(k))\| \\
&\quad + \frac{1}{\mu(n,\rho)} \\
&\quad \times \sum_{k\in([-n,n]\cap\mathbb{Z})\setminus E_u(n,\delta)}\rho(k)\|f(k,u(k+\omega)) - f(k,u(k))\| \\
&\le 2\|\psi\|_d\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in E_u(n,\delta)}\rho(k) \\
&\quad + \frac{1}{\mu(n,\rho)}\sum_{k\in([-n,n]\cap\mathbb{Z})\setminus E_u(n,\delta)}\rho(k)L_f\varepsilon \\
&\le 2\|\psi\|_d\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in E_u(n,\delta)}\rho(k) + \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)L_f\varepsilon \\
&\le 2\|\psi\|_d\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in E_u(n,\delta)}\rho(k) + L_f\varepsilon,
\end{aligned} \tag{29}$$

where $\|\psi\|_d = \sup_{n\in\mathbb{Z}}\|\psi(n)\|$.

For $n > N$, one has

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|f(k+\omega,u(k+\omega))-f(k,u(k))\right\| \\
&\le \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|f(k+\omega,u(k+\omega))\right. \\
&\qquad\qquad \left.-f(k,u(k+\omega))\right\| \\
&\quad+\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|f(k,u(k+\omega))-f(k,u(k))\right\| \\
&\le \varepsilon+2\|\psi\|_d\cdot\frac{1}{\mu(n,\rho)}\sum_{k\in E_u(n,\delta)}\rho(k)+L_f\varepsilon.
\end{aligned} \tag{30}$$

Due to the arbitrariness of $\varepsilon$, one has

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|f(k+\omega,u(k+\omega))-f(k,u(k))\right\| \\
&\quad=0,
\end{aligned} \tag{31}$$

which implies that $\psi(\cdot)\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$. □

**Corollary 13.** *Let $\rho\in U_\infty$. Assume that $f\in WP\mathcal{S}AP_\omega(\mathbb{Z}\times X,X,\rho)$ and there exists a constant $L_f>0$ such that*

$$\left\|f(k,u)-f(k,v)\right\|\le L_f\|u-v\|,\quad \forall k\in\mathbb{Z},\ u,v\in X. \tag{32}$$

*Then $\psi(\cdot)=f(\cdot,u(\cdot))\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ if $u\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$.*

## 3. Nonautonomous Semilinear Difference Equations

In this section, consider the following nonautonomous semilinear difference equations:

$$u(n+1)=A(n)u(n)+f(n,u(n)),\quad n\in\mathbb{Z}. \tag{33}$$

Its associated homogeneous linear difference equation is given by

$$u(n+1)=A(n)u(n),\quad n\in\mathbb{Z}. \tag{34}$$

To establish our results, we introduce the following conditions.

$(H_1)$ Let $f\in WP\mathcal{S}AP_\omega(\mathbb{Z}\times X,X,\rho)$, $\rho\in U_T$.

$(H_2)$ There exists a constant $L_f>0$ such that

$$\left\|f(k,u)-f(k,v)\right\|\le L_f\|u-v\|,\quad \forall k\in\mathbb{Z},\ u,v\in X. \tag{35}$$

$(H_2')$ There exists a linear nondecreasing function $\Phi:[0,\infty)\to[0,\infty)$ such that

$$\left\|f(k,u)-f(k,v)\right\|\le \Phi(\|u-v\|),\quad \forall k\in\mathbb{Z},\ u,v\in X. \tag{36}$$

$(H_3)$ Equation (34) admits an exponential dichotomy on $\mathbb{Z}$ with projection $P(n)$, positive constants $\eta,\nu,\alpha,\beta$.

$(H_4)$ For all $x\in X$, $\mathcal{A}(m+\omega,n+\omega)P(n+\omega)x=\mathcal{A}(m,n)P(n)x$.

**Theorem 14.** *Assume that $(H_1)$–$(H_4)$ hold and $\vartheta L_f<1$, where*

$$\vartheta:=\frac{\eta}{1-e^{-\alpha}}+\frac{\nu e^{-\beta}}{1-e^{-\beta}}; \tag{37}$$

*then (33) has a unique solution $u(n)\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ which is given by*

$$\begin{aligned}
u(n)&=\sum_{j=-\infty}^{n-1}\mathcal{A}(n,j+1)P(j+1)f(j,u(j)) \\
&\quad-\sum_{j=n}^{\infty}\mathcal{A}(n,j+1)Q(j+1)f(j,u(j)),\quad n\in\mathbb{Z}.
\end{aligned} \tag{38}$$

*Proof.* Similarly as the proof of [26, 27], it can be shown that $u(\cdot)$ given by (38) is the solution of (33). Define the operator $\Gamma: WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)\to WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ as follows:

$$\begin{aligned}
(\Gamma u)(n)&:=\sum_{j=-\infty}^{n-1}\mathcal{A}(n,j+1)P(j+1)f(j,u(j)) \\
&\quad-\sum_{j=n}^{\infty}\mathcal{A}(n,j+1)Q(j+1)f(j,u(j)),\quad n\in\mathbb{Z}.
\end{aligned} \tag{39}$$

Since $u\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ and $(H_2)$ holds, then $\Psi(\cdot)=f(\cdot,u(\cdot))\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ by Corollary 13.

Next, we show that $\Gamma u\in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$. In fact, let

$$(\Gamma u)(n):=(\Gamma_1 u)(n)-(\Gamma_2 u)(n), \tag{40}$$

where

$$\begin{aligned}
(\Gamma_1 u)(n)&=\sum_{j=-\infty}^{n-1}\mathcal{A}(n,j+1)P(j+1)\Psi(j), \\
(\Gamma_2 u)(n)&=\sum_{j=n}^{\infty}\mathcal{A}(n,j+1)Q(j+1)\Psi(j);
\end{aligned} \tag{41}$$

then,

$$\begin{aligned}
&\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|(\Gamma_1 u)(k+\omega)-(\Gamma_1 u)(k)\right\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|\sum_{j=-\infty}^{k+\omega-1}\mathscr{A}(k+\omega,j+1)P(j+1)\Psi(j)\right. \\
&\qquad\left.-\sum_{j=-\infty}^{k-1}\mathscr{A}(k,j+1)P(j+1)\Psi(j)\right\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|\sum_{j=-\infty}^{k-1}\mathscr{A}(k,j+1)P(j+1)\right. \\
&\qquad\left.\times\left[\Psi(j+\omega)-\Psi(j)\right]\right\| \\
&\le \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\sum_{j=-\infty}^{k-1}\eta e^{-\alpha(k-j-1)}\rho(k)\left\|\Psi(j+\omega)-\Psi(j)\right\| \\
&= \frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\sum_{j=0}^{\infty}\eta e^{-\alpha j}\rho(k)\left\|\Psi(k-1-j+\omega)\right. \\
&\qquad\left.-\Psi(k-1-j)\right\| \\
&= \sum_{j=0}^{\infty}\eta e^{-\alpha j}\left(\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|\Psi(k-1-j+\omega)\right.\right. \\
&\qquad\left.\left.-\Psi(k-1-j)\right\|\right).
\end{aligned} \tag{42}$$

Since $\Psi \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$, $\rho \in U_T$, then $\Psi(\cdot-1-j) \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ for each $j \in \mathbb{Z}$ by Lemma 10; hence

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|\Psi(k-1-j+\omega)-\Psi(k-1-j)\right\| \\
&\quad = 0.
\end{aligned} \tag{43}$$

By Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|(\Gamma_1 u)(k+\omega)-(\Gamma_1 u)(k)\right\| \\
&\le \sum_{j=0}^{\infty}\eta e^{-\alpha j}\left(\lim_{n\to\infty}\frac{1}{\mu(n,\rho)}\sum_{k=-n}^{n}\rho(k)\left\|\Psi(k-1-j+\omega)\right.\right. \\
&\qquad\left.\left.-\Psi(k-1-j)\right\|\right) \\
&= 0;
\end{aligned} \tag{44}$$

hence $\Gamma_1 u \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$. Similarly, one can prove $\Gamma_2 u \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$. So $\Gamma$ is well defined.

For $u,v \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$, by $(H_2)$ and exponential dichotomy, one has

$$\begin{aligned}
&\left\|(\Gamma u)(n)-(\Gamma v)(n)\right\| \\
&= \left\|\sum_{j=-\infty}^{n-1}\mathscr{A}(n,j+1)P(j+1)\left[f(j,u(j))-f(j,v(j))\right]\right. \\
&\qquad\left.-\sum_{j=n}^{\infty}\mathscr{A}(n,j+1)Q(j+1)\left[f(j,u(j))-f(j,v(j))\right]\right\| \\
&\le \sum_{j=-\infty}^{n-1}\eta e^{-\alpha(n-j-1)}L_f\left\|u(j)-v(j)\right\| \\
&\qquad+\sum_{j=n}^{\infty}\nu e^{-\beta(j+1-n)}L_f\left\|u(j)-v(j)\right\| \\
&\le \eta L_f\|u-v\|_d\sum_{j=0}^{\infty}e^{-\alpha j}+\nu L_f\|u-v\|_d\sum_{j=1}^{\infty}e^{-\beta j} \\
&\le \frac{\eta L_f}{1-e^{-\alpha}}\|u-v\|_d+\frac{\nu L_f e^{-\beta}}{1-e^{-\beta}}\|u-v\|_d=\vartheta L_f\|u-v\|_d;
\end{aligned} \tag{45}$$

hence $\Gamma$ is a contraction. By the Banach contraction mapping principle, $\Gamma$ has a unique fixed point $u \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$, which is the unique $WP\mathcal{S}AP_\omega$ solution of (33). The proof is completed. □

**Theorem 15.** *Assume that $(H_1)$, $(H_2')$, $(H_3)$, and $(H_4)$ hold; then (33) has a unique solution $u(n) \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$ if $(\vartheta\Phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$.*

*Proof.* Define the operator $\Gamma$ as in (39), so $\Gamma$ is well defined. For $u,v \in WP\mathcal{S}AP_\omega(\mathbb{Z},X,\rho)$, one has

$$\begin{aligned}
&\left\|(\Gamma u)(n)-(\Gamma v)(n)\right\| \\
&\le \sum_{j=-\infty}^{n-1}\eta e^{-\alpha(n-j-1)}\left\|f(j,u(j))-f(j,v(j))\right\| \\
&\qquad+\sum_{j=n}^{\infty}\nu e^{-\beta(j+1-n)}\left\|f(j,u(j))-f(j,v(j))\right\| \\
&\le \sum_{j=-\infty}^{n-1}\eta e^{-\alpha(n-j-1)}\Phi\left(\left\|u(j)-v(j)\right\|\right) \\
&\qquad+\sum_{j=n}^{\infty}\nu e^{-\beta(j+1-n)}\Phi\left(\left\|u(j)-v(j)\right\|\right) \\
&\le \Phi(\|u-v\|)\sum_{j=0}^{\infty}\eta e^{-\alpha j}+\Phi(\|u-v\|)\sum_{j=1}^{\infty}\nu e^{-\beta j} \\
&\le \vartheta\Phi(\|u-v\|).
\end{aligned} \tag{46}$$

Since $(\vartheta\Phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$, by Matkowski fixed point theorem (Theorem 1), $\Gamma$ has a unique fixed point $u \in WP\mathcal{S}AP_\omega(\mathbb{Z}, X, \rho)$, which is the unique $WP\mathcal{S}AP_\omega$ solution of (33). □

*Example 16.* Consider the system

$$u(n+1) = Au(n) + h(n)g(u), \quad n \in \mathbb{Z}, \tag{47}$$

where $A$ is a nonsingular $k \times k$ matrix such that $\sigma_p(A) \cap S^1 = \emptyset$, $h \in WP\mathcal{S}AP_\omega(\mathbb{Z}, \mathbb{R}^k, \rho)$, and $\rho \in U_T$ and there exists a constant $L_g > 0$ such that

$$\|g(u) - g(v)\| \le L_g \|u - v\|, \quad u, v \in \mathbb{R}^k. \tag{48}$$

Since $\sigma_p(A) \cap S^1 = \emptyset$, the system

$$u(n+1) = Au(n), \quad n \in \mathbb{Z}, \tag{49}$$

admits an exponential dichotomy with positive constants $\eta, \nu, \alpha, \beta$ [25] and $(\mathrm{H}_2)$ holds with $L_f = L_g\|h\|_d$. By Theorem 14, if we suppose that $\vartheta L_g\|h\|_d < 1$, then (47) has a unique discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic solution.

## 4. Scalar Second Order Difference Equations

Let $X = \mathbb{R}$; we study the existence and uniqueness of discrete weighted pseudo-$\mathcal{S}$-asymptotically $\omega$-periodic solutions to a scalar second order difference equation given by

$$\begin{aligned} &u(n+2) + b(n)u(n+1) + a(n)u(n) \\ &= f(n, u(n)), \quad n \in \mathbb{Z}, \end{aligned} \tag{50}$$

where $f \in WP\mathcal{S}AP_\omega(\mathbb{Z} \times \mathbb{R}, \mathbb{R}, \rho)$ and $\rho \in U_T$ and satisfies the following.

$(A_1)$ The function $(n, x) \to f(n, x)$ is Lipschitz in $x \in \mathbb{R}^2$ uniformly in $n \in \mathbb{Z}$; that is, there exists a constant $L_f > 0$ such that

$$\|f(n, x) - f(n, y)\| \le L_f \|x - y\| \quad x, y \in \mathbb{R}^2, \ n \in \mathbb{Z}. \tag{51}$$

$(A_2)$ $a, b : \mathbb{Z} \to \mathbb{R}$ are periodic functions, in the sense that there exists $\omega \in \mathbb{Z}^+$ such that

$$a(n+\omega) = a(n), \quad b(n+\omega) = b(n), \quad n \in \mathbb{Z}. \tag{52}$$

$(A_3)$ There exist $a_0, b_0 > 0$ such that $\inf_{n\in\mathbb{Z}} a(n) = a_0$ and $\inf_{n\in\mathbb{Z}} b(n) = b_0$.

$(A_4)$ Let $b(n) \ne 2\sqrt{a(n)}$ for all $n \in \mathbb{Z}$.

Now, let

$$A(n) = \begin{pmatrix} 0 & 1 \\ -a(n) & -b(n) \end{pmatrix}, \quad n \in \mathbb{Z}; \tag{53}$$

then (50) can be rewritten as the abstract form (33). It is not difficult to see that

$$\det\left(A(n) - \lambda I_{\mathbb{R}^2}\right) = \lambda^2 + b(n)\lambda + a(n), \quad n \in \mathbb{Z}. \tag{54}$$

Let $D(n) = b^2(n) - 4a(n)$ for all $n \in \mathbb{Z}$; then $(A_4)$ implies that either $D(n) > 0$ or $D(n) < 0$ for all $n \in \mathbb{Z}$. By $(A_3)$ and $(A_4)$, one has what follows.

(1) If $D(n) > 0$ for all $n \in \mathbb{Z}$, then the eigenvalues of $A(n)$ are given by

$$\begin{aligned} \lambda_1(n) &= \frac{-b(n) + \sqrt{b^2(n) - 4a(n)}}{2}, \\ \lambda_2(n) &= \frac{-b(n) - \sqrt{b^2(n) - 4a(n)}}{2}. \end{aligned} \tag{55}$$

Moreover, it can be easily shown that $\lambda_1(n) < 0, \lambda_2(n) < 0$ for all $n \in \mathbb{Z}$.

(2) If $D(n) < 0$ for all $n \in \mathbb{Z}$, then the eigenvalues of $A(n)$ are given by

$$\begin{aligned} \lambda_1(n) &= \frac{-b(n) + i\sqrt{4a(n) - b^2(n)}}{2}, \\ \lambda_2(n) &= \frac{-b(n) - i\sqrt{4a(n) - b^2(n)}}{2}. \end{aligned} \tag{56}$$

Moreover, it can be easily shown that $\mathfrak{Re}\lambda_1(n) < 0, \mathfrak{Re}\lambda_2(n) < 0$ for all $n \in \mathbb{Z}$.

In view of the above, it follows that the homogeneous linear difference equation

$$u(n+1) = A(n)u(n), \quad n \in \mathbb{Z}, \tag{57}$$

has an exponential dichotomy on $\mathbb{Z}$ [28]. Since $A(n+\omega) = A(n)$, $(H_4)$ holds. By Theorem 14, one has the following.

**Theorem 17.** *Under assumptions $(A_1)$–$(A_4)$, (50) has a unique solution $u(n) \in WP\mathcal{S}AP_\omega(\mathbb{Z}, \mathbb{R}, \rho)$ whenever $L_f$ is small enough.*

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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